

RANK-ONE DECOMPOSITION OF OPERATORS AND CONSTRUCTION OF FRAMES

KERI A. KORNELSON AND DAVID R. LARSON

ABSTRACT. The construction of frames for a Hilbert space \mathcal{H} can be equated to the decomposition of the frame operator as a sum of positive operators having rank one. This realization provides a different approach to questions regarding frames with particular properties and motivates our results. We find a necessary and sufficient condition under which any positive finite-rank operator B can be expressed as a sum of rank-one operators with norms specified by a sequence of positive numbers $\{c_i\}$. Equivalently, this result proves the existence of a frame with B as its frame operator and with vector norms $\{\sqrt{c_i}\}$. We further prove that, given a non-compact positive operator B on an infinite dimensional separable real or complex Hilbert space, and given an infinite sequence $\{c_i\}$ of positive real numbers which has infinite sum and which has supremum strictly less than the essential norm of B , there is a sequence of rank-one positive operators, with norms given by $\{c_i\}$, which sum to B in the strong operator topology.

These results generalize results by Casazza, Kovačević, Leon, and Tremain, in which the operator is a scalar multiple of the identity operator (or equivalently the frame is a tight frame), and also results by Dykema, Freeman, Kornelson, Larson, Ordower, and Weber in which $\{c_i\}$ is a constant sequence.

INTRODUCTION

The existence and characterization of frames with a variety of additional properties is an area of active research which has resulted in recent papers including [1], [2], and [6]. These questions can also be phrased in terms of the expression of positive operators as sums of rank-one positive operators. Therefore, although motivated by applications in signal processing, the results have independent interest to operator theory.

Throughout this paper, \mathcal{H} will be a real or complex separable Hilbert space. For \mathbb{J} a real or countably infinite index set, the collection of vectors

2000 *Mathematics Subject Classification*. Primary 42C15, 47N40; Secondary 47C05, 46B28.

Key words and phrases. frames, tight frames, rank-one operators, positive operators, rank-one decomposition.

The research of the first author was supported in part by an NSF VIGRE postdoctoral fellowship. The research of the second author was supported in part by a grant from the NSF.

$\{x_j\}_{j \in \mathbb{J}} \subset \mathcal{H}$ is a *frame* if there exist positive constants $D \geq C > 0$ such that for every $x \in \mathcal{H}$:

$$C\|x\|^2 \leq \sum_{j \in \mathbb{J}} |\langle x, x_j \rangle|^2 \leq D\|x\|^2$$

A frame is called a *tight frame* if $C = D$, and a *Parseval frame* if $C = D = 1$.

We use elementary tensor notation for a rank-one operator on \mathcal{H} . Given $u, v, x \in \mathcal{H}$, the operator $u \otimes v$ is defined by $(u \otimes v)x = \langle x, v \rangle u$ for $x \in \mathcal{H}$. The operator $u \otimes u$ is a projection if and only if $\|u\| = 1$.

Given a frame $\{x_i\}$ for \mathcal{H} , the frame operator is defined to be the map $S : \mathcal{H} \rightarrow \mathcal{H}$ taking $x \rightarrow \sum_i \langle x, x_i \rangle x_i$. We can write S in the form:

$$(1) \quad S = \sum_i x_i \otimes x_i$$

where the convergence is in the strong operator topology (SOT). It is a well-known result that $\{x_i\}$ is a tight frame with frame bound λ if and only if $S = \lambda I$, where I is the identity operator on \mathcal{H} . (see [8]) Therefore, finding a tight frame is equivalent to decomposing a scalar multiple of the identity as a sum of rank-one operators.

Given \mathcal{H} a finite dimensional Hilbert space, and $\{c_i\}$ a sequence of positive real numbers, Casazza, Kovačević, Leon, and Tremain found in [2] a necessary and sufficient condition (called the Fundamental Frame Inequality) for the existence of a tight frame for \mathcal{H} with the norms of the vectors given by $\{c_i\}$. As stated above, this is equivalent to decomposing a scalar multiple of the identity as a sum of rank-one positive operators with prescribed norms. In Section 1, we generalize this result to the case where the scalar is replaced with an arbitrary positive invertible operator on \mathcal{H} (in fact, we state the result for positive operators which are not necessarily invertible), thus obtaining the condition for the existence of non-tight frames with specified norms.

