



## Nest Algebras and Similarity Transformations

David R. Larson

*The Annals of Mathematics*, 2nd Ser., Vol. 121, No. 2. (Mar., 1985), pp. 409-427.

Stable URL:

<http://links.jstor.org/sici?sici=0003-486X%28198503%292%3A121%3A2%3C409%3ANAAST%3E2.0.CO%3B2-8>

*The Annals of Mathematics* is currently published by Annals of Mathematics.

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/annals.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

---

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# Nest algebras and similarity transformations

By DAVID R. LARSON\*

## Introduction

In recent years the theory of algebras of operators on Hilbert space has been stimulated by developments in the theory of quasitriangularity. Andersen [1] has shown that up to unitary equivalence there is only one “continuous” quasitriangular algebra. We use this to provide the following answer to a question posed by J. R. Ringrose approximately 20 years ago: Similar continuous nests on separable Hilbert space can fail to be unitarily equivalent (Theorem 2.2). A consequence is the existence of a nonhyperintransitive compact operator (Corollary 2.3), which answers a question of Kadison and Singer [12] and of Gohberg and Krein [11].

We extend our initial theorem to show that arbitrary continuous nests are similar (Theorem 2.10), and that every maximal nest is similar to a multiplicity-one nest (Theorem 2.11). A consequence is that every compact operator is similar to a hyperintransitive compact operator (Corollary 2.12). The similarity transformation can be induced by an arbitrarily small compact perturbation of a unitary operator.

The methods of Section 2 apply only to the continuous parts of nests. For general results the atomic core part must be dealt with. In Section 3 different methods are developed for this purpose, again utilizing Andersen’s results. These are used in Section 4 to prove that a complete nest  $\mathcal{N}$  admits an Arveson factorization for every positive invertible operator if and only if  $\mathcal{N}$  is countable as a family of subspaces (Theorem 4.7). A consequence is that if  $\mathcal{N}$  is an uncountable nest with atomic core then some similarity transformation of  $\mathcal{N}$  has a continuous part. This could not be deduced from Section 2. These methods also yield a weak factorization result which concludes the paper.

It should be noted that a negative resolution to the Ringrose question was conjectured in recent years by several mathematicians, including W. Arveson and J. Ringrose. Also, many of the results presented in this paper were announced in an A.M.S. Bulletin article [16]. Finally, we wish to thank the referee for suggesting that the original manuscript could be condensed and improved.

---

\*This work was partially supported by NSF.

**1. Notation and basic relationships**

A *nest*, or *chain*, is a family of closed subspaces of a Hilbert space  $H$  totally ordered by inclusion.  $\mathcal{N}$  is *complete* if it contains  $(0)$  and  $H$  and contains the join (closed linear span) and meet (intersection) of every subfamily. The associated *nest algebra* is  $\text{alg } \mathcal{N} = \{A \in L(H) : AN \subseteq N, N \in \mathcal{N}\}$ . In this paper nests will be complete and Hilbert space will be separable. The *core*  $\mathcal{C}_{\mathcal{N}}$  is the von Neumann algebra generated by the projections onto the members of  $\mathcal{N}$  and the *diagonal* is  $\mathcal{D}_{\mathcal{N}} = (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^* = (\mathcal{C}_{\mathcal{N}})'$ .

Nests  $\mathcal{N}, \mathcal{M}$  are *unitarily equivalent* if there is a unitary  $U$  such that  $\mathcal{M} = \{UN : N \in \mathcal{N}\}$ . A nest is *continuous* if its core is a nonatomic von Neumann algebra.  $\mathcal{N}$  has *multiplicity one* (is multiplicity free) if its core is a maximal abelian selfadjoint algebra. From [12], continuous multiplicity-one nests are unitarily equivalent. From general multiplicity theory for nests [6] it follows that continuous nests of uniform infinite multiplicity are unitarily equivalent.

If  $\mathcal{N}$  is a complete nest and  $T$  is an invertible operator then  $T\mathcal{N} = \{TN : N \in \mathcal{N}\}$  is a complete nest order isomorphic to  $\mathcal{N}$  via the correspondence  $N \rightarrow TN$ . It is clear that  $T\mathcal{N}$  is maximal if and only if  $\mathcal{N}$  is maximal, and that  $T\mathcal{N}$  is continuous if and only if  $\mathcal{N}$  is continuous. We say that the nests  $\mathcal{N}$  and  $T\mathcal{N}$  are *similar*. We have  $\text{alg}(T\mathcal{N}) = T(\text{alg } \mathcal{N})T^{-1}$  so that  $T(\cdot)T^{-1}$  is a similarity transformation between the associated nest algebras.

Upper-lower triangular factorization results are related to similarity questions. We will utilize two easily proved lemmas.

**LEMMA 1.1.** *Let  $\mathcal{N}$  be a nest and let  $T$  be an invertible operator. The following are equivalent:*

- (i) *There exists a unitary operator  $U$  such that  $TN = UN, N \in \mathcal{N}$ .*
- (ii) *There exists a unitary operator  $U$  such that  $UT \in (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^{-1}$ .*
- (iii)  *$T^*T = A^*A$  for some  $A \in (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^{-1}$ .*

**LEMMA 1.2.** *Let  $\mathcal{N}$  be a nest and let  $T$  be a positive invertible operator. If  $T = A^*B$  with  $A, B \in (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^{-1}$ , then  $T = C^*C$  for some  $C \in (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^{-1}$ .*

An important theorem of Gohberg-Krein [11, Chapter 4, Theorem 6.2] easily yields the following as a corollary, where  $C_{\omega}$  denotes the Macaev ideal. The first part is nontrivial, and useful, even when  $T - I$  is assumed to be of finite-rank.

**THEOREM 1.3.** *Let  $\mathcal{N}$  be an arbitrary nest.*

- (i) *If  $T > 0$  with  $T - I \in C_{\omega}$ , then  $T = A^*A$  for some*

$$A \in (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^{-1}.$$

(ii) If  $T$  is an invertible operator of the form  $T = U + K$  with  $U$  unitary and  $K \in C_\omega$ , then  $T$  satisfies (i), (ii), (iii) of Lemma 1.1 relative to  $\mathcal{N}$ .

In an important paper Niels Andersen proved strong results [1, Theorem 3.5.5, Corollary 3.5.6] concerning compact perturbations of nest algebras. We state the somewhat weaker forms required in this paper. The ideal of compact operators in  $L(H)$  will be denoted by  $\mathcal{K}$ .

**THEOREM 1.4** (Andersen). *Let  $\mathcal{N}, \mathcal{M}$  be continuous nests of projections and let  $\phi: \mathcal{N} \rightarrow \mathcal{M}$  be an order isomorphism. There exists a unitary  $U$  such that  $\text{alg}(U\mathcal{N}) + \mathcal{K} = \text{alg } \mathcal{M} + \mathcal{K}$ , and  $UNU^* - \phi(N) \in \mathcal{K}$ ,  $N \in \mathcal{N}$ .*

**COROLLARY 1.5** (Andersen). *Let  $\mathcal{N}, \mathcal{M}$  be continuous nests. There exists a unitary  $U$  such that  $\text{alg } \mathcal{N} + \mathcal{K} = U(\text{alg } \mathcal{M} + \mathcal{K})U^*$ .*

For additional exposition on nest algebras and related topics we refer the reader to the excellent survey article by J. Erdős [7].

## 2. The similarity problem

We prove that arbitrary continuous nests are similar, and that every maximal nest is similar to a multiplicity-one nest. The similarity transformation can be taken to be a small compact perturbation of a unitary operator.

For a continuous nest  $\mathcal{N}$  every compact operator in  $\text{alg } \mathcal{N}$  is quasinilpotent [18]; hence  $\text{alg } \mathcal{N} \cap \mathcal{K}$  is a topologically nil ideal in  $\text{alg } \mathcal{N}$  and so is contained in  $\text{rad}(\text{alg } \mathcal{N})$ . The following lemma is a generalization of a standard result on lifting idempotents from a Banach algebra modulo its radical and will be applied with  $\mathcal{R}_0 = \text{alg } \mathcal{N} \cap \mathcal{K}$ .

**LEMMA 2.1.** *Let  $\mathcal{A}$  be a unital Banach algebra with Jacobson radical  $\mathcal{R}$ , and let  $\mathcal{R}_0$  be a (not necessarily closed) two-sided ideal of  $\mathcal{A}$  contained in  $\mathcal{R}$ . If  $A \in \mathcal{A}$  satisfies  $A^2 - A \in \mathcal{R}_0$  then there exists an idempotent  $P \in \mathcal{A}$  with  $A - P \in \mathcal{R}_0$  and with  $PA = AP$ .*

*Proof.* We show that the proof given in [17, Thm. 2.3.9] for the case  $\mathcal{R}_0 = \mathcal{R}$  generalizes; we use notation as in that proof. Let  $q = A - A^2$  and  $q_0 = -q(I - 4q)^{-1}$ . Then  $q, q_0 \in \mathcal{R}_0$ . Since  $\mathcal{R}_0 \subseteq \mathcal{R}$ , by [17, Lemma 2.3.8] there exists  $x \in \mathcal{R}$  such that  $x^2 - x + q_0 = 0$  and such that  $x$  commutes with every element of  $\mathcal{A}$  which commutes with  $q_0$ . In particular  $xA = Ax$ . We have  $x = -q_0(x - I)^{-1}$  so that in fact  $x \in \mathcal{R}_0$ . Also,  $x^2 - x = q(I - 4q)^{-1}$  so that  $(I - 4q)(x^2 - x) = q$ . Since  $(2A - I)^2 = I - 4q$  we have  $(2A - I)^2(x^2 - x) = q$ . Let  $z = -(2A - I)x$ . Then  $z^2 + (2A - I)z = q$ . Since  $zA = Az$  and  $q = A - A^2$  we have  $A^2 + 2zA + z^2 = A + z$ , or simply  $(A + z)^2 = A + z$ .

Also,  $z \in \mathcal{R}_0$  since  $x \in \mathcal{R}_0$ . Now let  $P = A + z$ . □

**THEOREM 2.2.** *Let  $\mathcal{N}$  be a continuous nest of multiplicity one on separable Hilbert space  $H$ . Then given  $\varepsilon > 0$  there exists a positive invertible operator  $T \in L(H)$  with  $T - I$  compact and  $\|T - I\| < \varepsilon$  such that  $T\mathcal{N} = \{TN: N \in \mathcal{N}\}$  fails to have multiplicity one.*

*Proof.* By Corollary 1.5 (Andersen) there exists a continuous nest  $\mathcal{M}$  not of multiplicity one such that  $\text{alg } \mathcal{M} + \mathcal{X} = \text{alg } \mathcal{N} + \mathcal{X}$ . The diagonal  $\mathcal{D}_{\mathcal{M}} = \text{alg } \mathcal{M} \cap (\text{alg } \mathcal{M})^*$  is a nonabelian von Neumann algebra and so contains a nonzero partial isometry  $V$  with orthogonal initial and final spaces. Let  $\tilde{S} = V + V^* - VV^* - V^*V + I$  and  $\tilde{P} = VV^*$ . Then  $\tilde{S}^2 = I$ ,  $\tilde{P}^2 = \tilde{P} \neq 0$ ,  $\tilde{P}\tilde{S}\tilde{P} = 0$ . Write  $\tilde{Q} = \frac{1}{2}(I - \tilde{S})$ . Then  $\tilde{P}, \tilde{Q}$  are nonzero selfadjoint projections in  $\mathcal{D}_{\mathcal{M}}$ .

