MULTIWAVELETS ASSOCIATED WITH COUNTABLE GROUPS OF UNITARY OPERATORS IN HILBERT SPACES

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Abstract. Let $\mathcal{G}$ be a countable group of unitary operators on a complex separable Hilbert space H. We give a characterization of biorthogonality among Riesz multiwavelets in terms of certain invariant properties of their associated core spaces. A large family of non-biorthogonal Riesz multiwavelets is exhibited. We also discuss some results on linear perturbation of orthonormal multiwavelets.

1. Introduction

In this paper, we consider wavelet-type problems associated with countable groups of unitary operators on a separable complex Hilbert space. Other results in such and similar settings can be found in [1, 2, 4, 5, 7, 8, 12, 13, 14, 15, 17].

In section 2 of this paper, we use operator-theoretic methods to obtain and clarify results related to "orthonormalization" of frames and Riesz bases in Hilbert spaces. In section 3, we give a characterization of biorthogonality among Riesz multiwavelets in terms of certain invariant properties of their associated core spaces. A large family of non-biorthogonal Riesz multiwavelets is exhibited. In section 4, we discuss some results on linear perturbation of orthonormal multiwavelets.

Let us set up some notations and terminologies. Throughout this paper, let H denote a separable complex Hilbert space. The inner product of two vectors x and y in H is denoted by $\langle x, y\rangle$. A countable indexed family $\{v_n\}_{n\in J}$ of vectors in H is a *frame* for its closed linear span $V = \overline{span}\{v_n\}_{n\in J}$ if there exist positive constants

Date: March 30, 2005.

1991 *Mathematics Subject Classification.* 46C99, 47B99, 46B15.

Key words and phrases. Biorthogonal systems, Riesz bases, multiwavelets, unitary operators.

A and B such that

$$A\|f\|^2 \le \sum |\langle f, v_n\rangle|^2 \le B\|f\|^2, \quad \forall f \in V; \tag{1.1}$$

the family $\{v_n\}$ is a *tight frame* (respectively *normalized tight frame*) for V if the above condition holds with $A = B$ (respectively $A = B = 1$). A countable indexed family $\{v_n\}_{n\in J}$ is a *Riesz basis* for its closed linear span V if there exist positive constants A and B such that

$$A \sum |a_n|^2 \le \| \sum a_n v_n\|^2 \le B \sum |a_n|^2, \quad \forall \{a_n\} \in \ell^2(J). \tag{1.2}$$

It is well known that a Riesz basis for a Hilbert space is a frame for the same space. Two families $\{v_n\}$ and $\{\tilde{v}_n\}$ in H are *biorthogonal* if

$$\langle v_n, \tilde{v}_m\rangle = \delta_{n,m} \quad \forall n,\, m\,. \tag{1.3}$$

If V and W are closed linear subspaces of H such that $V \cap W = \{0\}$ and the vector sum $V_1 = V + W$ is closed, then we write $V_1 = V \oplus W$ and call this a *direct sum*. In this case, the map $P : V_1 \longrightarrow V_1$ defined by

$$P(v + w) = v, \quad v \in V,\, w \in W, \tag{1.4}$$

is called the (oblique) projection of V_1 on V along W. For the special case when V and W are orthogonal, we shall write $V_1 = V \oplus^{\perp} W$ and call this an *orthogonal* direct sum. We write $V^{\perp}$ for the orthogonal complement of V in H.

The space of all bounded linear maps on H is denoted by $B(H)$. A unitary system $\mathcal{U}$ in $B(H)$ is a set of unitary operators on H which contains the identity operator on H.

Let G be a discrete group. For every g in G, let χ_g denote the characteristic function of $\{g\}$. Then $\{\chi_g : g \in G\}$ is an orthonormal basis for $\ell^2(G)$. For each g in G, define $l_g : \ell^2(G) \longrightarrow \ell^2(G)$ by $(l_g a)(h) = a(g^{-1}h),\ h \in G$. Then $l_g(\chi_h) = \chi_{gh}$ for all g, h in G. For an indexed set J, the space $\ell^2(G \times J)$ can be identified with the spaces $\ell^2(G) \otimes \ell^2(J)$ and $\ell^2(G, \ell^2(J))$. For each g in G, define $L_g : \ell^2(G, \ell^2(J)) \longrightarrow \ell^2(G, \ell^2(J))$ by $(L_g a)(h) = a(g^{-1}h),\ h \in G$. Then $L_g = l_g \otimes I$, where I is the identity operator on $\ell^2(J)$.

2. Orthonormalization and polar decomposition

In this section, we show how operator theory has important implications in results related to "orthonormalization" of frames and Riesz bases.

Let $X = \{x_n\}_{n\in J}$ be a frame for $V = \overline{span}\{x_n\}_{n\in J} \subseteq H$. Let $T = T_X : \ell^2(J) \longrightarrow V$ be defined by

$$T(\{a_n\}) = \sum_n a_n x_n, \quad \forall \{a_n\} \in \ell^2(J). \tag{2.1}$$

It is well known that in this case T is a surjective bounded linear operator. Also, its adjoint $T^* = T_X^* : V \longrightarrow \ell^2(J)$, given by

$$T^*(f) = \{\langle f, x_n\rangle\}, \quad \forall f \in V, \tag{2.2}$$

is bounded below on V and so has closed range. Moreover, the bounded operator $S = TT^* : V \longrightarrow V$, given by

$$Sf = \sum_n \langle f, x_n\rangle x_n, \quad \forall f \in V, \tag{2.3}$$

is invertible. (For the case that $X = \{x_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for V, then both T and T^* are invertible.)

Let $T = U|T|$ and $T^* = Y|T^*| = YS^{1/2}$ be the polar decompositions of T and T^* respectively, where U and Y are partial isometries with $kerU = kerT$ and $kerY = kerT^*$, and $|A| = (A^*A)^{1/2}$ for an operator A (see [6, Problem 134]). We elucidate the following folk-result by appealing to operator-theoretic properties of polar decompositions. (See also Proposition 1.5, Proposition 1.10 and Remark 1.12 in [7] for other related results.)

Proposition 2.1. *Let $\{x_j\}_{j\in J}$ be a frame (respectively a Riesz basis) for a closed linear subspace V of H. Then*

(i) $U = Y^* = S^{-1/2}T$;

(ii) $\{S^{-1/2}x_j\}_{j\in J} = \{Ue_j\}_{j\in J}$ *is a normalized tight frame (respectively an orthonormal basis) for V, where $e_j = \{\delta_{jk}\}_{k\in J}$, $j \in J$.*

Proof. Since T^* is injective, $Y : V \longrightarrow \ell^2(J)$ is isometric. As $Y = T^*S^{-1/2}$, we have $|T|Y = (T^*T)^{1/2}T^*(TT^*)^{-1/2} = T^*(TT^*)^{1/2}(TT^*)^{-1/2} = T^*$. Taking adjoints, we obtain $T = Y^*|T|$. Since $kerY^* = kerT = kerU$ (as $Y^* = S^{-1/2}T$), by uniqueness of polar decomposition, $Y^* = U$. Since Y is isometric, for every f in V,

$$\sum_n |\langle f, Y^*e_n\rangle|^2 = \sum_n |\langle Yf, e_n\rangle|^2 = \|Yf\|^2 = \|f\|^2.$$

Therefore, $\{Y^*e_n\}_{n\in J}$ is a normalized tight frame for V. Moreover, for each n, $Y^*e_n = S^{-1/2}Te_n = S^{-1/2}x_n$.