In Section 2, \mathcal{H} is an infinite dimensional separable real or complex Hilbert space. Recall that the essential norm of an operator on \mathcal{H} is defined to be $\|B\|_{ess} = \inf\{\|B - K\|\}$ where K is a compact operator on \mathcal{H} . We prove that, given B a non-compact positive operator on \mathcal{H} and given $\{c_i\}$ a sequence of positive real numbers which sum to infinity and for which $\sup_i \{c_i\} < \|B\|_{ess}$, B can be expressed as a sum of rank-one operators having norms given by $\{c_i\}$, with convergence in the SOT. This implies that every positive invertible operator with essential norm strictly greater than $\sup_i \{c_i\}$ is the frame operator for a frame with prescribed norms $\{\sqrt{c_i}\}$. This result is the generalization of the infinite-dimensional result in [2], which restricts to tight frames. It also generalizes the results in [6] regarding ellipsoidal tight frames, in which the sequence $\{c_i\}$ is a constant sequence.

The authors would like to thank Eric Ricard for his helpful comments during the Linear Analysis Workshop at Texas A&M University in August, 2003.

1. FINITE DIMENSIONS

The following lemma was shown to us by Marc Ordower. It was a key step in the proof of Proposition 6 in [6].

Lemma 1. *Let B be a positive operator with rank n and nonzero eigenvalues $b_1 \geq b_2 \geq \dots \geq b_n > 0$ and with a corresponding orthonormal set of eigenvectors $\{e_i\}_{i=1}^n$. If c is a positive number with $b_j \geq c \geq b_{j+1}$ for some $1 \leq j \leq n-1$, then there is a unit vector x in $\text{span}\{e_j, e_{j+1}\}$ such that $B - c(x \otimes x)$ is positive and has rank $n-1$.*

Proof. If A is a finite-rank self-adjoint operator and $c \geq \|A\|$ is a constant, then the least eigenvalue of A is given by $\|(A + cI)^{-1}\|^{-1} - c$. It follows that the function giving the least eigenvalue is continuous on the set of finite-rank self-adjoint operators. Let $A_x = B - c(x \otimes x)$ for x an arbitrary unit vector in $\text{span}\{e_j, e_{j+1}\}$. The least eigenvalue of A_x is nonnegative for $x = e_j$ and nonpositive for $x = e_{j+1}$, hence $\text{span}\{e_j, e_{j+1}\}$ contains a unit vector x for which the least eigenvalue of A_x is zero. Therefore, A_x is positive of rank $n-1$. □

Remark 1. Since the trace of A_x is $(\sum_1^n b_i) - c$, and eigenvalues other than b_j and b_{j+1} are unchanged, the remaining nonzero eigenvalue must be

$$\tilde{b} = b_j + b_{j+1} - c$$

Moreover, the new eigenvalue maintains the j^{th} position in the ordering of all the eigenvalues:

$$b_1 \geq \dots \geq b_{j-1} \geq \tilde{b} \geq b_{j+2} \geq \dots \geq b_n$$

Theorem 2. *Let B be a positive operator with rank n and nonzero eigenvalues $b_1 \geq b_2 \geq \dots \geq b_n > 0$. Let $\{c_i\}_{i=1}^k$; $k \geq n$ be a nonincreasing sequence of positive numbers such that $\sum_{i=1}^k c_i = \sum_{j=1}^n b_j$. There exist unit vectors $\{x_i\}_{i=1}^k$ such that*

$$B = \sum_{i=1}^k c_i (x_i \otimes x_i)$$

if and only if for each p , $1 \leq p \leq n-1$,

$$(2) \quad \sum_{i=1}^p c_i \leq \sum_{i=1}^p b_i$$

Note: When a positive invertible operator B can be written in the form of this theorem, we say that B admits a *rank-one decomposition corresponding to $\{c_i\}$* .

Remark 2. The hypotheses in Theorem 2 reduces to exactly the Fundamental Frame Inequality from [2] when $b_1 = b_2 = \dots = b_n$. Thus, an immediate corollary of Theorem 2 is the result from [2] that the Fundamental Frame Inequality is a necessary and sufficient condition for the existence of a tight frame for \mathbb{R}^n consisting of vectors with norms $\sqrt{c_i}$; $i = 1, \dots, k$. We include this as Corollary 5.

Proof of Theorem 2. Assuming the property (2) holds, we use strong induction on the rank n of the operator. The case $n = 1$ when B is already a rank-one operator is clear. Let $\{e_i\}_1^n$ be an orthonormal set of eigenvectors for b_1, b_2, \dots, b_n , respectively.