Choose  $A, B \in \text{alg } \mathcal{N}$  with  $A - \tilde{P} \in \mathcal{X}$  and  $B - \tilde{Q} \in \mathcal{X}$ . Then  $A^2 - A$  and  $B^2 - B$  are elements of  $\text{alg } \mathcal{N} \cap \mathcal{X}$ . Since  $\mathcal{M}$  is continuous  $\mathcal{D}_{\mathcal{M}} \cap \mathcal{X} = (0)$ , so  $\tilde{P}, \tilde{Q} \notin \mathcal{X}$ , and hence  $A, B \notin \mathcal{X}$ . Also, since  $\mathcal{N}$  is continuous  $\text{alg } \mathcal{N} \cap \mathcal{X} \subset \text{rad}(\text{alg } \mathcal{N})$ , so that Lemma 2.1 implies the existence of idempotents  $P, Q \in \text{alg } \mathcal{N}$  with  $P - A \in \mathcal{X}$  and  $Q - B \in \mathcal{X}$ . Let  $S_1 = I - 2Q$ . Then  $S_1^2 = I$ , and  $PS_1P \in \text{alg } \mathcal{N} \cap \mathcal{X}$  since  $P - \tilde{P} \in \mathcal{X}$  and  $S_1 - \tilde{S} \in \mathcal{X}$ . Let  $S_2 = S_1 - PS_1P$ . Then  $S_2$  is invertible in  $\text{alg } \mathcal{N}$  since  $S_1$  is invertible in  $\text{alg } \mathcal{N}$  and  $PS_1P \in \text{rad}(\text{alg } \mathcal{N})$ . We have  $PS_2P = 0$ . Now let

$$S = S_2P + PS_2^{-1}(I - P) - S_2PS_2^{-1}(I - P) + I - P.$$

We have  $PSP = 0$ , and it can be verified that  $S^2 = I$ . (Let  $\alpha$  denote the sum of the first two terms,  $\beta$  the sum of the remaining terms, and compute  $\beta^2 = \beta$ ,  $\alpha\beta = \beta\alpha = 0$ ,  $\alpha^2 = I - \beta$ .) We have  $S_2 - \tilde{S} \in \mathcal{X}$ ; so also  $S_2^{-1} - \tilde{S} \in \mathcal{X}$ . Since in addition  $P - \tilde{P} \in \mathcal{X}$  it follows by direct computation that  $S - \tilde{S} \in \mathcal{X}$ .

Let  $R = I - 2P$ . Then  $R^2 = I$ ,  $S^2 = I$ ,  $RS \neq SR$ . Let  $\mathcal{G}$  be the group generated by  $R, S$ . We have  $SRS = I - 2SPS$ , and  $PSP = 0$ , and  $PSRS = P = SRSP$ . Hence  $R$  commutes with  $SRS$  since  $P$  does. It easily follows that

$$\mathcal{G} = \{I, S, R, RS, SR, SRS, RSR, SRSR\}.$$

So  $\mathcal{G}$  is a finite noncommutative group contained in  $\text{alg } \mathcal{N}$ . ( $\mathcal{G}$  is in fact a representation of  $D_4$ —the dihedral group of order 8.) Set  $T = (1/8\sum_{g \in \mathcal{G}} g^*g)^{1/2}$ . Then  $T\mathcal{G}T^{-1} = \{TgT^{-1}: g \in \mathcal{G}\}$  is a noncommutative group of unitary operators contained in the diagonal of the nest algebra  $\text{alg}(T\mathcal{N})$ , and thus  $T\mathcal{N}$  fails to have multiplicity 1.

If we set  $\tilde{R} = I - 2\tilde{P}$ , then  $\tilde{R}, \tilde{S}$  are unitary. Since  $S - \tilde{S}, R - \tilde{R} \in \mathcal{X}$ , it follows that  $g^*g - I \in \mathcal{X}$ ,  $g \in \mathcal{G}$ , and hence  $T - I \in \mathcal{X}$ .

Write  $T = I + K$  and choose  $K_0$  to be a finite rank operator with  $I + K_0$  invertible and  $\|(I + K_0)^{-1}(K - K_0)\| < \varepsilon/3$ . Let

$$T_0 = I + (I + K_0)^{-1}(K - K_0).$$

We have  $T = (I + K_0)T_0$ . By Theorem 1.3, the nests  $T_0\mathcal{N}$  and  $T\mathcal{N}$  are unitarily equivalent; thus  $T_0\mathcal{N}$  fails to have multiplicity one and so  $|T_0|\mathcal{N}$  fails to have multiplicity one. We have  $|T_0| - I \in \mathcal{X}$  and  $\||T_0| - I\| < \varepsilon$ . Replace  $T$  with  $|T_0|$  if necessary.  $\square$

We next answer a question of Kadison-Singer [12] and independently of Gohberg-Krein [11].

**COROLLARY 2.3.** *There exists a non-hyperintransitive compact operator.*

*Proof.* Let  $V$  be the Volterra operator. It is well-known that  $\text{Lat}(V)$  is a continuous multiplicity one nest. Let  $\mathcal{N} = \text{Lat}(V)$  and let  $T$  be an invertible operator such that  $T\mathcal{N}$  does not have multiplicity one. Since  $\text{Lat}(TVT^{-1}) = T\mathcal{N}$  and since  $T\mathcal{N}$  is a maximal nest the similarity  $TVT^{-1}$  is not hyperintransitive.  $\square$

The observation that a negative resolution of the Ringrose question would yield the above result was, to our knowledge, first made by J. Erdős and was first pointed out to the author by W. Arveson.

To prove that arbitrary continuous nests are similar (Theorem 2.10) we require several lemmas.

**LEMMA 2.4.** *Let  $\mathcal{N}$  be a continuous nest and let  $\varepsilon > 0$  be given. Then  $\text{alg } \mathcal{N}$  contains an operator  $B$  such that for some positive invertible operator  $T \in L(H)$  with  $T - I \in \mathcal{X}$  and  $\|T - I\| < \varepsilon$  the operator  $W = TBT^{-1}$  is a partial isometry with  $WW^* + W^*W = I$ , and with  $W, W^*$  in the diagonal of  $\text{alg}(T\mathcal{N})$ .*

*Proof.* We adapt the proof of Theorem 2.2. By Andersen’s theorem there exists a nest  $\mathcal{M}$  of uniform infinite multiplicity such that  $\text{alg } \mathcal{M} + \mathcal{X} = \text{alg } \mathcal{N} + \mathcal{X}$ . The diagonal  $\mathcal{D}_{\mathcal{M}}$  then contains a p.i. (partial isometry)  $V$  with  $VV^* + V^*V = I$ . Mimic the proof of (2.2) obtaining  $\tilde{S}, \tilde{P}, S, P$  as in that proof. As in (2.2) we have  $P - \tilde{P} \in \mathcal{X}$  and  $S - \tilde{S} \in \mathcal{X}$ . Also, since  $V^*V + VV^* = I$ , we have  $\tilde{S} = V + V^*$ . Also  $V = \tilde{P}\tilde{S}, V^* = \tilde{S}\tilde{P}$ . Let  $B = PS$ . Then  $B - V \in \mathcal{X}$ . Define  $T$  as in the proof of (2.2). Then  $W = TBT^{-1}$  is a p.i. in  $\mathcal{D}_{T\mathcal{N}}$ . Since  $T - I \in \mathcal{X}$  and  $V^*V + VV^* = I$  we have  $W^*W + WW^* - I \in \mathcal{X}$ . But this latter is in  $\mathcal{D}_{T\mathcal{N}}$  and  $\mathcal{D}_{T\mathcal{N}} \cap \mathcal{X} = \{0\}$  since  $T\mathcal{N}$  is continuous; so  $W^*W + WW^* = I$ .

We have obtained  $T, B$  as required except for the condition that  $\|T - I\| < \varepsilon$ . For this, let  $T_0, \mathcal{X}_0$  be as in the last paragraph of the proof of

(2.2). Since  $T > 0$  we have  $T = T_0^*(I + \mathcal{X}_0^*)$ . Write  $T_0 = U|T_0|$  for  $U$  unitary, so that  $T_0^* = |T_0|U^*$ , and write  $T_1 = |T_0|$ ,  $T_2 = U^*(I + \mathcal{X}_0^*)$ . By (1.5)  $\mathcal{N}$  and  $T_2\mathcal{N}$  are unitarily equivalent. Write  $\hat{\mathcal{N}} = T_2\mathcal{N}$  and  $\hat{B} = T_2BT_2^{-1}$ . Now replace  $\mathcal{N}$ ,  $B$  with  $\hat{\mathcal{N}}$ ,  $\hat{B}$  and  $T$  with  $T_1$ . The lemma is proved.  $\square$

LEMMA 2.5. *A continuous nest  $\mathcal{N}$  has uniform infinite multiplicity if and only if  $\mathcal{D}_{\mathcal{N}}$  contains an infinite sequence of mutually orthogonal projections each with central support  $I$  in  $\mathcal{D}_{\mathcal{N}}$ .*

*Proof.* This follows from the multiplicity theory of nests in [6].

*Definition.* Let  $\langle W_1, \dots, W_n \rangle$  be an ordered  $n$ -tuple of partial isometries. We say  $\langle W_1, \dots, W_n \rangle$  is a *directed system* if  $W_iW_i^* + W_i^*W_i = W_{i-1}W_{i-1}^*$ ,  $i = 2, \dots, n$ , and  $W_iW_i^* \neq 0$ ,  $i = 1, \dots, n$ . An infinite sequence  $\{W_i\}$  satisfying this relation is called an *infinite directed system*. A system is *proper* if  $W_1W_1^* + W_1^*W_1 = I$ .