If $X = \{x_n\}_{n\in J}$ is a Riesz basis for V, then U, Y and Y^* are all unitary. Hence $\{Y^*e_n\}_{n\in J}$ is an orthonormal basis for V. □

For the rest of this section, let $\mathcal{G}$ be a countable (hence discrete) group of unitary operators on H, let $Y = \{y_j : j \in J\}$ be a countable indexed family of vectors in H, and let $V = \overline{span}\mathcal{G}(Y)$.

Proposition 2.2. *Suppose that $\mathcal{G}(Y) = \{gy_j : g \in \mathcal{G}, j \in J\}$ is a frame (respectively a Riesz basis) for V. Then there exists a countable indexed family $\widehat{Y} = \{\hat{y}_j : j \in J\}$ of vectors in V such that $\mathcal{G}(\widehat{Y}) = \{g\hat{y}_j : g \in \mathcal{G}, j \in J\}$ is a normalized tight frame (respectively an orthonormal basis) for V.*

Proof. Suppose that $\mathcal{G}(Y)$ is a frame (respectively a Riesz basis) for $V = \overline{span}\mathcal{G}(Y)$. By (2.3), the operator $S = TT^* : V \longrightarrow V$ is of the form

$$Sf = \sum_{j\in J}\sum_{g\in\mathcal{G}} \langle f, gy_j\rangle gy_j, \quad \forall f \in V. \tag{2.4}$$

Since $\mathcal{G}$ is a group of unitary operators, by routine calculations we have

$$g\mid_V S = Sg\mid_V \quad \forall g \in \mathcal{G}. \tag{2.5}$$

Hence $g\mid_V S^{-1/2} = S^{-1/2}g\mid_V$ for every g in $\mathcal{G}$. It follows from Proposition 2.1 that $\mathcal{G}(\widehat{Y})$ is a normalized tight frame (respectively an orthonormal basis) for V, where $\hat{y}_j = S^{-1/2}y_j$, $j \in J$, and $\widehat{Y} = \{\hat{y}_j : j \in J\}$. □

Suppose that $\mathcal{G}(Y)$ is a frame for V. Identifying the space $\ell^2(\mathcal{G}\times J)$ with the spaces $\ell^2(\mathcal{G})\otimes\ell^2(J)$ and $\ell^2(\mathcal{G},\ell^2(J))$, the operator $T:\ell^2(\mathcal{G},\ell^2(J))\longrightarrow V$ in (2.1) becomes

$$T(a)=\sum_{j\in J}\sum_{g\in\mathcal{G}}a_j(g)gy_j,\quad \forall\, a=\sum a_j(g)\chi_g\otimes e_j. \tag{2.6}$$

It can be easily checked that

$$gT=TL_g\quad \forall\, g\in\mathcal{G}. \tag{2.7}$$

Taking adjoints above with g replaced by g^{-1}, we have

$$T^*g=L_gT^*\quad \forall\, g\in\mathcal{G}. \tag{2.8}$$

Recall that $T^*:V\longrightarrow\ell^2(\mathcal{G},\ell^2(J))$ is bounded below and $T^*=YS^{1/2}$, where $Y:V\longrightarrow\ell^2(\mathcal{G},\ell^2(J))$ is an isometry with $ranY=ranT^*$.

Proposition 2.3. *Let P be the orthogonal projection of $\ell^2(\mathcal{G},\ell^2(J))$ onto $ranY$. Then $L_gP=PL_g$ for every g in $\mathcal{G}$.*

Proof. First observe that $PT^*=PY|PT^*|$. Considering as operators from V to $ranY$, the above is the polar decomposition of PT^* and PY is unitary. By (2.8),

$$PT^*g=L_gPT^*\quad \forall\, g\in\mathcal{G}.$$

Hence for every $g\in\mathcal{G}$, we have

$$PYg=L_gPY,$$

and so

$$YgY^*=PYgY^*=L_gPYY^*=L_gPP=L_gP,\ \forall\, g\in\mathcal{G}.$$

Taking adjoints above (with g replaced by g^{-1}), we get

$$YgY^*=PL_g,\quad \forall\, g\in\mathcal{G}.$$

Hence $L_gP=PL_g$ for every g in $\mathcal{G}$. ∎

3. Characterization of biorthogonality among Riesz multiwavelets

Let $\mathcal{U}$ be a unitary system in $B(H)$, and let r be a positive integer. A vector $\psi = (\psi_1, ..., \psi_r)$ in $H^r := \underbrace{H \oplus \ldots \oplus H}_{r-fold}$ is an *orthonormal multiwavelet* (respectively, a *Riesz multiwavelet*) of multiplicity r for $\mathcal{U}$ if $\{U\psi_i : U \in \mathcal{U}, i = 1, ..., r\}$ is an orthonormal basis (respectively, a Riesz basis) for H. Denote by $\mathcal{W}^r(\mathcal{U})$ (respectively by $\mathcal{R}^r(\mathcal{U})$) the set of all orthonormal multiwavelets (respectively Riesz multiwavelets) of multiplicity r for $\mathcal{U}$. Obviously $\mathcal{W}^r(\mathcal{U}) \subseteq \mathcal{R}^r(\mathcal{U})$. A vector $\eta = (\eta_1, ..., \eta_r)$ in $\mathcal{R}^r(\mathcal{U})$ is a *biorthogonal* Riesz multiwavelet if there exists a vector $\tilde{\eta} = (\tilde{\eta}_1, ..., \tilde{\eta}_r)$ in H^r such that

$$\langle U\eta_i, V\tilde{\eta}_j\rangle = \delta_{U,V}\delta_{i,j}, \quad U, V \in \mathcal{U},\ i, j = 1, ..., r. \tag{3.1}$$

It is easy to see that such a vector $\tilde{\eta}$ is necessarily unique and is in $\mathcal{R}^r(\mathcal{U})$.

Let $\psi = (\psi_1, ..., \psi_r) \in H^r$. Following [4], we define the *local commutant* of $\mathcal{U}$ at ψ to be the set

$$\mathcal{C}^r_\psi(\mathcal{U}) = \{A \in B(H) : AU\psi_i = UA\psi_i,\ U \in \mathcal{U}, i = 1, ..., r\}. \tag{3.2}$$

For any operator A on H, write $A\psi = (A\psi_1, ..., A\psi_r)$. We have several occasions to use the next simple lemma (cf. [4, Lemma 1.1]). We leave its proof to the reader.