Case 1: If $b_1 \geq c_1 \geq b_n$, there exists an index l ; $1 \leq l \leq n - 1$ such that $b_l \geq c_1 \geq b_{l+1}$. By Lemma 1, there exists a unit vector x in the span of e_l and e_{l+1} such that $B - c_1(x \otimes x)$ has rank $n - 1$ and eigenvalues in nonincreasing order $b_1, \dots, b_{l-1}, \tilde{b}, b_{l+2}, \dots, b_n$, where $\tilde{b} = b_l + b_{l+1} - c_1$. Let $x_1 = x$. It remains to check that the inequalities (2) hold for the operator $B - c_1(x_1 \otimes x_1)$ and the sequence $\{c_2, \dots, c_k\}$. Clearly, for $1 \leq r \leq l$, we have

$$c_2 + \dots + c_r \leq c_1 + \dots + c_{r-1} \leq b_1 + \dots + b_{r-1}$$

For $l + 1 \leq r \leq n - 1$,

$$\begin{aligned} c_1 + \dots + c_l + c_{l+1} + \dots + c_r &\leq b_1 + \dots + b_l + b_{l+1} + \dots + b_r \\ \Rightarrow c_2 + \dots + c_l + c_{l+1} + \dots + c_r &\leq b_1 + \dots + (b_l + b_{l+1} - c_1) + \dots + b_r \\ \Rightarrow c_2 + \dots + c_l + c_{l+1} + \dots + c_r &\leq b_1 + \dots + \tilde{b} + \dots + b_r \end{aligned}$$

Since we have reduced to an operator of rank $n - 1$ which satisfies the hypotheses of the theorem, induction gives the remaining elements of the rank-one decomposition of B corresponding to the sequence $\{c_i\}_1^k$.

Case 2: If $c_1 < b_n$, then we are unable to use Lemma 1. We select rank-one operators to subtract from B which preserve the rank and decrease the smallest eigenvalue until the Case 1 property is attained.

Let p be the largest integer such that $c_1 + c_2 + \dots + c_p < b_n$. Note that Case 2 can only occur for $k > n$. Since the sums of each sequence are equal, $c_{p+1} + \dots + c_k > b_1 + \dots + b_{n-1}$, but because $c_{p+1} \leq c_1 \leq b_n \leq b_{n-1}$, the sum on the left must have more terms. Therefore, $k - p > n - 1$, or alternatively, $1 \leq p \leq k - n$. Let $x_i = e_n$ for each $i = 1, 2, \dots, p$. The operator $B - \sum_{i=1}^p c_i(x_i \otimes x_i)$ still has rank n and the eigenvalues in decreasing order are $b_1, b_2, \dots, b_{n-1}, \tilde{b}$, where $\tilde{b} = b_n - \sum_{i=1}^p c_i$. By the selection of p , c_{p+1} exceeds or equals \tilde{b} , the smallest eigenvalue of $B - \sum_{i=1}^p c_i(x_i \otimes x_i)$. The method of Case 1 can now be applied to the operator $B - \sum_{i=1}^p c_i(x_i \otimes x_i)$ and the remaining sequence c_{p+1}, \dots, c_k .

Conversely, assume B has a rank-one decomposition corresponding to $\{c_i\}$. Given a fixed j with $1 \leq j \leq k$, define P to be the orthogonal projection onto the span of $\{x_i\}_{i=1}^j$. Clearly, $\text{rank } P \leq j$. For each i , define

$P_i = x_i \otimes x_i$. We then have:

$$PBP = \sum_{i=1}^k c_i P P_i P \geq \sum_{i=1}^j c_i P P_i P = \sum_{i=1}^j c_i P_i$$

and therefore $\text{trace}(PBP) \geq \text{trace}\left(\sum_{i=1}^j c_i P_i\right) = c_1 + c_2 + \cdots + c_j$. We next show that $\text{trace}(PBP) \leq b_1 + b_2 + \cdots + b_n$, which will complete the proof.

Let $\{e_i\}_{i=1}^n$ be as above, and for notational purposes, set $b_{n+1} = 0$. For $1 \leq i \leq n$, define $Q_i = e_1 \otimes e_1 + \cdots + e_i \otimes e_i$. A calculation verifies that B can be written

$$B = \sum_{i=1}^n (b_i - b_{i+1}) Q_i$$

Then, $PBP = \sum_{i=1}^n (b_i - b_{i+1}) P Q_i P$. Observe that $\text{trace}(P Q_i P) = \text{trace}(P Q_i) \leq \min\{\text{rank } P, \text{rank } Q_i\} \leq \min\{j, i\}$. Therefore,

$$\begin{aligned} \text{trace}(PBP) &= \sum_{i=1}^n (b_i - b_{i+1}) \text{trace}(P Q_i P) \\ &\leq \sum_{i=1}^j (b_i - b_{i+1}) i + j \sum_{i=j+1}^n (b_i - b_{i+1}) \\ &= (b_1 + b_2 + \cdots + b_j - j b_{j+1}) + j (b_{j+1} - b_{n+1}) \\ &= b_1 + b_2 + \cdots + b_j \end{aligned}$$

□

Theorem 2 will be used in the proof of Proposition 7. To demonstrate the theorem, we have included the following example for the case where $\mathcal{H} = \mathbb{R}^2$.