LEMMA 2.6. *Let  $\mathcal{B}$  be a von Neumann algebra which contains an infinite proper directed system. Then  $\mathcal{B}$  contains an infinite sequence of mutually orthogonal projections all of which have central support  $I$  in  $\mathcal{B}$ .*

*Proof.* Let  $\{W_i\}$  be the system. Let  $P_i = W_iW_i^*$ . Then  $P_1 > P_2 > \dots$ , and  $P_i \neq P_{i+1}$  for all  $i$ . Since the system is proper,  $P_1$  and  $I - P_1$  have central support  $I$  in  $\mathcal{B}$ . In general, if  $E, F$  are projections in  $\mathcal{B}$  with  $E > F$  such that  $E$  has central support  $I$  in  $\mathcal{B}$  and  $EF|_{EH}$  has central support  $I_{EH}$  in  $E\mathcal{B}|_{EH}$ , then  $F$  has central support  $I$  in  $\mathcal{B}$ . It follows that each  $P_i$  has central support  $I$  in  $\mathcal{B}$ , and also that each difference projection  $P_{i+1} - P_i$  has central support  $I$  in  $\mathcal{B}$ .  $\square$

The following construction will be used: Let  $\langle W_1, \dots, W_n \rangle$  be a directed system, and let  $P_n = W_nW_n^*$  and  $P_0 = W_1W_1^* + W_1^*W_1$ . If  $A \in L(H)$  with  $A = P_nAP_n$  then  $A$  will not commute with  $\{W_1, W_2, \dots, W_n\}$  unless  $A = 0$ . However,  $A$  can be “extended” to an operator  $\tilde{A}$  supported on  $P_0$  which commutes with  $\{W_1, \dots, W_n\}$ . Let  $\alpha_1 = A + W_n^*AW_n$ , and define  $\alpha_i$ ,  $2 \leq i \leq n$ , inductively by  $\alpha_i = W_{n-i+1}^*(\alpha_1 + \dots + \alpha_{i-1})W_{n-i+1}$ . Let  $\tilde{A} = \alpha_1 + \alpha_2 + \dots + \alpha_n$ . In the case where  $P_0 = I$ ,  $\tilde{A}$  can be viewed as a direct sum of  $2^n$  copies of  $P_nA|_{P_nH}$ , the equivalences between the direct summand Hilbert spaces being determined by the  $W_i$ . If  $P_0 \neq I$ , an additional 0 direct summand is included. To verify that  $\tilde{A}$  commutes with  $\{W_i\}$ , note that we have  $W_n\tilde{A} = W_n\alpha_1 = W_nW_n^*AW_n = AW_n$ , and  $\tilde{A}W_n = \alpha_1W_n = AW_n$ ; so  $\tilde{A}W_n = W_n\tilde{A}$ , and for  $1 \leq j \leq n - 1$  we have  $\tilde{A}W_j = (\alpha_1 + \dots + \alpha_{n-j})W_j = W_j\alpha_{n-j+1}W_j^*W_j = W_j\tilde{A}$ . Similarly,  $\tilde{A}$  commutes with  $\{W_i^*\}$ . It is clear that “ $\sim$ ” determines an isometric positive linear map from  $P_nL(H)|_{P_nH}$  to  $L(H)$  which maps compacts to compacts.

If  $\mathcal{N}$  is a nest and  $P$  a projection in  $\mathcal{D}_{\mathcal{N}}$  we write  $\mathcal{N}_P$  for the nest  $\{PN: N \in \mathcal{N}\}$  of subspaces of  $PH$ . We have  $\text{alg}(\mathcal{N}_P) = P(\text{alg}(\mathcal{N}))|_{PH}$ .

LEMMA 2.7. *Let  $\mathcal{N}$  be a continuous nest and let  $\varepsilon > 0$  be given. There exists an infinite sequence of positive invertible operators  $S_n$ , and partial isometries  $W_n$ , with the following properties:*

- (i)  $S_n - I \in \mathcal{K}$  and  $\|S_n - I\| < \varepsilon/2^n, n \geq 1$ .
- (ii)  $\langle W_1, \dots \rangle$  is an infinite, proper, directed system.
- (iii)  $\{W_1, W_2, \dots, W_n\}$  is contained in the diagonal of  $\text{alg}(S_n S_{n-1} \dots S_1 \mathcal{N})$ ,  $1 \leq n < \infty$ .
- (iv)  $S_{n+1}$  commutes with  $\{W_1, \dots, W_n\}$ ,  $n \geq 1$ .

*Proof.* Apply Lemma 2.4 to  $\mathcal{N}$  obtaining  $B_1 \in \text{alg} \mathcal{N}$ ,  $T_1 \in L^{-1}(H)$ ,  $\|T_1 - I\| < \varepsilon/2$ , as in that lemma. Let  $V_1 = T_1 B_1 T_1^{-1}$ , so that  $V_1$  is a p.i. contained in  $\mathcal{D}_{T_1 \mathcal{N}}$  with  $V_1 V_1^* + V_1^* V_1 = I$ . Let  $\mathcal{N}_1$  denote  $T_1 \mathcal{N}$ , let  $P_1 = V_1 V_1^*$ , and let  $S_1 = T_1$ ,  $W_1 = V_1$  for this first case.

Another application of Lemma 2.4 shows the existence of  $B_2 \in \text{alg}(\mathcal{N}_1|_{P_1 H})$  and a positive invertible operator  $T_2$  in  $L(P_1 H)$  which differs from the identity by a compact of norm  $< \varepsilon/4$  such that  $V_2 = T_2 B_2 T_2^{-1}$  is a p.i. with  $V_2 V_2^* + V_2^* V_2$  the identity on  $L(P_1 H)$  and with  $V_2, V_2^*$  both in the nest algebra of the image nest of  $\mathcal{N}_1|_{P_1 H}$  under  $T_2$ . Extend  $V_2$  to a p.i.  $W_2$  in  $L(H)$  by  $W_2 = V_2 \oplus 0$ , so that  $W_2 W_2^* + W_2^* W_2 = P_1$ . Extend  $T_2$  to a positive invertible operator  $S_2$  in  $L(H)$  via the map “ $\sim$ ” defined above, applied to the directed system  $\langle W_1 \rangle$ . (In this case, identifying  $T_2$  with  $T_2 \oplus 0$ , we have  $S_2 = T_2 + W_1^* T_2 W_1$ .)

We have  $\|S_2 - I\| < \varepsilon/4$ ,  $S_2 - I \in \mathcal{K}$ ,  $\langle W_1, W_2 \rangle$  is a proper directed system,  $S_2$  commutes with  $W_1$ . Let  $\mathcal{N}_2 = S_2 \mathcal{N}_1 = S_2 S_1 \mathcal{N}$ . Since  $W_1, W_1^* \in \text{alg} \mathcal{N}_1$  and  $S_2$  commutes with  $W_1, W_1^*$ , we have  $W_1, W_1^* \in \text{alg} \mathcal{N}_2$ . Also,  $B \oplus 0$  is in  $\text{alg} \mathcal{N}_1$  and  $S_2(B_2 \oplus 0)S_2^{-1} = W_2$  by our construction; so  $W_2 \in \text{alg} \mathcal{N}_2$ . Moreover,  $W_2^* \in \text{alg} \mathcal{N}_2$  also. Indeed,  $W_2^* = V_2^* \oplus 0$  and  $V_2^* = T_2 C T_2^{-1}$  for some  $C \in \text{alg}(\mathcal{N}_1|_{P_1 H})$ , so that  $C \oplus 0 \in \text{alg} \mathcal{N}_1$ , and

$$S_2(C \oplus 0)S_2^{-1} = T_2 C T_2^{-1} \oplus 0 = V_2^* \oplus 0 = W_2^*.$$

Now, inductively, assume  $S_i, W_i, 1 \leq i \leq n$  have been obtained satisfying our requirements. Let  $P_n = W_n W_n^*$  and repeat the previous paragraph with “ $n$ ” in place of “ $1$ ” and with the map “ $\sim$ ” applied to the directed system  $\langle W_1, \dots, W_n \rangle$ , obtaining  $S_{n+1}, W_{n+1}$ . Here  $\varepsilon/4$  is replaced with  $\varepsilon/2^{n+1}$ . Then  $S_i, W_i, 1 \leq i \leq n + 1$ , satisfy our requirements. □

LEMMA 2.8. *If  $\{K_n\}$  is a sequence of operators with  $\sum_1^\infty \|K_n\| < \infty$ , then the infinite product  $\prod_1^\infty (I + K_n)$  converges in norm to an operator  $S$  with  $\|S - I\| < \exp(\sum \|K_n\|) - 1$ .*

*Proof.* Let  $\alpha = \sum_1^\infty \|K_n\|$  and  $S_n = \prod_1^n (I + K_i)$ . The inequality  $\|S_n - I\| \leq \prod_1^n (1 + \|K_i\|) - 1$  is obtained by expanding both sides. Since  $1 + t \leq e^t$ ,  $t \geq 0$ , we have  $\prod_1^n (1 + \|K_i\|) \leq \exp(\sum_1^n \|K_i\|) \leq e^\alpha$ . Thus  $\|S_n - I\| \leq e^\alpha - 1$ , and so  $\|S_n\| \leq e^\alpha$  for each  $n$ . If  $m > n$ , we have  $(S_m - S_n) = S_n[\prod_{n+1}^m (I + K_i) - I]$  so that

$$\|S_m - S_n\| \leq e^\alpha \left[ \prod_{n+1}^m (1 + \|K_i\|) - 1 \right] \leq e^\alpha \left[ \exp\left(\sum_{n+1}^m \|K_i\|\right) - 1 \right].$$

It follows that  $\{S_n\}$  is a norm-Cauchy sequence and so converges to an operator  $S$ . We have

$$\|S - I\| = \lim \|S_n - I\| \leq e^\alpha - 1,$$

as required. □

**LEMMA 2.9.** *Let  $\mathcal{N}$  be a continuous nest. Given  $0 < \eta < 1$ , there exists an invertible operator  $S$  with  $S - I$  compact and  $\|S - I\| < \eta$  such that  $S\mathcal{N}$  has uniform infinite multiplicity.*

*Proof.* First apply Lemma 2.7 with  $\varepsilon = \ln(1 + \eta)$  and obtain  $S_n, W_n$  as in that lemma. Let  $K_n = S_n - I$ . Then  $\sum_1^\infty \|K_n\| < \varepsilon$ ; so by Lemma 2.8 the infinite product  $\prod_1^\infty S_n^*$  converges in norm to an operator  $S^*$  with  $\|S^* - I\| < e^\varepsilon - 1 = \eta$ . The sequence of partial products  $R_n = S_n S_{n-1} \cdots S_1$  then converges in norm to  $S$ . We have  $S - I \in \mathcal{X}$  since each  $S_i - I \in \mathcal{X}$ , and  $\|S - I\| < \eta < 1$ , so that  $S$  is invertible. Also,  $R_n^{-1} \rightarrow S^{-1}$  in norm. Let  $\mathcal{N}_\infty = S\mathcal{N}$ , and let  $\mathcal{N}_n = R_n \mathcal{N}$ ,  $1 \leq n < \infty$ . We claim that  $\text{alg } \mathcal{N}_\infty$  contains  $W_n, W_n^*$  for all  $n$ . Fix  $n$ . Then  $W_n, W_n^* \in \text{alg } \mathcal{N}_n$ . Let  $a_n = R_n^{-1} W_n R_n$ ,  $b_n = R_n^{-1} W_n^* R_n$ . Then  $a_n, b_n \in \text{alg } \mathcal{N}$ . Since  $S_l$  commutes with  $W_n, W_n^*$  for  $l \geq n + 1$  we have  $R_l a_n R_l^{-1} = R_n a_n R_n^{-1} = W_n$ ,  $l \geq n + 1$ , so that  $S a_n S^{-1} = \lim_l R_l a_n R_l^{-1} = W_n$ , and similarly  $S b_n S^{-1} = W_n^*$ . So  $\text{diag}(\text{alg } \mathcal{N}_\infty)$  contains the infinite proper directed system  $\langle W_1, W_2, \dots \rangle$ . Thus by Lemmas 2.5 and 2.6 the nest  $\mathcal{N}_\infty$  has uniform infinite multiplicity. □