Lemma 3.1. *Let $A \in \mathcal{C}^r_\psi(\mathcal{U})$ and $B \in B(H)$. Then $B \in \mathcal{C}^r_{A\psi}(\mathcal{U})$ if and only if $BA \in \mathcal{C}^r_\psi(\mathcal{U})$. Moreover if $A \in \mathcal{C}^r_\psi(\mathcal{U})$ is invertible, then $\mathcal{C}^r_{A\psi}(\mathcal{U}) = \mathcal{C}^r_\psi(\mathcal{U})A^{-1}$.*

Let $\eta = (\eta_1, ..., \eta_r) \in \mathcal{R}^r(\mathcal{U})$. In this case, the frame operator $S : H \longrightarrow H$ as given by (2.3) takes the form

$$Sf = \sum_{i=1}^{r} \sum_{U\in\mathcal{U}} \langle f, U\eta_i\rangle U\eta_i, \quad f \in H, \tag{3.3}$$

and it is an invertible positive operator on H. For every $j = 1, ..., r$, $V \in \mathcal{U}$, we have

$$V\eta_j = S(S^{-1}V\eta_j) = \sum_{i=1}^{r} \sum_{U\in\mathcal{U}} \langle S^{-1}V\eta_j, U\eta_i\rangle U\eta_i.$$

Therefore

$$\langle S^{-1}V\eta_j, U\eta_i\rangle = \delta_{U,V}\delta_{i,j}, \quad U,V\in\mathcal{U},\ i,j=1,...,r, \tag{3.4}$$

i.e., $\{S^{-1}U\eta_i : U\in\mathcal{U},\ i=1,...,r\}$ is biorthogonal to $\{U\eta_i : U\in\mathcal{U},\ i=1,...,r\}$. It follows from (3.4) and (3.1) that

Proposition 3.2. *η is biorthogonal if and only if $S^{-1}\in\mathcal{C}^r_\eta(\mathcal{U})$.*

If $\mathcal{W}^r(\mathcal{U})\neq\emptyset$, there is another characterization of biorthogonality. The first two parts of the next result are analogous to [4, Proposition 1.3].

Proposition 3.3. *Assume that $\mathcal{W}^r(\mathcal{U})\neq\emptyset$. Fix a vector $\psi=(\psi_1,...,\psi_r)\in\mathcal{W}^r(\mathcal{U})$. Let $\Phi : B(H)\longrightarrow H^r$ be defined by*

$$\Phi(A) = (A\psi_1,...,A\psi_r), \quad A\in B(H).$$

(i) *Φ maps the set of all unitary operators in $\mathcal{C}^r_\psi(\mathcal{U})$ bijectively onto $\mathcal{W}^r(\mathcal{U})$.*

(ii) *Φ maps the set of all invertible operators in $\mathcal{C}^r_\psi(\mathcal{U})$ bijectively onto $\mathcal{R}^r(\mathcal{U})$.*

(iii) *Let $\eta=(\eta_1,...,\eta_r)\in\mathcal{R}^r(\mathcal{U})$ and let $A\in\mathcal{C}^r_\psi(\mathcal{U})$ be invertible such that $A\psi_i = \eta_i$, $i=1,...,r$. Then η is biorthogonal if and only if $A^{*-1}\in\mathcal{C}^r_\psi(\mathcal{U})$.*

Proof. Parts (i) and (ii) follow from similar arguments as in the proof of [4, Proposition 1.3].

(iii): First observe that by (3.3) and the assumption that $A\in\mathcal{C}^r_\psi(\mathcal{U})$, the frame operator S satisfies $S=AA^*$, since

$$\begin{aligned} Sf &= \sum_{i=1}^r\sum_{U\in\mathcal{U}}\langle f, U\eta_i\rangle U\eta_i \\ &= \sum_{i=1}^r\sum_{U\in\mathcal{U}}\langle f, AU\psi_i\rangle AU\psi_i \\ &= A\left(\sum_{i=1}^r\sum_{U\in\mathcal{U}}\langle A^*f, U\psi_i\rangle U\psi_i\right) \\ &= AA^*f \end{aligned}$$

for every $f \in H$. Since $A^{*-1} = S^{-1}A$, the desired result follows from Proposition 3.2 and Lemma 3.1 $\square$

For the rest of this section, let $\mathcal{U}$ be a unitary system of the form $\mathcal{U} = \mathcal{U}_0\mathcal{G}$ such that

(1) $\mathcal{U}_0 = \{D^n : n \in \mathbb{Z}\}$ for some unitary operator D on H and $\mathcal{G}$ is a countable (but not necessarily abelian) group of unitary operators on H,
(2) there exists a *non-surjective* map $\sigma : \mathcal{G} \longrightarrow \mathcal{G}$ satisfying $gD = D\sigma(g)$ for every g in $\mathcal{G}$, and
(3) $D^n g = I$ only if $n = 0$ and $g = I$.

We remark that this set-up includes, as a special case, the usual setting in the wavelet literature: for $H = L^2(\mathbb{R}^d)$, x in $\mathbb{R}^d$ and f in $L^2(\mathbb{R}^d)$, the dilation operator D is defined by

$$(Df)(x) = |det(M)|^{\frac{1}{2}} f(Mx),$$

where M is a $d \times d$ matrix with integer entries and $|det(M)| > 1$, and the (abelian) group $\mathcal{G}$ is generated by the translation operators $U_1, \dots, U_d$, given by

$$(U_k f)(x) = f(x - e_k),$$

where $e_k = (\delta_{k,j})_{j=1,\dots,d}$ for $k = 1, \dots, d$.

For the time being, we do *not* assume that $\mathcal{W}^r(\mathcal{U}) \neq \emptyset$. Note that condition (2) implies that σ is an injective homomorphism and

$$gD^j = D^j\sigma^j(g), \quad g \in \mathcal{G}, j \geq 0. \tag{3.5}$$

For a vector $\phi = (\phi_1, ..., \phi_r) \in H^r$, let

$$V_n(\phi) = \overline{span}\{D^j g\phi_i : j < n,\, g \in \mathcal{G},\, i = 1, ..., r\}, \quad n \in \mathbb{Z}. \tag{3.6}$$

Obviously $V_n(\phi) \subseteq V_{n+1}(\phi)$ and $V_{n+1}(\phi) = D(V_n(\phi))$ for every n in $\mathbb{Z}$.