Example 1. Let $B = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$. We find the rank-one decomposition of B corresponding to $\{c_i\}_{i=1}^4 = 3, 3, 2, 1$.

Since $c_1 < b_2$ (Case 2), we take $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This leaves $B - 3(x_1 \otimes x_1) = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ to decompose. We now have c_2 between the two eigenvalues 5 and 1 (Case 1). Let $x(t) = \sqrt{1-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sqrt{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{1-t} \\ \sqrt{t} \end{bmatrix}$. We wish to find t such that $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} - 3(x(t) \otimes x(t))$ has rank one, i.e. the determinant is zero. Straightforward calculation shows the solution $t = \frac{1}{6}$. Take $x_2 = x(\frac{1}{6}) = \begin{bmatrix} \sqrt{\frac{5}{6}} \\ \sqrt{\frac{1}{6}} \end{bmatrix}$. The remainder $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} - 3(x_2 \otimes x_2) = \begin{bmatrix} \frac{5}{2} & -\frac{\sqrt{5}}{2} \\ -\frac{\sqrt{5}}{2} & \frac{1}{2} \end{bmatrix}$ has rank one

and it's range is spanned by the unit vector $z = \begin{bmatrix} \sqrt{\frac{5}{6}} \\ -\sqrt{\frac{1}{6}} \end{bmatrix}$. We then must have $x_3 = x_4 = z$, which completes the decomposition.

$$3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} \frac{5}{6} & \frac{\sqrt{5}}{6} \\ \frac{\sqrt{5}}{6} & \frac{1}{6} \end{bmatrix} + (2+1) \begin{bmatrix} \frac{5}{6} & -\frac{\sqrt{5}}{6} \\ -\frac{\sqrt{5}}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} = B$$

The next example demonstrates that a rank-one decomposition is impossible when property (2) does not hold.

Example 2. The operator $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ does not have a rank-one decom-

position corresponding to the sequence $\{c_i\}_{i=1}^3 = \{4, 4, 1\}$. To see this, assume that we can find unit vectors x_1, x_2, x_3 such that $B = 4(x_1 \otimes x_1) + 4(x_2 \otimes x_2) + (x_3 \otimes x_3)$. Let P be the projection onto the span of $\{x_1, x_2\}$. Then $\text{trace}(PBP) = \sum_{i=1}^3 c_i \text{trace} P(x_i \otimes x_i)P \geq 4 + 4 = 8$, because $P(x_i \otimes x_i)P = x_i \otimes x_i$, which has trace 1.

Given P any projection with rank two, using the argument from the last paragraph of Theorem 2, we have $\text{trace}(PBP)$ less than or equal to 7, i.e. the sum of the largest two eigenvalues. The contradiction implies B does not have a rank-one decomposition corresponding to $\{4, 4, 1\}$.

The following proposition lends insight into the underlying geometry of tight frames. We include it for independent interest.

Proposition 3 (Alternate Version of \mathbb{R}^2 Case). *Let B be a positive invertible operator on \mathbb{R}^2 , and express B as $\text{diag}(b_1, b_2)$, where $b_1 \geq b_2 > 0$. Let $\{c_i\}_{i=1}^k$ be positive constants such that $c_1 \geq c_2 \geq \dots \geq c_k$ and*

$$\sum_{i=1}^k c_i = b_1 + b_2$$

If $c_1 \leq b_1$, then there exist unit vectors $\{x_i\}_{i=1}^k$ such that:

$$B = \sum_{i=1}^k c_i (x_i \otimes x_i)$$

Proof. Let

$$x_i = \begin{bmatrix} \cos \theta_i \\ \sin \theta_i \end{bmatrix} \quad \text{so} \quad x_i \otimes x_i = \begin{bmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ \cos \theta_i \sin \theta_i & \sin^2 \theta_i \end{bmatrix}$$