**THEOREM 2.10.** *Let  $\mathcal{N}, \mathcal{M}$  be arbitrary continuous nests. Then given  $\varepsilon > 0$  there exists a unitary  $U$  and a positive invertible operator  $T$  with  $T - I$  compact and  $\|T - I\| < \varepsilon$  such that  $\mathcal{M} = UT\mathcal{N}$ .*

*Proof.* By Lemma 2.9, given  $0 < \eta < 1$ , there exist  $S_1, S_2$  with  $S_i - I$  compact and  $\|S_i - I\| < \eta$  such that  $S_1\mathcal{N}$  and  $S_2\mathcal{M}$  have uniform infinite multiplicity, hence are unitarily equivalent. Thus there is a unitary  $V$  such that  $VS_1\mathcal{N} = S_2\mathcal{M}$ , and hence  $S_2^{-1}VS_1\mathcal{N} = \mathcal{M}$ . If  $S_2^{-1}VS_1 = UT$  is the polar decomposition, then  $T - I \in \mathcal{X}$ . For sufficiently small  $\eta$  we have  $\|T - I\| < \varepsilon$ . □

**THEOREM 2.11.** *Let  $\mathcal{N}$  be a maximal nest. Then given  $\varepsilon > 0$  there exists a positive invertible operator  $T$  with  $T - I \in \mathcal{K}$ ,  $\|T - I\| < \varepsilon$ , such that  $T\mathcal{N}$  has multiplicity one.*

*Proof.* Let  $\{E_n\}$  be the set of minimal  $\mathcal{N}$ -intervals. Since  $\mathcal{N}$  is maximal, these have rank 1. Let  $E = \sum E_n$ . Then  $\mathcal{N}_E$  has purely atomic core and is maximal, and if  $E \neq I$  then  $\mathcal{N}_{E^\perp}$  is a continuous nest since its core has no nonzero minimal projection.  $\mathcal{N}_E$  has multiplicity one, so if  $E = I$  we are finished. Otherwise, let  $T_0$  be a positive invertible operator in  $L(E^\perp H)$  which differs from the identity on  $E^\perp H$  by a compact of norm  $< \varepsilon$  such that  $T_0\mathcal{N}_{E^\perp}$  has multiplicity one. Regard  $T_0$  as an operator in  $L(H)$  with  $T_0 = E^\perp T_0 E^\perp$  and let  $T = T_0 + E$ . We claim  $T\mathcal{N}$  has multiplicity one. Each  $E_n$  commutes with  $T$  so that  $E_n, E \in \text{alg } T\mathcal{N}$ . Since  $E_n$  is semi-invariant for  $\text{alg } \mathcal{N}$  and  $E_n = TE_n T^{-1}$ , it follows that  $E_n$  is also semi-invariant for  $\text{alg } T\mathcal{N} = T(\text{alg } \mathcal{N})T^{-1}$ , so is a  $T\mathcal{N}$ -interval, hence is in  $\mathcal{C}_{T\mathcal{N}}$ . Thus  $E \in \mathcal{C}_{T\mathcal{N}}$ . Since  $(T\mathcal{N})_E = \mathcal{N}_E$  and  $(T\mathcal{N})_{E^\perp} = T_0\mathcal{N}_{E^\perp}$ , the cores of both  $(T\mathcal{N})_E$  and  $(T\mathcal{N})_{E^\perp}$  are maximal abelian; hence since  $\mathcal{C}_{T\mathcal{N}}$  is the direct sum of these, it is maximal abelian also. Thus  $T\mathcal{N}$  has multiplicity one. □

**COROLLARY 2.12.** *An operator whose lattice of invariant subspaces contains a maximal nest is similar to a hyperintransitive operator. In particular, every compact operator is similar to a hyperintransitive compact operator.*

### 3. Absolute continuity and the ideal $\mathcal{R}_\mathcal{N}^\infty$

We develop methods for dealing with the atomic core parts of nests needed in the proof of our general result, Theorem 4.7, which is given as a factorization result. We first establish Proposition 3.6 which gives a means of transfer of information from a continuous special case to the atomic core case, and then to the general case. Theorem 3.7 then gives the continuous special case result needed. The information transferred concerns the idempotent structure related to a concept of absolute continuity of a similarity transformation, and this is related to factorization.

Let  $H, K$  be Hilbert spaces with  $T \in L(H, K)$  invertible. Let  $\mathcal{N}$  be a nest in  $H$  and let  $\tilde{\mathcal{N}} = T\mathcal{N}$ . We write  $P(N)$  for the projection onto a subspace  $N$ . An  $\mathcal{N}$ -interval is a difference projection  $P(M) - P(N)$ ,  $M \supset N$ ,  $M, N \in \mathcal{N}$ . If  $E$  is an  $\mathcal{N}$ -interval with endpoints  $M, N$  let  $\tilde{E}$  denote the  $\tilde{\mathcal{N}}$ -interval with endpoints  $TM$  and  $TN$ . (In general  $\tilde{E} \neq P(TEH)$ .)  $E \leftrightarrow \tilde{E}$  is a one-to-one correspondence between the family of  $\mathcal{N}$ -intervals and the family of  $\tilde{\mathcal{N}}$ -intervals.

LEMMA 3.1. *Let  $K = \|T\|\|T^{-1}\|$ . Then for all  $\mathcal{N}$ -intervals  $E$ , for all  $A \in \text{alg } \mathcal{N}$  we have  $\|EAE\| \leq K\|\tilde{E}(TAT^{-1})\tilde{E}\| \leq K^2\|EAE\|$ . In particular,  $EAE = 0$  if and only if  $\tilde{E}(TAT^{-1})\tilde{E} = 0$ .*

*Proof.* Let  $\hat{E} = T^{-1}\tilde{E}T$ . Then  $\hat{E} \in \text{alg } \mathcal{N}$ . If  $A, B \in \text{alg } \mathcal{N}$  then  $\tilde{E}(TAT^{-1})\tilde{E}(TBT^{-1})\tilde{E} = \tilde{E}(TABT^{-1})\tilde{E}$  since  $\tilde{E}$  is semi-invariant for  $\text{alg } \tilde{\mathcal{N}}$ , so that  $\hat{E}A\hat{E}B\hat{E} = \hat{E}AB\hat{E}$ . We claim that  $E\hat{E}E = E$ ,  $\hat{E}E\hat{E} = \hat{E}$ . Let  $M, N$  be the upper and lower endpoints of  $E$ , let  $\tilde{M} = TM$ ,  $\tilde{N} = TN$ , and let  $S_M = T^{-1}P(\tilde{M})T$ ,  $S_N = T^{-1}P(\tilde{N})T$ . Then  $S_M, S_N$  are idempotents with ranges  $M, N$  respectively, and  $\hat{E} = S_M - S_N$ . So  $E\hat{E}E = ES_ME - ES_NE$ . Note that  $S_ME = E$  since  $\text{ran}(S_M) = M$  and  $M \supset E$ . Also  $ES_N = 0$  since  $\text{ran}(S_N) = N$  and  $N \perp E$ . So  $E\hat{E}E = E$ . A similar argument shows that  $\hat{E}E\hat{E} = \hat{E}$ . So if  $A \in \text{alg } \mathcal{N}$  then  $\hat{E}A\hat{E} = \hat{E}\hat{E}A\hat{E} = \hat{E}EA\hat{E} = \hat{E}A\hat{E}$ . Similarly,  $EAE = E\hat{E}A\hat{E}E$ . So

$$\|EAE\| \leq \|\hat{E}A\hat{E}\| \leq K\|\tilde{E}(TAT^{-1})\tilde{E}\|$$

and

$$\|\tilde{E}(TAT^{-1})\tilde{E}\| = \|\tilde{E}(TEAET^{-1})\tilde{E}\| \leq K\|EAE\|. \quad \square$$

Let  $\mathcal{N}$  be a complete nest. We define an  $\mathcal{N}$ -partition to be a family  $\{E_\lambda\}$  of mutually orthogonal  $\mathcal{N}$ -intervals with  $\sum E_\lambda = I$ . Since we are considering separable Hilbert space every  $\mathcal{N}$ -partition is finite or countably infinite. Hereafter we will use the notation  $\mathcal{R}_{\mathcal{N}}$  to denote the Jacobson radical of a nest algebra  $\text{alg } \mathcal{N}$ .

THEOREM 3.2 (Ringrose). *Let  $\mathcal{N}$  be a complete nest. If  $T \in \text{alg } \mathcal{N}$  then  $T \in \mathcal{R}_{\mathcal{N}}$  if and only if given  $\varepsilon > 0$  there exists a finite  $\mathcal{N}$ -partition  $E_1, \dots, E_n$  such that  $\|E_iTE_i\| < \varepsilon$ ,  $i = 1, \dots, n$ .*

Definition 3.3. Let  $\mathcal{N}$  be a complete nest on a separable Hilbert space. We define  $\mathcal{R}_{\mathcal{N}}^\infty$  to be the class of all operators  $A$  in  $\text{alg } \mathcal{N}$  with the property that given  $\varepsilon > 0$  there exists a (perhaps infinite)  $\mathcal{N}$ -partition  $\{E_n\}$  with  $\|E_nAE_n\| < \varepsilon$ , for all  $n$ .

The following is a statement of some basic properties of  $\mathcal{R}_{\mathcal{N}}^\infty$ . Proofs are not difficult and are left to the reader.

PROPOSITION 3.4.  *$\mathcal{R}_{\mathcal{N}}^\infty$  is a norm-closed 2-sided ideal in  $\text{alg } \mathcal{N}$  containing  $\mathcal{R}_{\mathcal{N}}$ , with equality if and only if  $\mathcal{N}$  has only a finite number of members.  $\mathcal{R}_{\mathcal{N}}^\infty$  is contained in the strong closure of  $\mathcal{R}_{\mathcal{N}}$ .  $\mathcal{R}_{\mathcal{N}}^\infty$  is diagonal-disjoint in the sense that  $\mathcal{R}_{\mathcal{N}}^\infty \cap \mathcal{D}_{\mathcal{N}} = \{0\}$ . The algebraic sum  $\mathcal{R}_{\mathcal{N}}^\infty + \mathcal{D}_{\mathcal{N}}$  is a norm-closed subalgebra of  $\text{alg } \mathcal{N}$  with the property that for every  $D \in \mathcal{D}_{\mathcal{N}}$ ,  $R \in \mathcal{R}_{\mathcal{N}}^\infty$ , we have  $\|D + R\| \geq \|D\|$ . All expectations from  $\mathcal{L}(H)$  onto  $\mathcal{D}_{\mathcal{N}}$  annihilate  $\mathcal{R}_{\mathcal{N}}^\infty$  and so agree on  $\mathcal{R}_{\mathcal{N}}^\infty + \mathcal{D}_{\mathcal{N}}$ ; their common restriction to  $\mathcal{R}_{\mathcal{N}}^\infty + \mathcal{D}_{\mathcal{N}}$  is a homomorphism. If  $\mathcal{N} \supset \mathcal{M}$  are nests, then  $\mathcal{R}_{\mathcal{N}} \supset \mathcal{R}_{\mathcal{M}}$  and  $\mathcal{R}_{\mathcal{N}}^\infty \supset \mathcal{R}_{\mathcal{M}}^\infty$ .*

The following answers a question of Erdős [7].