Let $\eta = (\eta_1, ..., \eta_r) \in \mathcal{R}^r(\mathcal{U})$. In this case, the frame operator $S : H \longrightarrow H$ as given by (3.3) now takes the form

$$Sf = \sum_{i=1}^{r}\sum_{n\in\mathbb{Z}}\sum_{g\in\mathcal{G}}\langle f, D^n g\eta_i\rangle D^n g\eta_i, \quad f \in H. \tag{3.7}$$

and

(3.8)

$$\langle S^{-1}D^l h\eta_j, D^n g\eta_i\rangle = \delta_{l,n}\delta_{h,g}\delta_{j,i}, \quad l,n\in\mathbb{Z},\ g,h\in\mathcal{G},\ i,j=1,...,r,$$

i.e., $\{S^{-1}D^n g\eta_i : n\in\mathbb{Z},\ g\in\mathcal{G},\ i=1,...,r\}$ is biorthogonal to $\{D^n g\eta_i : n\in\mathbb{Z},\ g\in\mathcal{G},\ i=1,...,r\}$. Note also that $SD=DS$, since for every f in H, we have

$$SDf = \sum_{i=1}^{r}\sum_{n\in\mathbb{Z}}\sum_{g\in\mathcal{G}}\langle Df, D^n g\eta_i\rangle D^n g\eta_i = D\left(\sum_{i=1}^{r}\sum_{n\in\mathbb{Z}}\sum_{g\in\mathcal{G}}\langle f, D^{n-1}g\eta_i\rangle D^{n-1}g\eta_i\right) = DSf.$$

By (3.7), for every f in H,

$$f = S^{-1}Sf = \sum_{i=1}^{r}\sum_{n\in\mathbb{Z}}\sum_{g\in\mathcal{G}}\langle f, D^n g\eta_i\rangle S^{-1}D^n g\eta_i. \tag{3.9}$$

It follows from (3.6), (3.8) and (3.9) that for every n in $\mathbb{Z}$,

$$V_n(\eta)^\perp = \overline{span}\{S^{-1}D^j g\eta_i : j\geq n,\ g\in\mathcal{G},\ i=1,...,r\}. \tag{3.10}$$

We need the following two elementary results, which are of independent interest. We omit the proof of the first lemma.

Lemma 3.4. *Let M and N be linear subspaces of a vector space X such that $X = M\oplus N$ (i.e., $X=M+N$ and $M\cap N=\{0\}$). Let P be the (oblique) projection of X on M along N, and $A : X\longrightarrow X$ be a linear map. Then $AP=PA$ if and only if $A(M)\subseteq M$ and $A(N)\subseteq N$.*

Lemma 3.5. *Let M, M' and N be linear subspaces of a vector space X such that*

$$X = M\oplus N = M'\oplus N.$$

Let P be the projection of X on M along N and let Q be the projection of X on M' along N. Then $P_1 = P|_{M'} : M'\longrightarrow M$ and $Q_1 = Q|_M : M\longrightarrow M'$ are invertible, and $P_1^{-1} = Q_1$.

Proof. If $f'\in M'$ and $Pf'=0$, then $f'\in M'\cap N=\{0\}$. Therefore P_1 is injective. Also, $P_1(M') = P(M'+N) = P(X) = M$. The same arguments hold for Q_1. For every f' in M', $f' = u+v$ for some $u\in M$, $v\in N$. Then $u = f'-v\in M'+N$, and $Q_1P_1f' = Q_1u = f'$. Therefore $Q_1P_1 = id_{M'}$. Similarly, $P_1Q_1 = id_M$. □

The next result gives characterizations of biorthogonality among Riesz multiwavelets η in terms of certain invariant properties of the associated core spaces $V_0(\eta)$.

Theorem 3.6. *Let $\eta = (\eta_1, ..., \eta_r) \in \mathcal{R}^r(\mathcal{U})$. The following conditions are equivalent:*

(a) *η is biorthogonal;*

(b) *$g(V_0(\eta)) = V_0(\eta), \quad g \in \mathcal{G}$;*

(c) *there exists $\mu = (\mu_1, ..., \mu_r) \in \mathcal{W}^r(\mathcal{U})$ such that $V_0(\eta) = V_0(\mu)$.*

Proof. We shall first prove the equivalence (a) $\Longleftrightarrow$ (b). Suppose that η is biorthogonal. By Proposition 3.2 and (3.10),

$$V_0(\eta)^\perp = \overline{span}\{D^j g\tilde{\eta}_i : j \geq 0, \, g \in \mathcal{G}, \, i = 1, ..., r\},$$

where $\tilde{\eta}_i = S^{-1}\eta_i$, $i = 1, ..., r$. Hence by (3.5), we have

$$g(V_0(\eta)^\perp) \subseteq V_0(\eta)^\perp, \quad \forall g \in \mathcal{G}.$$

It follows that (b) holds.

Conversely, suppose that (b) holds. We claim that

$$(*) := \langle gS^{-1}\eta_i, D^n h\eta_j \rangle = \delta_{0,n}\delta_{g,h}\delta_{i,j}, \; n \in \mathbb{Z}, \; g, h \in \mathcal{G}, \; i, j = 1, ..., r.$$

If $n < 0$, then for all $h \in \mathcal{G}$ and $j = 1, ..., r$, $D^n h\eta_j \in V_0(\eta)$. Hence by (b), $g^{-1}D^n h\eta_j \in V_0(\eta)$ for every $g \in \mathcal{G}$. By (3.10), $S^{-1}\eta_i \in V_0(\eta)^\perp$ for every $i = 1, ..., r$. In this case $(*) = 0$.

For $n \geq 0$, by (3.5),

$$(*) = \langle \eta_i, S^{-1}g^{-1}D^n h\eta_j \rangle = \langle \eta_i, S^{-1}D^n\sigma^n(g^{-1})h\eta_j \rangle.$$

If $n > 0$, then by (3.8), $(*) = 0$. For $n = 0$, again by (3.8),

$$(*) = \langle \eta_i, S^{-1}g^{-1}h\eta_j \rangle = \delta_{g,h}\delta_{i,j}.$$

This completes the proof of the claim. Since $span\{D^n h\eta_j : n \in \mathbb{Z}, h \in \mathcal{G}, j = 1, ..., r\}$ is dense in H and

$$\langle S^{-1}g\eta_i, D^n h\eta_j \rangle = \delta_{0,n}\delta_{g,h}\delta_{i,j}, \; n \in \mathbb{Z}, \; g, h \in \mathcal{G}, \; i, j = 1, ..., r,$$

comparing the above with (∗), we conclude that

$$gS^{-1}\eta_i = S^{-1}g\eta_i, \quad g \in \mathcal{G},\ i = 1, ..., r. \tag{3.11}$$

Since S commutes with D,

$$D^n gS^{-1}\eta_i = S^{-1}D^n g\eta_i, \quad n \in \mathbb{Z},\ g \in \mathcal{G},\ i = 1, ..., r.$$

Hence $S^{-1} \in \mathcal{C}^r_\eta(\mathcal{U})$. By Proposition 3.2, η is biorthogonal.