PROPOSITION 3.5. *If  $\mathcal{N}$  is a continuous nest the commutator ideal of  $\text{alg } \mathcal{N}$  is all of  $\text{alg } \mathcal{N}$ .*

*Proof.* Let  $\mathcal{M}$  be a continuous nest of uniform multiplicity 2 such that  $\text{alg } \mathcal{M} + \mathcal{K} = \text{alg } \mathcal{N} + \mathcal{K}$ . Then  $\mathcal{D}_{\mathcal{M}}$  contains a partial isometry  $V$  with orthogonal initial and final spaces such that  $VV^* + V^*V = I$ . Let  $P = VV^*$ . Then  $VV^* - V^*V = P - P^\perp = 2P - I$ , a symmetry. Choose  $A, B \in \text{alg } \mathcal{N}$  such that  $A - V$  and  $B - V^*$  are compact. Then  $(AB - BA)^2 - I$  is a compact operator in  $\text{alg } \mathcal{N}$  and so is in  $\mathcal{R}_{\mathcal{N}}$  since  $\mathcal{N}$  is continuous; hence  $(AB - BA)$  is invertible in  $\text{alg } \mathcal{N}$ .  $\square$

Let  $\mathcal{N}, \tilde{\mathcal{N}}, T, H, K, E, \tilde{E}$  be as in (3.1) and let  $\mathcal{E}(\cdot)$  and  $\tilde{\mathcal{E}}(\cdot)$  denote the projection-valued measures on  $\mathcal{N}, \tilde{\mathcal{N}}$ , respectively, investigated by J. Erdős [6]. Here  $\mathcal{N}, \tilde{\mathcal{N}}$  are regarded as compact metrizable spaces with the order topology, and the PVM relates order intervals  $[0, N]$  to the projections (or subspaces)  $N \in \mathcal{N}$ . Let  $\phi_T(\cdot)$  denote the order isomorphism  $N \rightarrow TN$ . If the composition  $\tilde{E}(\phi_T(\cdot))$  is absolutely continuous with respect to  $\mathcal{E}(\cdot)$  we will say that  $T$  acts *absolutely continuously* on  $\mathcal{N}$ . It follows from [6] that this term is well defined. If  $\phi_T(\cdot)$  is implemented by a unitary then  $T$  acts absolutely continuously on  $\mathcal{N}$ . We examine deviation from this behavior. If  $\delta$  is a closed subset of  $\mathcal{N}$  with  $\mathcal{E}(\delta) = 0$  then the complement  $\delta^c$  can be written as a countable disjoint union of open order intervals  $\{\omega_n\}$ , so that  $\{\mathcal{E}(\omega_n)\}$  is an  $\mathcal{N}$ -partition. We will have  $\tilde{\mathcal{E}}(\phi_T(\delta)) = 0$  if and only if  $\{\tilde{\mathcal{E}}(\phi_T(\omega_n))\}$  is an  $\tilde{\mathcal{N}}$ -partition. By regularity and reversing this argument it follows that  $T$  acts absolutely continuously on  $\mathcal{N}$  if and only if  $T$  *preserves partitions* for  $\mathcal{N}$  in the sense that if  $E_n$  is an  $\mathcal{N}$ -partition then  $\tilde{E}_n$  is an  $\tilde{\mathcal{N}}$ -partition.

If  $P$  is an idempotent in  $L(H)$ , by a *normalizer* for  $P$  we mean an invertible operator  $T$  such that  $TPT^{-1}$  is selfadjoint. In general, an idempotent has many normalizers.

PROPOSITION 3.6. *Let  $\mathcal{N}$  be a complete nest and let  $T$  be an invertible operator. The following are equivalent:*

- (i)  $T$  acts absolutely continuously on  $\mathcal{N}$ .
- (ii)  $T$  acts absolutely continuously on every subnest of  $\mathcal{N}$ .
- (iii)  $T$  acts absolutely continuously on every subnest of  $\mathcal{N}$  with purely atomic core.
- (iv)  $\mathcal{R}_{\mathcal{N}}^\infty$  does not contain a nonzero idempotent  $P$  which is normalized by  $T$ .
- (v)  $\mathcal{R}_{T\mathcal{N}}^\infty \supseteq T\mathcal{R}_{\mathcal{N}}^\infty T^{-1}$ .

*Proof.* (i)  $\rightarrow$  (iv). Suppose  $P$  is an idempotent in  $\mathcal{R}_{\mathcal{N}}^\infty$  for which  $TPT^{-1}$  is selfadjoint. Let  $\{E_n\}$  be an  $\mathcal{N}$ -partition with  $\|E_n P E_n\| < 1$ , for all  $n$ . Each

$E_n P E_n$  is an idempotent by semi-invariance of  $E_n$ , and has norm less than 1, and so is 0. Let  $\tilde{\mathcal{N}} = T\mathcal{N}$  and let  $\tilde{E}_n$  be the  $\tilde{\mathcal{N}}$ -interval corresponding to  $E_n$ . Let  $E = T P T^{-1}$ . By Lemma 3.1 we have  $\tilde{E}_n E \tilde{E}_n = 0$  for all  $n$ . Since  $E \in \mathcal{D}_{\tilde{\mathcal{N}}}$ ,  $E$  commutes with all  $\tilde{\mathcal{N}}$ -intervals. Since  $T$  acts absolutely continuously on  $\mathcal{N}$  we have  $\sum \tilde{E}_n = I$ . Thus  $E = 0$ , and hence  $P = 0$ .

(iv)  $\rightarrow$  (i). Let  $\{E_n\}$  be an arbitrary  $\mathcal{N}$ -partition. Suppose  $\{\tilde{E}_n\}$  is not an  $\tilde{\mathcal{N}}$ -partition. Then  $\sum \tilde{E}_n \neq I$ . Let  $F = I - \sum \tilde{E}_n$  and let  $P = T^{-1} F T$ . Then  $P \in \text{alg } \mathcal{N}$ ,  $T$  normalizes  $P$ , and by Lemma 3.1,  $E_n P E_n = 0$  for all  $n$  so that  $P \in \mathcal{R}_{\mathcal{N}}^\infty$ , a contradiction.

(i)  $\rightarrow$  (v). If  $A \in \mathcal{R}_{\mathcal{N}}^\infty$  then given  $\varepsilon > 0$  there exists an  $\mathcal{N}$ -partition  $\{E_n\}$  with  $\|E_n A E_n\| < \varepsilon$  for all  $n$ . Lemma 3.1 then implies that  $\|\tilde{E}_n (T A T^{-1}) \tilde{E}_n\| \leq \|T\| \|T^{-1}\| \varepsilon$  for all  $n$ . Since (i) is assumed,  $\{\tilde{E}_n\}$  is an  $\tilde{\mathcal{N}}$ -partition. Thus  $T A T^{-1} \in \mathcal{R}_{\tilde{\mathcal{N}}}^\infty$ .

(v)  $\rightarrow$  (iv). If  $P$  is an idempotent in  $\mathcal{R}_{\mathcal{N}}^\infty$  normalized by  $T$ , then  $T P T^{-1} \in \mathcal{D}_{\tilde{\mathcal{N}}} \cap \mathcal{R}_{\tilde{\mathcal{N}}}^\infty = 0$ , so that  $P = 0$ .

(iv)  $\rightarrow$  (ii). If  $\mathcal{M}$  is a subnest of  $\mathcal{N}$  then  $\mathcal{R}_{\mathcal{N}}^\infty \supseteq \mathcal{R}_{\mathcal{M}}^\infty$ ; so if we assume (iv),  $\mathcal{R}_{\mathcal{M}}^\infty$  contains no nonzero idempotent normalized by  $T$ ; so by (iv)  $\rightarrow$  (i) above,  $T$  acts absolutely continuously on  $\mathcal{M}$ .

We have shown that (i), (ii), (iv), (v) are equivalent. For (iii), if  $T$  fails to act absolutely continuously on  $\mathcal{N}$  then  $\mathcal{R}_{\mathcal{N}}^\infty$  contains a nonzero idempotent  $P$  normalized by  $T$  and from the above argument there exists an  $\mathcal{N}$ -partition  $\{E_n\}$  with  $E_n P E_n = 0$  for all  $n$ . Let  $\mathcal{M}$  be the subnest consisting of all members of  $\mathcal{N}$  spanned by the  $\{E_n\}$ . Then  $\mathcal{M}$  is complete, has purely atomic core, and since  $\{E_n\}$  is an  $\mathcal{M}$ -partition,  $P \in \mathcal{R}_{\mathcal{M}}^\infty$ . □

**THEOREM 3.7.** *Let  $\mathcal{N}$  be a continuous nest of multiplicity one. Then for each  $\varepsilon > 0$ ,  $\mathcal{R}_{\mathcal{N}}^\infty$  contains a nonzero idempotent  $Q$  with  $Q - Q^*$  compact such that  $\|Q - Q^*\| < \varepsilon$ .*

*Proof.* Let  $\mu$  be Lebesgue measure on  $[0, 1]$ , and let  $\Omega$  be an open dense subset of  $[0, 1]$  with  $\mu(\Omega) < 1$ . Define the finite Borel measure  $\nu$  on  $[0, 1]$  by  $\nu(G) = \mu(G \cap \Omega)$ . Then  $\mu, \nu$  are nonatomic Borel measures, and the nests  $\mathcal{N}$  and  $\mathcal{M}$  whose elements are  $N_t = L^2([0, t], \mu)$  and  $M_t = L^2([0, t], \nu)$ ,  $0 \leq t \leq 1$ , in the Hilbert spaces  $H = L^2([0, 1], \mu)$ ,  $H' = L^2([0, 1], \nu)$ , respectively, are continuous, complete nests of multiplicity one. The map  $N_t \rightarrow M_t$  is an order isomorphism of nests. By Andersen [1] there exists a unitary  $U: H' \rightarrow H$  such that the nests  $N_t$  and  $\tilde{M}_t = U M_t$  are a *compactly perturbed pair* in the sense of [1]. Using the same notation for the subspaces  $N_t, M_t, \tilde{M}_t$  as for the orthogonal projections onto them, we have in particular that  $N_t - \tilde{M}_t \in \mathcal{K}$  for all  $t$  and that  $\text{alg } \mathcal{N} + \mathcal{K} = \text{alg } \tilde{\mathcal{M}} + \mathcal{K}$ , where here  $\tilde{\mathcal{M}} = \{\tilde{M}_t; 0 \leq t \leq 1\}$ .

Now write  $\Omega$  as the union of a sequence  $F_n$  of disjoint, relatively open intervals in  $[0, 1]$ , and let  $t_n, s_n$  be the upper and lower endpoints, respectively, of  $F_n$ . Let  $E_n = N_{t_n} - N_{s_n}$ , the sequence of projections corresponding to multiplication by the characteristic functions  $\chi_{F_n}$  on  $H$ . Let  $\hat{E}_n = M_{t_n} - M_{s_n}$  and  $\tilde{E}_n = \tilde{M}_{t_n} - \tilde{M}_{s_n}$ . Then  $\tilde{E}_n = U\hat{E}_nU^*$ . We have  $\sum E_n \neq I$  while  $\sum \hat{E}_n = I$  and hence  $\sum \tilde{E}_n = I$ . From the preceding paragraph we have  $E_n - \tilde{E}_n \in \mathcal{K}$ .

Let  $E = \sum E_n$ . Since  $\text{alg } \mathcal{N} + \mathcal{K} = \text{alg } \tilde{\mathcal{M}} + \mathcal{K}$  we may choose  $A \in \text{alg } \tilde{\mathcal{M}}$  with  $E^\perp - A$  compact. Then  $A^2 - A \in \text{alg } \tilde{\mathcal{M}} \cap \mathcal{K}$ . By Lemma 2.1 there exists an idempotent  $P \in \text{alg } \tilde{\mathcal{M}}$  with  $A - P \in \text{alg } \tilde{\mathcal{M}} \cap \mathcal{K}$ . Since  $E^\perp$  has infinite rank,  $P \neq 0$ . Since  $E_n - \tilde{E}_n \in \mathcal{K}$  and  $P - E^\perp \in \mathcal{K}$ , we have  $\tilde{E}_n P \tilde{E}_n \in \text{alg } \tilde{\mathcal{M}} \cap \mathcal{K}$  for all  $n$ . By semi-invariance,  $\tilde{E}_n P \tilde{E}_n$  is an idempotent, and since  $\text{alg } \tilde{\mathcal{M}} \cap \mathcal{K} \subseteq \mathcal{R}_{\tilde{\mathcal{M}}}$  it must be quasinilpotent. Hence  $\tilde{E}_n P \tilde{E}_n = 0$  for all  $n$ . So  $P \in \mathcal{R}_{\tilde{\mathcal{M}}}^\infty$  since  $\{\tilde{E}_n\}$  is an  $\tilde{\mathcal{M}}$ -partition. Since  $\tilde{\mathcal{M}}$  and  $\mathcal{N}$  are continuous nests of multiplicity one, they are unitarily equivalent so that  $\mathcal{R}_{\mathcal{N}}^\infty$  also contains a nonzero idempotent  $Q$  which is a compact perturbation of a selfadjoint projection.

Now let  $K = Q - Q^*$ . Since  $\mathcal{N}$  is continuous, an elementary construction using compactness of  $\mathcal{N}$  in its order topology yields a finite subnest  $\mathcal{N}_0$  such that the compression of  $K$  to each  $\mathcal{N}_0$ -interval has norm  $< \epsilon$ . Let  $F_1, \dots, F_n$  denote the  $\mathcal{N}_0$ -intervals. Let  $Q_0 = \sum F_i Q F_i$ . Then  $Q - Q_0 \in \text{rad}(\text{alg } \mathcal{N})$ , and  $Q_0 \notin \text{rad}(\text{alg } \mathcal{N})$  since  $Q_0$  is not quasinilpotent; so  $Q_0 \neq 0$ . We have  $Q_0 \in \mathcal{R}_{\mathcal{N}}^\infty$  since the latter is an ideal,  $\|Q_0 - Q_0^*\| < \epsilon$ , and  $Q_0 - Q_0^*$  is compact. Now replace  $Q$  with  $Q_0$  if necessary. □

#### 4. Factorization problems and absolute continuity

A complete nest  $\mathcal{N}$  is said to have the *factorization property* if every positive invertible operator  $T$  factors  $T = A^*A$  for some invertible operator  $A \in \text{alg } \mathcal{N}$  with  $A^{-1}$  also in  $\text{alg } \mathcal{N}$ . By Lemma 1.1,  $\mathcal{N}$  has the factorization property if and only if every invertible operator  $T$  factors  $T = US$  for some  $S \in (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^{-1}$ . In [3], Arveson showed that a complete nest order isomorphic to a subset of the extended integers has the factorization property. We first generalize this, establishing a portion of Theorem 4.7. Then using Theorem 3.7, we show that the atomic core nest  $\mathcal{N}(Q)$  has the same idempotent property as does the continuous special case (Proposition 4.5). The nest  $\mathcal{N}(Q)$  is a model (Lemma 4.4) in the sense that it can be embedded, after compression, in every atomic core complete uncountable nest. This yields the requisite idempotent result for the general atomic core case. The case where  $\mathcal{N}$  has a continuous part is dealt with separately in Lemma 4.3. These prove Theorem 4.6 which is the complete generalization of Theorem 3.7, and which yields our main factoriza-

tion Theorem 4.7. We conclude with the weak factorization Theorem 4.9 which generalizes that in [5].

**PROPOSITION 4.1.** *A complete countable nest has the factorization property.*

*Proof.* Let  $\mathcal{N}$  be complete and countable, and let  $T$  be an invertible operator. Let  $\tilde{\mathcal{N}} = T\mathcal{N}$ . Let  $\{E_n\}$  be the set of minimal  $\mathcal{N}$ -intervals and  $\{\tilde{E}_n\}$  the corresponding  $\tilde{\mathcal{N}}$ -intervals. Then  $\sum \tilde{E}_n = I$ , since otherwise, if  $\tilde{E} = \sum \tilde{E}_n$ , then  $\tilde{F} = I - \tilde{E}$  would be a nonzero core projection for  $\tilde{\mathcal{N}}$  for which  $\tilde{\mathcal{N}}_{\tilde{F}}$  is a continuous nest, contradicting the countability of  $\tilde{\mathcal{N}}$ . Let  $N_n, M_n$  be the upper and lower endpoints of  $E_n$ . The range of  $E_n$  is  $N_n \ominus M_n$  and the range of  $\tilde{E}_n$  is  $TN_n \ominus TM_n$ , so it follows that  $E_n$  and  $\tilde{E}_n$  have the same rank. For each  $n$  let  $U_n$  be a partial isometry with initial projection  $E_n$  and final projection  $\tilde{E}_n$ , and let  $U = \sum U_n$ , the sum converging strongly. Then  $U$  is a unitary operator. For each  $N \in \mathcal{N}$ ,  $P(N)$  is a sum of minimal  $\mathcal{N}$ -intervals and  $P(TN)$  is the sum of the corresponding  $\tilde{\mathcal{N}}$ -intervals, so we have  $UN = TN$ ,  $N \in \mathcal{N}$ . Let  $S = U^*T$ . Then  $S$  is invertible, and  $SN = N$ ,  $S^{-1}N = N$ ,  $N \in \mathcal{N}$ , so  $S \in (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^{-1}$ . We have  $T = US$ . □

**LEMMA 4.2.** *Let  $\mathcal{N}$  be a complete nest and  $F$  a projection in  $\mathcal{D}_{\mathcal{N}}$ . If  $\mathcal{R}_{\mathcal{N}_F}^\infty$  contains a nonzero idempotent then so does  $\mathcal{R}_{\mathcal{N}}$ .*

*Proof.* Let  $P$  be an idempotent in  $\mathcal{R}_{\mathcal{N}_F}^\infty$ . Extend to an idempotent in  $L(H)$  by  $P_1 = P \oplus 0$ . Then  $P_1 \in \text{alg } \mathcal{N}$ . Let  $F_n$  be a sequence of  $\mathcal{N}_F$ -intervals with  $\sum F_n$  the identity in  $L(FH)$  such that  $F_n P F_n = 0$  for all  $n$ . A simple argument shows that there exist  $\mathcal{N}$ -intervals  $\{E_n\}$  with  $\sum E_n = I$  and  $F_n = F E_n|_{FH}$ . Then since  $E_n$  commutes with  $F$  we have  $E_n P_1 E_n = 0$  for all  $n$ . So  $P_1 \in \mathcal{R}_{\mathcal{N}}^\infty$ . □

**LEMMA 4.3.** *Let  $\mathcal{N}$  be a continuous nest. Then for each  $\varepsilon > 0$ ,  $\mathcal{R}_{\mathcal{N}}^\infty$  contains a nonzero idempotent  $Q$  with  $Q - Q^*$  compact and  $\|Q - Q^*\| < \varepsilon$ .*

*Proof.* Fix  $\varepsilon > 0$ . Let  $F$  be a projection in  $\mathcal{D}_{\mathcal{N}}$  such that  $F, F^\perp$  have central support  $I$  and such that  $F\mathcal{D}_{\mathcal{N}}|_{FH}$  is abelian. Then  $\mathcal{N}_F$  is a continuous nest of multiplicity one; so by Theorem 3.7,  $\mathcal{R}_{\mathcal{N}_F}^\infty$  contains a nonzero idempotent  $Q$  with  $Q - Q^*$  compact and  $\|Q - Q^*\| < \varepsilon$ . Now by Lemma 4.2 and the construction in the proof,  $\mathcal{R}_{\mathcal{N}}^\infty$  contains such an idempotent. □

Let  $\mathbf{Q}$  denote the rational numbers with the usual linear order. A countable, linearly ordered set with the property that no element has an immediate predecessor or immediate successor and with no first or last element is order isomorphic to  $\mathbf{Q}$  (cf. [22, § 6]). Let  $\{e_r: r \in \mathbf{Q}\}$  be an orthonormal basis for separable Hilbert space, and for each  $r \in \mathbf{R}$  let  $N_r$  be the closed linear span of  $\{e_s: s \in \mathbf{Q}, s < r\}$  and let  $M_r$  be the closed span of  $\{e_s: s \in \mathbf{Q}, s \leq r\}$ . Then the nest consisting of  $N_r, M_r, r \in \mathbf{R}$  together with  $\{0\}$  and  $H$  is a complete

uncountable nest. Denote this by  $\mathcal{N}(\mathbf{Q})$ . Let  $E_r$  denote the rank-1 projection onto the span of  $e_r$ . The core of  $\mathcal{N}(\mathbf{Q})$  is generated by these projections so is purely atomic, and  $\mathcal{N}(\mathbf{Q})$  is multiplicity free. The set of minimal intervals for any nest has a natural linear order “ $\ll$ ” induced by the nest order. Here  $E \ll F$  means the upper endpoint of  $E$  is contained in the lower endpoint of  $F$ . If  $\mathcal{M}$  is any multiplicity-free nest with purely atomic core, let  $\mathcal{E}_{\mathcal{M}}$  denote the set of nonzero minimal  $\mathcal{M}$ -intervals. If  $(\mathcal{E}_{\mathcal{M}}, \ll)$  is order isomorphic to  $\mathbf{Q}$ , then an obvious construction shows that  $\mathcal{M}$  is unitarily equivalent to  $\mathcal{N}(\mathbf{Q})$ . Every uncountable multiplicity free nest with purely atomic core is in a fundamental way related to  $\mathcal{N}(\mathbf{Q})$ .