Obviously (c) $\Longrightarrow$ (b), for in this case μ is in particular biorthogonal and we can use the implication (a) $\Longrightarrow$ (b) proven earlier. Suppose then (b) holds. For every $n \in \mathbb{Z}$, let

$$W_n(\eta) = \overline{span}\{D^n g\eta_i : g \in \mathcal{G},\ i = 1, ..., r\}, \tag{3.12}$$

and

$$L_n(\eta) = V_{n+1}(\eta) \cap V_n(\eta)^\perp. \tag{3.13}$$

We have the decompositions

$$V_1(\eta) = V_0(\eta) \oplus W_0(\eta) = V_0(\eta) \oplus^\perp L_0(\eta).$$

By (2) and (b),

$$g(V_1(\eta)) = g(D(V_0(\eta))) = D(\sigma(g)(V_0(\eta))) = D(V_0(\eta)) = V_1(\eta), \quad g \in \mathcal{G}.$$

It is obvious that both $W_0(\eta)$ and $L_0(\eta)$ are invariant under all g in $\mathcal{G}$. Let $P : V_1(\eta) \longrightarrow V_1(\eta)$ be the orthogonal projection of $V_1(\eta)$ on $L_0(\eta)$. By Lemma 3.4 and Lemma 3.5, P commutes with $g|_{V_1(\eta)}$ for every $g \in \mathcal{G}$ and $P|_{W_0(\eta)} : W_0(\eta) \longrightarrow L_0(\eta)$ is invertible. Since $\{g\eta_i : g \in \mathcal{G},\ i = 1, ..., r\}$ is a Riesz basis for $W_0(\eta)$, $\{gP\eta_i : g \in \mathcal{G},\ i = 1, ..., r\}$ is a Riesz basis for $L_0(\eta)$. By Proposition 2.2, there exist $\mu_1, ..., \mu_r$ in $L_0(\eta)$ such that $\{g\mu_i : g \in \mathcal{G},\ i = 1, ..., r\}$ is an orthonormal basis for $L_0(\eta)$. It follows from standard Hilbert space arguments that $L_0(\eta)$ is a complete wandering subspace of H for D, $\mu = (\mu_1, ..., \mu_r) \in \mathcal{W}^r(\mathcal{U})$ and for all $n \in \mathbb{Z}$,

$$V_n(\eta) = \sum_{j<n} \oplus^\perp L_j(\eta) = \sum_{j<n} \oplus^\perp D^j L_0(\eta) = V_n(\mu).$$

□

Remark. After we had obtained Theorem 3.6, we received from Professsor H. O. Kim the preprint [10], where he and his coauthors had also proved independently a special case of Theorem 3.6. Another special case can be found in [11].

Corollary 3.7. *$\mathcal{W}^r(\mathcal{U}) \neq \emptyset$ if and only if there exists a biorthogonal Riesz multiwavelet in $\mathcal{R}^r(\mathcal{U})$.*

The following example is motivated by the discussion in [9, pp. 415-417]. It shows that under our setting and provided that $\mathcal{W}^r(\mathcal{U}) \neq \emptyset$, there is an abundance of non-biorthogonal Riesz multiwavelets.

Example 3.8. Let $\psi = (\psi_1, ..., \psi_r) \in \mathcal{W}^r(\mathcal{U})$. Define an operator $V : H \longrightarrow H$ by

$$V(D^n g\psi_i) = D^{n+1}\sigma(g)\psi_i, \quad n \in \mathbb{Z},\, g \in \mathcal{G},\, i = 1, ..., r. \tag{3.14}$$

It is routine to check that V is an isometry in the local commutant $\mathcal{C}^r_\psi(\mathcal{U})$ of $\mathcal{U}$ at ψ,

$$V^*(D^n\sigma(g)\psi_i) = D^{n-1}g\psi_i, \quad n \in \mathbb{Z},\, g \in \mathcal{G},\, i = 1, ..., r, \tag{3.15}$$

and

$$V^*(D^n h\psi_i) = 0, \quad n \in \mathbb{Z},\, h \in \mathcal{G} \setminus \sigma(\mathcal{G}),\, i = 1, ..., r. \tag{3.16}$$

Note that by (3.16), $V^* \notin \mathcal{C}^r_\psi(\mathcal{U})$. Let $t \in \mathbb{C}$ with $0 < |t| < 1$. Then $A = I - tV$ is an invertible operator in $\mathcal{C}^r_\psi(\mathcal{U})$. Let $\eta_i = A\psi_i$, $i = 1, ..., r$. By Proposition 3.3, $\eta = (\eta_1, ..., \eta_r) \in \mathcal{R}^r(\mathcal{U})$. We will show that η is *not* biorthogonal.

First recall that the frame operator $S : H \longrightarrow H$ associated with the Riesz basis $\{D^n g\eta_i : n \in \mathbb{Z},\, g \in \mathcal{G},\, i = 1, ..., r\}$ satisfies $S = AA^*$, and the dual basis of $\{D^n g\eta_i : n \in \mathbb{Z},\, g \in \mathcal{G},\, i = 1, ..., r\}$ is given by $\{S^{-1}D^n g\eta_i : n \in \mathbb{Z},\, g \in \mathcal{G},\, i = 1, ..., r\}$. For every $n \in \mathbb{Z}$, $g \in \mathcal{G}$, $i = 1, ..., r$, we have

$$S^{-1}D^n g\eta_i = (AA^*)^{-1}AD^n g\psi_i = (A^*)^{-1}D^n g\psi_i = \sum_{p=0}^{\infty} \bar{t}^p V^{*p}(D^n g\psi_i).$$

We claim that for every $i = 1, ..., r$, there exists no μ_i in H such that

$$D^n g\mu_i = S^{-1}D^n g\eta_i, \quad n \in \mathbb{Z},\, g \in \mathcal{G}.$$

Suppose on the contrary that for some $i = 1, ..., r$, such an μ_i exists. Then in particular

$$\mu_i = S^{-1}\eta_i = \sum_{p=0}^{\infty} \bar{t}^p V^{*p}\psi_i = \sum_{p=0}^{\infty} \bar{t}^p D^{-p}\psi_i$$

by (3.15). Take any $h \in \mathcal{G} \setminus \sigma(\mathcal{G})$. By (3.16),

$$h\mu_i = S^{-1}h\eta_i = \sum_{p=0}^{\infty} \bar{t}^p V^{*p}(h\psi_i) = h\psi_i.$$

Then $\psi_i = \mu_i = \psi_i + \sum_{p=1}^{\infty} \bar{t}^p D^{-p}\psi_i$. Hence $\sum_{p=1}^{\infty} \bar{t}^p D^{-p}\psi_i = 0$, which contradicts the orthonormality of $\{D^n\psi_i : n \in \mathbb{Z}\}$.

For some special types of frames, we have the following result related to Theorem 3.6.

Proposition 3.9. *Suppose $\eta = (\eta_1, \dots, \eta_r)$ is a frame wavelet, i.e. $\{D^n g\eta_i : n \in \mathbb{Z},\ g \in \mathcal{G},\ i = 1, \dots, r\}$ is a frame for H, and S is the associated frame operator given by (3.7) such that $S^{-1} \in \mathcal{C}_\eta^r(\mathcal{U})$. Then*

$$g(V_0(\eta)) = V_0(\eta), \quad g \in \mathcal{G}. \tag{3.17}$$

Proof. Define operators X and Y by

$$X = \sum_{i=1}^{r} \sum_{n<0} \sum_{g\in\mathcal{G}} \langle S^{-1}\cdot, D^n g\eta_i\rangle D^n g\eta_i \tag{3.18}$$

$$Y = \sum_{i=1}^{r} \sum_{n\geq 0} \sum_{g\in\mathcal{G}} \langle S^{-1}\cdot, D^n g\eta_i\rangle D^n g\eta_i \tag{3.19}$$