**LEMMA 4.4.** *Let  $\mathcal{N}$  be a complete uncountable nest with purely atomic core with the property that for all  $N \in \mathcal{N}$  different from  $\{0\}$  and  $H$  the nests  $\mathcal{N}_N$  and  $\mathcal{N}_{N^\perp}$  are also uncountable. Then  $\mathcal{N}$  has a complete subnest  $\mathcal{N}_0$  for which the set of nonzero minimal intervals  $(\mathcal{E}_{\mathcal{N}_0}, \ll)$  is order isomorphic to  $\mathbf{Q}$ . ( $\mathcal{N}_0$  will not in general be multiplicity free.)*

*Proof.* Define an equivalence relation “ $\sim$ ” on  $\mathcal{N}$  by writing  $N \sim M$  if the closed order interval with endpoints  $N, M$  contains at most a countable number of members of  $\mathcal{N}$ . It is easily verified that the equivalence classes are closed order intervals, many of which may be singleton sets. Let  $\mathcal{N}_1$  be the subset of  $\mathcal{N}$  consisting of  $\{0\}$ ,  $H$ , and the upper and lower endpoints of these equivalence classes, and let  $\mathcal{N}_0$  be its completion. Note that the set of minimal intervals for  $\mathcal{N}_1$  and  $\mathcal{N}_0$  are identical and are precisely those projections of the form  $P(M) - P(N)$  where  $M, N$  are the upper and, respectively, lower endpoints of an equivalence class in  $\mathcal{N}$  under “ $\sim$ ” whose upper and lower endpoints are different. The nest  $\mathcal{N}_0$  is uncountable, since otherwise an obvious argument would show that  $\mathcal{N}$  must be countable. The definition of “ $\sim$ ” and consideration of the relationship of endpoints show that no element of  $(\mathcal{E}_{\mathcal{N}_0}, \ll)$  has an immediate predecessor or immediate successor. Also, our hypothesis guarantees that  $(\mathcal{E}_{\mathcal{N}_0}, \ll)$  has no first or last element. Since  $(\mathcal{E}_{\mathcal{N}_0}, \ll)$  is a countable, linearly ordered set with this property, it is order isomorphic to  $\mathbf{Q}$ . □

**PROPOSITION 4.5.** *For each  $\varepsilon > 0$ ,  $\mathcal{R}_{\mathcal{N}(\mathbf{Q})}^\infty$  contains a nonzero idempotent  $Q$  with  $Q - Q^*$  compact and  $\|Q - Q^*\| < \varepsilon$ .*

*Proof.* Let  $\mathcal{N}$  be a multiplicity free, continuous nest. By Theorem 3.7 and the last part of the proof of (3.6), there exists a complete subnest  $\mathcal{M}$  with purely atomic core such that  $\mathcal{R}_{\mathcal{M}}^\infty$  contains a nonzero idempotent  $Q_0$  with  $Q_0 - Q_0^*$  compact and  $\|Q_0 - Q_0^*\| < \varepsilon$ . The minimal  $\mathcal{M}$ -intervals have infinite rank. We construct a nest unitarily equivalent to  $\mathcal{N}(\mathbf{Q})$  having the requisite properties. By Proposition 4.1, Proposition 3.6 and Lemma 1.1, the nest  $\mathcal{M}$  must be

uncountable. Let  $M_a = \vee \{N \in \mathcal{M}: \mathcal{M}_n \text{ is countable}\}$  and let  $M_b = \wedge \{N \in \mathcal{M}: \mathcal{M}_{N^\perp} \text{ is countable}\}$ . Since there are a countable number of minimal  $\mathcal{M}$ -intervals and since the projection  $P(M)$  of  $H$  onto  $M$  is a sum of minimal  $\mathcal{M}$ -intervals for each  $M \in \mathcal{M}$ , it follows that  $\mathcal{M}_{M_b^\perp}$  are countable. Also,  $M_a$  has no immediate successor and  $M_b$  has no immediate predecessor. Let  $E = P(M_b) - P(M_a)$ . Then  $\mathcal{M}_E$  is a complete uncountable nest with purely atomic core which satisfies the hypotheses of Lemma 4.4. Since  $E$  is an  $\mathcal{M}$ -interval,  $EQ_0E$  is an idempotent, and it is easily verified that  $EQ_0|_{EH}$  is in  $\mathcal{R}_{\mathcal{M}_E}^\infty$ . We must show that it is nonzero. We have  $P(M_a)Q_0P(M_a) = 0$  since its restriction to  $M_a$  is an idempotent in  $\mathcal{R}_{\mathcal{M}_{M_a}}^\infty$  and  $\mathcal{M}_{M_a}$  is countable, and similarly  $P(M_b)Q_0P(M_b) = 0$ . So since  $P(M_a)$  and  $P(M_b)$  commute with  $Q_0$  modulo the radical  $\mathcal{R}_{\mathcal{M}}$ , if  $EQ_0E = 0$  then necessarily  $Q_0 \in \mathcal{R}_{\mathcal{M}}$ . But then  $Q_0$  must be quasinilpotent, and a quasinilpotent idempotent is necessarily 0; so  $Q_0 = 0$ , a contradiction. Now  $EQ_0E \neq 0$ , and thus  $EQ_0|_{EH}$  is a nonzero idempotent in  $\mathcal{R}_{\mathcal{M}_E}^\infty$ .

Utilizing Lemma 4.4 and the construction in its proof, let  $\mathcal{L}$  denote a complete subnest of  $\mathcal{M}_E$  for which  $(\mathcal{E}_{\mathcal{L}}, \ll)$  is order isomorphic to  $\mathbf{Q}$  and for which the compression of  $\mathcal{M}_E$  to each minimal  $\mathcal{L}$ -interval is a countable nest. Let  $Q = EQ_0|_{EH}$ . An argument analogous to that in the above paragraph shows that the compression of  $Q$  to each minimal  $\mathcal{L}$ -interval is 0. (Here we are using the semi-invariance property of intervals. The compression of an idempotent to an interval is an idempotent.) So since  $\mathcal{L}$  has purely atomic core and  $Q \in \text{alg } \mathcal{L}$  we have  $Q \in \mathcal{R}_{\mathcal{L}}^\infty$ . From our construction we have  $Q - Q^*$  compact and  $\|Q - Q^*\| < \varepsilon$ . Also, since our construction began with a continuous nest every nonzero minimal  $\mathcal{L}$ -interval has infinite rank. Let  $K$  denote the Hilbert space  $EH$ . Let  $\{E_r: r \in \mathbf{Q}\}$  denote an indexing of  $\mathcal{E}_{\mathcal{L}}$  by the rational numbers, with  $E_r \ll E_s$  when  $r < s$ . For each  $r \in \mathbf{Q}$  let  $\{e_{rs}: s \in \mathbf{Q}\}$  denote an o.n. basis for the Hilbert space  $E_rK$ . Then  $\{e_{rs}: r, s \in \mathbf{Q}\}$  is an o.n. basis for  $K$ . Let  $E_{rs}$  denote the rank-1 projection onto the span of  $e_{rs}$ , and let  $\mathcal{L}_1$  be the complete nest generated by the subspaces  $L_{rs} = \overline{\text{sp}}\{e_{\lambda\mu}: \lambda < r \text{ or } \lambda = r \text{ and } \mu \leq s\}$ . Then  $\mathcal{L}_1$  is a multiplicity free nest with purely atomic core which contains  $\mathcal{L}$  as a subnest. The minimal  $\mathcal{L}_1$ -intervals are the  $E_{rs}$ .  $(\mathcal{E}_{\mathcal{L}_1}, \leq)$  has no first or last element and no element has an immediate predecessor or immediate successor, and so  $\mathcal{L}_1$  is unitarily equivalent to  $\mathcal{N}(\mathbf{Q})$ . We have  $\mathcal{R}_{\mathcal{L}_1}^\infty \supseteq \mathcal{R}_{\mathcal{L}}^\infty$ ; so  $Q$  is a nonzero idempotent in  $\mathcal{R}_{\mathcal{L}_1}^\infty$  with  $Q - Q^*$  compact,  $\|Q - Q^*\| < \varepsilon$ .  $\square$

**THEOREM 4.6.** *Let  $\mathcal{N}$  be a complete nest. Then  $\mathcal{R}_{\mathcal{N}}^\infty$  contains a nonzero idempotent if and only if  $\mathcal{N}$  is uncountable. If  $\mathcal{N}$  is a complete uncountable nest then for each  $\varepsilon > 0$ ,  $\mathcal{R}_{\mathcal{N}}^\infty$  contains a nonzero idempotent  $Q$  with  $Q - Q^*$  compact and  $\|Q - Q^*\| < \varepsilon$ .*

*Proof.* Let  $\mathcal{N}$  be a complete nest. If  $\mathcal{N}$  is countable then  $\mathcal{R}_{\mathcal{N}}^\infty$  fails to contain a nonzero idempotent by (1.1), (3.6), (4.1). Assume  $\mathcal{N}$  is uncountable.

Fix  $\varepsilon > 0$ . If  $\mathcal{E}_{\mathcal{N}}$  is not purely atomic there exists a core projection  $F$  such that  $\mathcal{N}_F$  is continuous. Then (4.2), (4.3), and the construction in the proof of (4.2), imply that  $\mathcal{R}_{\mathcal{N}}^{\infty}$  contains a nonzero idempotent  $Q$  with  $Q - Q^*$  compact,  $\|Q - Q^*\| < \varepsilon$ . If  $\mathcal{E}_{\mathcal{N}}$  is purely atomic, construct  $M_a, M_b, E$  as in the second paragraph of the proof of (4.5). Then  $\mathcal{N}_E$  satisfies the hypotheses of (4.4) and so contains a complete subnest  $\mathcal{M}$  such that  $(\mathcal{E}_{\mathcal{M}}, \ll)$  is order isomorphic to  $\mathbf{Q}$ . The elements of  $\mathcal{E}_{\mathcal{M}}$  here may have varying rank. Let  $G$  be a projection in  $\mathcal{D}_{\mathcal{M}}$  with the property that  $GE_rG$  has rank 1 ( $\neq 0$ ) for all  $E_r \in \mathcal{E}_{\mathcal{M}}$ . Then  $\mathcal{M}_G$  is multiplicity free, and  $(\mathcal{E}_{\mathcal{M}_G}, \ll)$  is order isomorphic to  $\mathbf{Q}$  also, so that  $\mathcal{M}_G$  is unitarily equivalent to  $\mathcal{N}(\mathbf{Q})$ . Now  $\mathcal{R}_{\mathcal{M}_G}^{\infty}$  contains a nonzero idempotent  $Q_0$  with  $Q_0 - Q_0^*$  compact,  $\|Q_0 - Q_0^*\| < \varepsilon$ ; hence by (4.2) and its proof, so does  $\mathcal{R}_{\mathcal{M}}^{\infty}$ . Since  $\mathcal{N}_E \supseteq \mathcal{M}$  we have  $\mathcal{R}_{\mathcal{N}_E}^{\infty} \supseteq \mathcal{R}_{\mathcal{M}}^{\infty}$  by (3.4) so that  $\mathcal{R}_{\mathcal{N}_E}^{\infty}$  contains such an idempotent. Another application of (4.2) and its proof shows that  $\mathcal{R}_{\mathcal{N}}^{\infty}$  contains the required idempotent.  $\square$

**THEOREM 4.7.** *A complete nest has the factorization property if and only if it is countable. If a complete nest  $\mathcal{N}$  is uncountable then for each  $\varepsilon > 0$  there exists a positive invertible operator  $T$  with  $T - I$  compact and with  $\|T - I\| < \varepsilon$  such that  $T^{1/2}$  fails to act absolutely continuously on  $\mathcal{N}$  so  $T$  does not equal  $A^*A$  for any invertible  $A \in \text{alg } \mathcal{N}$  with  $A^{-1} \in \text{alg } \mathcal{N}$  also.*

*Proof.* The “if” part was proved in (4.1). If  $\mathcal{N}$  is uncountable then given  $1 > \varepsilon > 0$  by (4.6),  $\mathcal{R}_{\mathcal{N}}^{\infty}$  contains a nonzero idempotent  $Q$  with  $Q - Q^*$  compact,  $\|Q - Q^*\| < \varepsilon/5$ . Let  $T = Q^*Q + (I - Q)^*(I - Q)$ . Then  $T^{1/2}$  normalizes  $Q$  so that  $T^{1/2}$  fails to act absolutely on  $\mathcal{N}$  by (3.6). If  $T$  factored  $T = A^*A$  for some  $A \in \text{alg } \mathcal{N} \cap (\text{alg } \mathcal{N})^{-1}$  then by (1.1) the order isomorphism  $N \rightarrow T^{1/2}N$  would be implemented by a unitary, a contradiction, since a unitary acts absolutely continuously on every nest.  $T - I$  is compact, and a computation shows  $T - I = (Q^* - Q)(2Q - I)$ . We have  $Q^*Q = Q + (Q^* - Q)Q$ , so that  $\|Q\|^2 = \|Q^*Q\| < (1 + \varepsilon/5)\|Q\|$ , and thus  $\|Q\| < 1 + \varepsilon/5$ . Hence  $\|T - I\| < (\varepsilon/5)(3 + 2\varepsilon/5) < \varepsilon$ .  $\square$

**LEMMA 4.8.** *If  $\mathcal{N}$  is a nest containing an infinite subnest  $\mathcal{N}_0$  of order type that of the extended integers such that the minimal  $\mathcal{N}_0$ -intervals have equal rank, then every invertible operator  $T \in L(H)$  admits a factorization  $T = US$  with  $U$  unitary and with  $S \in \mathcal{R}_{\mathcal{N}}^{\infty}$  and invertible in  $L(H)$ .*

*Proof.* Since  $\mathcal{N}_0$  is countable, by (4.1) and (1.1) there is a unitary  $V \in L(H)$  and an operator  $A \in (\text{alg } \mathcal{N}_0) \cap (\text{alg } \mathcal{N}_0)^{-1}$  such that  $T = VA$ . Let  $N_i, -\infty < i < \infty$ , denote the nontrivial members of  $\mathcal{N}_0$  and for each  $i$  let  $W_i$  denote a partial isometry with initial space  $N_{i+1} \oplus N_i$  and final space  $N_i \oplus N_{i-1}$ . Let  $W = \sum W_i$ . Then  $W_i, W \in \text{alg}(\mathcal{N}_0)$ ,  $W$  is unitary, and it is clear that  $W \in \mathcal{R}_{\mathcal{N}_0}^{\infty}$ . So  $WA \in \mathcal{R}_{\mathcal{N}_0}^{\infty} \subset \mathcal{R}_{\mathcal{N}}^{\infty} \subset \text{alg}(\mathcal{N})$  by (3.4). Now let  $S = WA$  and

$U = VW^*$ , which gives the required factorization. □

**THEOREM 4.9.** *Let  $\mathcal{N}$  be an arbitrary complete nest. Then every invertible operator  $T$  in  $L(H)$  admits a factorization  $T = US$  with  $U$  unitary and  $S \in \text{alg } \mathcal{N}$ , provided that we do not require  $S^{-1} \in \text{alg } \mathcal{N}$  also. Similarly every positive invertible operator  $T$  admits a factorization  $T = A^*A$  with  $A \in \text{alg } \mathcal{N}$ , provided we do not require  $A^{-1} \in \text{alg } \mathcal{N}$ .*

*Proof.* By (4.1) and (1.1) we may assume  $\mathcal{N}$  is uncountable. Let  $L = \vee \{N \in \mathcal{N} : \mathcal{N}_N \text{ is countable}\}$ ,  $M = \wedge \{N \in \mathcal{N} : \mathcal{N}_{N^\perp} \text{ is countable}\}$ . Then  $\mathcal{N}_L, \mathcal{N}_{M^\perp}$  are countable,  $L < M$ ,  $L = L_+$  and  $M = M_-$ . Also, each order interval  $(L, L')$  for  $L' > L$  is uncountable, and each  $(M', M)$ ,  $M' < M$ , is uncountable. Thus an obvious construction yields a doubly infinite sequence  $N_n \in \mathcal{N}$ ,  $-\infty < n < \infty$ ,  $L < N_n < N_{n+1} < M$ , with  $L = \lim_{n \rightarrow -\infty} N_n$  and  $M = \lim_{n \rightarrow +\infty} N_n$ , such that each order interval  $(N_n, N_{n+1})$  is uncountable. In particular, each difference space  $N_{n+1} \ominus N_n$  is infinite dimensional. Let  $\mathcal{N}_0$  denote the subnest of  $\mathcal{N}$  consisting of  $L$  and  $M$  together with the predecessors of  $L$ , the successors of  $M$ , and the elements of  $N_n$ ,  $-\infty < n < \infty$ .  $\mathcal{N}_0$  is complete and countable. If  $L = 0$  and  $M = H$  then (4.8) completes the proof. If not, fix an invertible  $T$  and modify the proof of (4.8): choose a unitary  $W$  and an operator  $A \in (\text{alg } \mathcal{N}_0) \cap (\text{alg } \mathcal{N}_0)^{-1}$  with  $T = WA$ . For  $-\infty < n < \infty$ , let  $V_n$  be a p.i. with support  $N_{n+1} \ominus N_n$  and range  $N_n \ominus N_{n-1}$ . Using the same notation for the projections onto  $L, M$  as for the subspaces, let  $V = L + M^\perp + \sum_n V_n$ , the sum converging strongly, so that  $V$  is unitary and  $V \in \text{alg } \mathcal{N}$ . We claim  $VA \in \text{alg } \mathcal{N}$ . Since  $A \in \text{alg } \mathcal{N}_0$  and  $(\mathcal{N}_0)_L = \mathcal{N}_L, (\mathcal{N}_0)_{M^\perp} = \mathcal{N}_{M^\perp}$ , we have  $LAL, M^\perp AM^\perp \in \text{alg } \mathcal{N}$ , and also  $LAL^\perp \in \text{alg } \mathcal{N}$  since  $L \in \mathcal{N}$ . We have

$$VA = LA + M^\perp A + \sum V_n A = LAL + LAL^\perp + M^\perp AM^\perp + \sum V_n A.$$

Also  $V_n \in \mathcal{R}_{\mathcal{N}_0}$  so that  $V_n A \in \mathcal{R}_{\mathcal{N}_0} \subset \mathcal{R}_{\mathcal{N}} \subset \text{alg } \mathcal{N}$ ; hence  $\sum V_n A \in \text{alg } \mathcal{N}$  since  $\text{alg } \mathcal{N}$  is strongly closed. Now let  $S = VA$  and  $U = WV^*$ , which gives the required factorization of  $T$ .

If  $R$  is a positive invertible operator then  $R^{1/2}$  factors  $US$  as above, where  $S \in \text{alg } \mathcal{N}$ , so that  $R = S^*S$ , as required. □

UNIVERSITY OF NEBRASKA, LINCOLN, NEBRASKA

REFERENCES

[1] N. T. ANDERSEN, Compact perturbations of reflexive algebras, *J. Funct. Anal.* **38** (1980), 366–400.  
 [2] W. B. ARVESON, Analyticity in operator algebras, *Amer. J. Math.* **89** (1967), 578–642.  
 [3] \_\_\_\_\_, Interpolation problems in nest algebras, *J. Funct. Anal.* **20** (1975), 208–233.

- [4] W. B. ARVESON, Operator algebras and invariant subspaces, *Ann. of Math.* **100** (1974), 433–532.
- [5] M. D. CHOI, and A. FEINTUCH, Operator factorization with respect to nest algebras, preprint.
- [6] J. A. ERDŐS, Unitary invariants for nests, *Pacific J. Math.* **23** (1967), 229–256.
- [7] \_\_\_\_\_, Non-selfadjoint operator algebras, *Proc. R. Irish Acad.* **81A** (1981), 127–145.
- [8] T. FALL, W. B. ARVESON, and P. S. MUHLY, Perturbations of nest algebras, *J. Operator Theory* **1** (1979), 137–150.
- [9] F. GILFEATHER, and D. R. LARSON, Nest subalgebras of von Neumann algebras, *Adv. in Math.* **46** (1982), 171–199.
- [10] I. C. GOHBERG, and M. G. KREIN, *Introduction to the Theory of Linear Non-Selfadjoint Operators*, Transl. Math. Monographs **18**, A.M.S., R.I. (1969).
- [11] \_\_\_\_\_, *Theory and Applications of Volterra Operators in Hilbert Space*, Transl. Math. Monographs **24**, A.M.S., R.I. (1970).
- [12] R. V. KADISON, and I. M. SINGER, Triangular operator algebras, *Amer. J. Math.* **82** (1960), 227–259.
- [13] E. C. LANCE, Cohomology and perturbations of nest algebras, *Proc. London Math. Soc.* (3) **43** (1981), 334–356.
- [14] D. R. LARSON, Doctoral dissertation, Univ. of California, Berkeley (1976).
- [15] \_\_\_\_\_, On the structure of certain reflexive operator algebras, *J. Funct. Anal.* **31** (1979), 275–292.
- [16] \_\_\_\_\_, A solution to a problem of J. R. Ringrose, *Bull. (New Series) A.M.S.* **7** (1982), 243–246.
- [17] C. RICKART, *General Theory of Banach Algebras*, Van Nostrand, New York (1960).
- [18] J. R. RINGROSE, Super-diagonal forms for compact linear operators, *Proc. London Math. Soc.* (3) **12** (1962), 367–384.
- [19] \_\_\_\_\_, Algebraic isomorphisms between ordered bases, *Amer. J. Math.* **83** (1961), 463–478.
- [20] \_\_\_\_\_, On some algebras of operators, *Proc. London Math. Soc.* (3) **15** (1965), 61–83.
- [21] \_\_\_\_\_, On some algebras of operators II, *Proc. London Math. Soc.* (3) **16** (1966), 385–402.
- [22] K. KURATOWSKI, and A. MOSTOWSKI, *Set Theory*, North-Holland, Amsterdam (1968).

(Received May 10, 1982)

(Revised March 27, 1985)

*Added in proof.* In subsequent work, K. Davidson has generalized several of our results to show that an arbitrary order isomorphism between nests which preserves the dimensions of minimal intervals can be implemented by a similarity transformation, and W. Arveson has generalized the perturbation results of N. Andersen for nests utilized in this paper to a much wider class of commutative subspace lattices.