Note that $X + Y = SS^{-1} = I$. Furthermore, for every $h \in \mathcal{G}$, since $S^{-1} \in \mathcal{C}_\eta^r(\mathcal{U})$,

$$\begin{aligned}
Yh &= \sum_{i=1}^{r} \sum_{n \geq 0} \sum_{g \in \mathcal{G}} \langle S^{-1} h \cdot, D^n g \eta_i \rangle D^n g \eta_i \\
&= \sum_{i=1}^{r} \sum_{n \geq 0} \sum_{g \in \mathcal{G}} \langle \cdot, h^{-1} D^n g S^{-1} \eta_i \rangle D^n g \eta_i \\
&= \sum_{i=1}^{r} \sum_{n \geq 0} \sum_{g \in \mathcal{G}} \langle \cdot, D^n \sigma^n(h^{-1}) g S^{-1} \eta_i \rangle D^n g \eta_i \\
&= \sum_{i=1}^{r} \sum_{n \geq 0} \sum_{g' \in \mathcal{G}} \langle \cdot, D^n g' S^{-1} \eta_i \rangle D^n \sigma^n(h) g' \eta_i \\
&= \sum_{i=1}^{r} \sum_{n \geq 0} \sum_{g' \in \mathcal{G}} \langle S^{-1} \cdot, D^n g' \eta_i \rangle h D^n g' \eta_i \\
&= hY,
\end{aligned}$$

by the reindexing of $\mathcal{G}$ via $g \to \sigma^n(h)g$. Therefore, for every $g \in \mathcal{G}$, we have $gX = Xg$ as well, whence it follows that the closure of the range of X is invariant under g. We claim that the closure of the range of X is $V_0(\eta)$. Clearly, by definition, the range of X is contained in $V_0(\eta)$. Now, suppose $v \in H$ is perpendicular to the range of X. Then

$$\begin{aligned}
0 &= \langle XSv, v \rangle \\
&= \left\langle \sum_{i=1}^{r} \sum_{n < 0} \sum_{g \in \mathcal{G}} \langle S^{-1} Sv, D^n g \eta_i \rangle D^n g \eta_i, v \right\rangle \\
&= \sum_{i=1}^{r} \sum_{n < 0} \sum_{g \in \mathcal{G}} |\langle v, D^n g \eta_i \rangle|^2.
\end{aligned}$$

Thus, v is perpendicular to $V_0(\eta)$, hence the range of X is dense in $V_0(\eta)$. It follows that (3.17) holds. □

The converse of Proposition 3.9 is false (see [3]).

Corollary 3.10. *Let* $\eta = (\eta_1, ..., \eta_r) \in H^r$. *Suppose that* $\{D^n g \eta_i : n \in \mathbb{Z},\ g \in \mathcal{G},\ i = 1, ..., r\}$ *is either*

(i) *a tight frame for H, or*

(ii) *a semi-orthogonal frame for H, i.e., it is a frame for H such that*

$$\langle D^n g\eta_i, D^m h\eta_j\rangle = 0, \quad n \neq m \in \mathbb{Z},\ g, h \in \mathcal{G},\ i, j = 1, ..., r.$$

Then

$$g(V_0(\eta)) = V_0(\eta), \quad g \in \mathcal{G}.$$

Proof. (i) Suppose first that $\{D^n g\eta_i : n \in \mathbb{Z},\ g \in \mathcal{G},\ i = 1, ..., r\}$ is a tight frame for H, with frame constant c. Then the frame operator S equals cI, a scalar multiple of the identity operator I on H. Hence the desired result follows from Proposition 3.9.

(ii) Suppose next that $\{D^n g\eta_i : n \in \mathbb{Z},\ g \in \mathcal{G},\ i = 1, ..., r\}$ is a semi-orthogonal frame for H. Then

$$H = \sum_{n\in\mathbb{Z}} \oplus^{\perp} W_n(\eta), \quad \text{where} \quad W_n(\eta) = \overline{span}\{D^n g\eta_i : g \in \mathcal{G},\ i = 1, ..., r\}.$$

Hence $\{g\eta_i : g \in \mathcal{G},\ i = 1, ..., r\}$ is a frame for $W_0(\eta)$. By Proposition 2.2, there exist $\psi_1, \dots, \psi_r \in W_0(\eta)$ such that $\{g\psi_i : g \in \mathcal{G},\ i = 1, ..., r\}$ is a normalized tight frame for $W_0(\eta)$. Then for every $n \in \mathbb{Z}$, $\{D^n g\psi_i : g \in \mathcal{G},\ i = 1, ..., r\}$ is a normalized tight frame for $W_n(\eta)$. Hence $\{D^n g\psi_i : n \in \mathbb{Z}, g \in \mathcal{G},\ i = 1, ..., r\}$ is a normalized tight frame for H, and

$$V_0(\psi) = \sum_{n<0} \oplus^{\perp} W_n(\eta) = V_0(\eta),$$

where $\psi = (\psi_1, \dots, \psi_r)$. By (i),

$$g(V_0(\eta)) = g(V_0(\psi)) = V_0(\psi) = V_0(\eta), \quad g \in \mathcal{G}.$$

□

4. Linear perturbation of orthonormal multiwavelets

In this section, consider first a unitary system $\mathcal{U}$ in $B(H)$ such that $\mathcal{W}^r(\mathcal{U}) \neq \emptyset$.

Lemma 4.1. *Suppose $\psi \in \mathcal{W}^r(\mathcal{U})$ and $B \in \mathcal{C}^r_\psi(\mathcal{U})$ but $B^{*n} \notin \mathcal{C}^r_\psi(\mathcal{U})$ for some positive integer n. Then there exists a $\gamma > 0$ such that for every complex ε with $0 < |\varepsilon| < \gamma$, the vector $\psi + \varepsilon B\psi$ is in $\mathcal{R}^r(\mathcal{U})$ but it is not biorthogonal.*

Proof. Without loss of generality, replace B with $-B$. Also, assume that n is the smallest positive integer such that $B^{*n} \notin \mathcal{C}_\psi^r(\mathcal{U})$. For every sufficently small nonzero ε, $I - \varepsilon B$ is invertible and we have the expansion

$$(I - \varepsilon B^*)^{-1} = I + \varepsilon B^* + (\varepsilon B^*)^2 + (\varepsilon B^*)^3 + \dots . \tag{4.1}$$

Write

$$(I - \varepsilon B^*)^{-1} - \sum_{k=0}^{n-1} (\varepsilon B^*)^k = (\varepsilon B^*)^n + (\varepsilon B^*)^{n+1}(I - \varepsilon B^*)^{-1},$$

so

$$B^{*n} = C_\varepsilon - \varepsilon B^{*n+1}(I - \varepsilon B^*)^{-1},$$

where

$$C_\varepsilon = \frac{(I - \varepsilon B^*)^{-1} - \sum_{k=0}^{n-1} (\varepsilon B^*)^k}{\varepsilon^n}.$$

Hence

$$\|B^{*n} - C_\varepsilon\| = \|\varepsilon B^{*n+1}(I - \varepsilon B^*)^{-1}\| \leq |\varepsilon| \|B\|^{n+1} / (1 - |\varepsilon| \|B\|).$$

Therefore, $C_\varepsilon \longrightarrow B^{*n}$ as $\varepsilon \longrightarrow 0$.

Since $\mathcal{C}_\psi^r(\mathcal{U})$ is closed and $B^{*n} \notin \mathcal{C}_\psi^r(\mathcal{U})$, then there exists a $\gamma > 0$ for which $(I - \varepsilon B^*)^{-1} \notin \mathcal{C}_\psi^r(\mathcal{U})$ for every nonzero ε with $|\varepsilon| < \gamma$. □

Proposition 4.2. *Let $\psi \in \mathcal{W}^r(\mathcal{U})$ and $B \in \mathcal{C}_\psi^r(\mathcal{U})$. The following conditions are equivalent:*

(i) *There exists a sequence of real (or complex) numbers ε_n such that $\varepsilon_n \to 0$ and $\psi + \varepsilon_n B\psi$ is biorthogonal for each n.*

(ii) *There exists $\gamma > 0$ such that for every real (or complex) ε with $|\varepsilon| < \gamma$, the vector $\psi + \varepsilon B\psi$ is biorthogonal.*

(iii) *$B^{*n} \in \mathcal{C}_\psi^r(\mathcal{U})$ for every positive integer n.*

Proof. The implication (ii)$\Longrightarrow$(i) is trivial and the implication (i)$\Longrightarrow$(iii) follows from Lemma 4.1.

Suppose that (iii) holds. For every ε with $|\varepsilon| < 1/\|B\|$, $I + \varepsilon B$ is invertible and by (4.1), $(I + \varepsilon B^*)^{-1} \in \mathcal{C}_\psi^r(\mathcal{U})$. Therefore (ii) holds. □

For the rest of this section, consider in particular a unitary system $\mathcal{U} = \mathcal{U}_0\mathcal{G}$ of the product type satisfying the same assumptions as in the previous section.

We say that $\eta = (\eta_1, ..., \eta_r) \in \mathcal{R}^r(\mathcal{U})$ is an *MRA multiwavelet* if there exist a positive integer s and $\phi_1, ..., \phi_s$ in the core space $V_0(\eta)$ such that $\{g\phi_j : g \in \mathcal{G}, j = 1, ..., s\}$ is a Riesz basis for $V_0(\eta)$. (By Proposition 2.2, we can choose $\phi_1, ..., \phi_s$ such that $\{g\phi_j : g \in \mathcal{G}, j = 1, ..., s\}$ is an *orthonormal* basis for $V_0(\eta)$.)

Two vectors ψ and η in $\mathcal{R}^r(\mathcal{U})$ are said to be *core equivalent* if there exists an invertible operator Y on H such that $Y(V_0(\psi)) = V_0(\eta)$ and $Yg = gY$ for every $g \in \mathcal{G}$. The following result shows that two core equivalent vectors share certain similar properties.

Proposition 4.3. *Let ψ and η in $\mathcal{R}^r(\mathcal{U})$ be core equivalent.*

(i) *If ψ is biorthogonal, then so is η.*

(ii) *If ψ is an MRA multiwavelet, then so is η.*

Proof. (i) follows from Theorem 3.6.

(ii) Suppose that ψ is an MRA multiwavelet. Then there exist a positive integer s and $\phi_1, ..., \phi_s \in V_0(\psi)$ such that $\{g\phi_j : g \in \mathcal{G}, j = 1, ..., s\}$ is an orthonormal basis for $V_0(\psi)$. By core equivalence of ψ and η, there exists an invertible operator B on H such that $B(V_0(\psi)) = V_0(\eta)$ and $Bg = gB$ for every $g \in \mathcal{G}$. It follows that $\{gB\phi_j : g \in \mathcal{G}, j = 1, ..., s\}$ is a Riesz basis for $V_0(\eta)$. Hence η is also an MRA multiwavelet. □

Theorem 4.4. *Let $\psi = (\psi_1, ..., \psi_r) \in \mathcal{W}^r(\mathcal{U})$ and $\eta = (\eta_1, ..., \eta_r) \in \mathcal{R}^r(\mathcal{U})$. Let A be an invertible operator in $\mathcal{C}^r_\psi(\mathcal{U})$ such that $A\psi_i = \eta_i$, $i = 1, ..., r$, and $w^*(A) \subset \mathcal{C}^r_\psi(\mathcal{U})$. Then the following statements hold:*

(i) *η is biorthogonal and core equivalent to ψ, i.e., there exists an invertible operator B on H such that $B(V_0(\psi)) = V_0(\eta)$ and $Bg = gB$ for every $g \in \mathcal{G}$.*

(ii) *If A is unitary, then the operator B in (i) can be chosen to be unitary.*

Assume that the conditions in Theorem 4.4 hold. We break up the proof of the main part of Theorem 4.4 into several lemmas. Let

$$\mathcal{I} = \{V \in w^*(A) : V \text{ is invertible}\} \tag{4.2}$$

be the group of all invertible elements of $w^*(A)$, and for any operator $V \in \mathcal{I}$, let $V\psi = (V\psi_1, ..., V\psi_r)$, which is in $\mathcal{R}^r(\mathcal{U})$.

Lemma 4.5. *For every $V \in \mathcal{I}$, $w^*(A) \subset \mathcal{C}^r_{V\psi}(\mathcal{U})$.*

Proof. Let $V \in \mathcal{I}$ and $B \in w^*(A)$. Then $BV \in w^*(A) \subset \mathcal{C}^r_\psi(\mathcal{U})$ and $V \in \mathcal{C}^r_\psi(\mathcal{U})$ too. By Lemma 3.1, $B \in \mathcal{C}^r_{V\psi}(\mathcal{U})$. □

Define a closed linear subspace E of H by

$$E = \bigcap_{V \in \mathcal{I}} V_0(V\psi). \tag{4.3}$$

Denote by P and Q the orthogonal projections of H onto E and $E^\perp$ respectively. For every $V \in \mathcal{I}$, since $V^{*-1} \in w^*(A) \subset \mathcal{C}^r_\psi(\mathcal{U})$, by Proposition 3.3 $V\psi$ is biorthogonal in $\mathcal{R}^r(\mathcal{U})$. By Theorem 3.6, for all $g \in \mathcal{G}$, we have $g(V_0(V\psi)) = V_0(V\psi)$ and so $g(E) = E$. Since g is unitary, P and Q both commute with g for all $g \in \mathcal{G}$.

Lemma 4.6. *For the projection Q as above, we have $AgQ = gAQ$ for all $g \in \mathcal{G}$.*

Proof. It suffices to establish the lemma at the generating vectors of $E^\perp$. Note that

$$E^\perp = \overline{span} \bigcup_{V \in \mathcal{I}} V_0(V\psi)^\perp, \tag{4.4}$$

whence $E^\perp$ is precisely the closed linear span of $\{D^n h V^{*-1}\psi_i : n \geq 0,\ h \in \mathcal{G}, i = 1, ..., r, V \in \mathcal{I}\}$. For every $V \in \mathcal{I}$, V^{*-1} is again in $\mathcal{I}$ and by Lemma 4.5, $A \in \mathcal{C}^r_{V^{*-1}\psi}(\mathcal{U})$. Hence for all $n \geq 0,\ h \in \mathcal{G}$ and $i = 1, ..., r$, by (3.5) we have

$$\begin{aligned} AgD^n hV^{*-1}\psi_i &= AD^n\sigma^n(g)hV^{*-1}\psi_i \\ &= D^n\sigma^n(g)hAV^{*-1}\psi_i \\ &= gD^n hAV^{*-1}\psi_i \\ &= gAD^n hV^{*-1}\psi_i. \end{aligned}$$

□

Proof of Theorem 4.4.
(i) Since $A \in \mathcal{I}$, we have A^*, A^{-1} and A^{*-1} all in $\mathcal{I} \subset \mathcal{C}_\psi^r(\mathcal{U})$. Hence η is biorthogonal. Also, as $\mathcal{I}$ is a group, $A(\mathcal{I}) = \mathcal{I}$ and $A^*(\mathcal{I}) = \mathcal{I}$. For every $V \in \mathcal{I}$, by Lemma 4.5 $A(V_0(V\psi)) = V_0(AV\psi)$ and $A^*(V_0(V\psi)) = V_0(A^*V\psi)$. It follows from (4.3) that $A(E) = \bigcap_{V\in\mathcal{I}} A(V_0(V\psi)) = \bigcap_{V\in\mathcal{I}} V_0(AV\psi) = E$. Similarly, $A^*(E) = E$ and so $A(E^\perp) = E^\perp$. Hence $AQ = QA$.

We define the operator

$$B = AQ + P \tag{4.5}$$

and claim that B satisfies the desired conditions in Theorem 4.4(i). First note that by Lemma 4.6, for every $g \in \mathcal{G}$,

$$Bg = AQg + Pg = AgQ + gP = gAQ + gP = gB.$$

Furthermore, we claim that B is invertible, with inverse $B^{-1} = A^{-1}Q + P$. Indeed, since $AQ = QA$ and $QP = PQ = 0$, we have

$$\begin{aligned}(A^{-1}Q + P)(AQ + P) &= A^{-1}QAQ + A^{-1}QP + PAQ + P^2\\ &= Q + 0 + 0 + P\\ &= I.\end{aligned}$$

A similar computation shows that $(AQ + P)(A^{-1}Q + P) = I$.

Finally, as noted above, A maps $V_0(\psi)$ onto $V_0(A\psi) = V_0(\eta)$. Since $E \subset V_0(\psi) \cap V_0(\eta)$, we have the orthogonal decompositions

$$V_0(\psi) = E + (E^\perp \cap V_0(\psi)), \quad V_0(\eta) = E + (E^\perp \cap V_0(\eta)).$$

Therefore by (4.5) and properties of P and Q,

$$\begin{aligned} B(V_0(\psi)) &= B(E) + B(E^\perp \cap V_0(\psi)) \\ &= P(E) + AQ(E^\perp \cap V_0(\psi)) \\ &= E + A(E^\perp \cap V_0(\psi)) \\ &= E + (A(E^\perp) \cap A(V_0(\psi))) \\ &= E + (E^\perp \cap V_0(\eta)) \\ &= V_0(\eta). \end{aligned}$$

(ii) Suppose that A is unitary. Then by (4.5) and the above discussion,

$$B^* = A^*Q + P = A^{-1}Q + P = B^{-1}. \qquad \square$$

Corollary 4.7. *Let $\psi = (\psi_1, ..., \psi_r)$ and $\eta = (\eta_1, ..., \eta_r)$ be vectors in $\mathcal{W}^r(\mathcal{U})$. Let V be the unitary operator in $\mathcal{C}_\psi^r(\mathcal{U})$ such that $V\psi_i = \eta_i$, $i = 1, ..., r$, and $V^n \in \mathcal{C}_\psi^r(\mathcal{U})$ for every $n \in \mathbb{Z}$. Then*

(i) *η is core equivalent to ψ;*

(ii) *for every $t \in \mathbb{C}$ with $|t| \neq 1$, the vector $\psi_t = \psi + t\eta$ is core equivalent to ψ.*

A special case of part (i) of the above corollary is in [16, Theorem 2].

Proof. Since V is unitary and $V^n \in \mathcal{C}_\psi^r(\mathcal{U})$ for every $n \in \mathbb{Z}$, we have $w^*(V) \subset \mathcal{C}_\psi^r(\mathcal{U})$. Hence (i) follows from Theorem 4.4.

Let $t \in \mathbb{C}$ with $|t| \neq 1$, and $\psi_t = \psi + t\eta$. Define $V_t = I + tV$. Then $V_t\psi = \psi_t$, and it is easy to check that V_t is an invertible operator in $w^*(V)$. Hence $w^*(V_t) \subset w^*(V) \subset \mathcal{C}_\psi^r(\mathcal{U})$. Therefore (ii) also follows from Theorem 4.4. $\square$

Theorem 4.8. *Let $\psi = (\psi_1, ..., \psi_r)$ and $\eta = (\eta_1, ..., \eta_r)$ be vectors in $\mathcal{W}^r(\mathcal{U})$, and let V be the unitary operator in $\mathcal{C}_\psi^r(\mathcal{U})$ such that $V\psi_i = \eta_i$, $i = 1, ..., r$. The following conditions are equivalent:*

(i) *There exist sequences $\varepsilon_n \to 0$ and $\delta_n \to 0$ such that $\psi + \varepsilon_n\eta$ and $\eta + \delta_n\psi$ are biorthogonal for all n.*

(ii) *$w^*(V) \subset \mathcal{C}_\psi^r(\mathcal{U})$.*

(iii) *$\psi + t\eta$ is core equivalent to ψ for every real (or complex) t with $|t| \neq 1$.*

Proof. Suppose that (i) holds. Then by Proposition 4.2, $V^{-n} = V^{*n} \in \mathcal{C}^r_\psi(\mathcal{U})$ and $V^n = (V^{-1*})^n \in \mathcal{C}^r_\eta(\mathcal{U})$ for all positive integers n. Then by Lemma 3.1, $V^n \in \mathcal{C}^r_\psi(\mathcal{U})$ for all positive integers n. Hence $w^*(V) \subset \mathcal{C}^r_\psi(\mathcal{U})$.

The implication (ii)$\Longrightarrow$(iii) is a consequence of Corollary 4.7. The implication (iii)$\Longrightarrow$(i) follows from Proposition 4.3(i) and Proposition 4.2(ii), since ψ is orthonormal (hence biorthogonal). □

Acknowledgement. *The first author was support in part by NSF grant DMS-0070796. The second author's research was supported in part by the Academic Research Fund No. R-146-000-025-112, National University of Singapore. The third author was supported in part by the NSF grant DMS-0200756. The second author wishes to thank the first and third authors and the Department of Mathematics, Texas A&M University for their hospitality during his visit in the fall semester of 2000.*

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