Then let $B = \text{diag}(b_1, b_2)$ and $\{c_i\}_{i=1}^k$ a sequence of positive numbers such that $\sum_{i=1}^k c_i = b_1 + b_2$ and $c_1 \geq c_2 \geq \dots \geq c_k$. Then $B = \sum_{i=1}^k c_i (x_i \otimes x_i)$ if

and only if the following equations hold:

$$\begin{aligned}
 c_1 \cos^2 \theta_1 + c_2 \cos^2 \theta_2 + \cdots + c_k \cos^2 \theta_k &= b_1 \\
 c_1 \sin^2 \theta_1 + c_2 \sin^2 \theta_2 + \cdots + c_k \sin^2 \theta_k &= b_2 \\
 (3) \quad c_1 \cos \theta_1 \sin \theta_1 + c_2 \cos \theta_2 \sin \theta_2 + \cdots + c_k \cos \theta_k \sin \theta_k &= 0
 \end{aligned}$$

These equations, under the condition that $\sum_1^k c_i = b_1 + b_2$, are equivalent to:

$$\begin{aligned}
 \sum_1^k c_i \cos(2\theta_i) &= b_1 - b_2 \\
 (4) \quad \sum_1^k c_i \sin(2\theta_i) &= 0
 \end{aligned}$$

By defining $y_i = c_i \begin{bmatrix} \cos 2\theta_i \\ \sin 2\theta_i \end{bmatrix}$, the above equations become the vector equation:

$$(5) \quad \sum_{i=1}^k y_i - \begin{bmatrix} b_1 - b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the “tip-to-tail” method of adding vectors, we can solve this equation by finding a polygon with $k+1$ sides having lengths $c_1, c_2, \dots, c_k, b_1 - b_2$. We must place the side with length $b_1 - b_2$ along the x-axis, but all of the angles between sides are variable. Such a polygon always exists for $k \geq 2$ provided the greatest side length is less than or equal to the sum of the lengths of the remaining sides. Since the longest vector has either length c_1 or $b_1 - b_2$, there is a solution when $c_1 \leq b_1$:

$$\begin{aligned}
 c_1 &\leq b_1 \\
 2c_1 &\leq (b_1 - b_2) + (b_1 + b_2) \\
 &= (b_1 - b_2) + \sum_{i=1}^k c_i \\
 c_1 &\leq (b_1 - b_2) + (c_2 + c_3 + \cdots + c_k) \\
 \text{and } b_1 - b_2 &\leq b_1 + b_2 \\
 &= c_1 + c_2 + \cdots + c_k
 \end{aligned}$$

□

The following is the frame version of Theorem 2.

Corollary 4. *Let \mathcal{H} be a Hilbert space with finite dimension n . Let B be a positive invertible operator on \mathcal{H} with eigenvalues $b_1 \geq b_2 \geq \cdots \geq b_n > 0$, and for $k \geq n$ let $\{c_i\}_{i=1}^k$ be a sequence of positive numbers with $c_1 \geq c_2 \geq$*

$\dots \geq c_k$ such that $\sum_{i=1}^k c_i = \sum_{j=1}^n b_j$. There exists a frame for \mathcal{H} with k vectors $\{x_i\}_{i=1}^k$ having frame operator B and such that $\|x_i\| = \sqrt{c_i}$, $1 \leq i \leq k$, if and only if condition (2) from Theorem 2 is satisfied.

Proof. B has a rank-one decomposition corresponding to $\{c_i\}_{i=1}^k$ if and only if there exist unit vectors $\{y_i\}_{i=1}^k$ such that

$$B = \sum_{i=1}^k c_i (y_i \otimes y_i) = \sum_{i=1}^k \sqrt{c_i} y_i \otimes \sqrt{c_i} y_i$$

From the expression (1) of the frame operator, $\{x_i\}_{i=1}^k = \{\sqrt{c_i} y_i\}_{i=1}^k$ is a frame with frame operator B and $\|x_i\| = \sqrt{c_i}$. \square

An immediate corollary is Theorem 5.1 from [2], which finds the Fundamental Frame Inequality as the necessary and sufficient condition for the existence of a tight frame with lengths of the vectors prescribed by the sequence. We state this result here.

Corollary 5. [2] *Let \mathcal{H} be a Hilbert space with finite dimension n , and for some $k \geq n$, let $\{a_i\}_{i=1}^k$ be a sequence of positive numbers such that $a_1 \geq a_2 \geq \dots \geq a_k > 0$. If*

$$a_1^2 \leq \frac{1}{n} \sum_{i=1}^k a_i^2 \quad (\text{Fund. Frame Inequality})$$

then there exist unit vectors $\{x_i\}_{i=1}^k$ such that the vectors

$$y_i = a_i x_i$$

form a tight frame for \mathcal{H} . The frame bound will be

$$\lambda = \frac{1}{n} \sum_{i=1}^k a_i^2$$

Proof. This case satisfies the hypotheses of Corollary 4 for $B = \lambda I$ and $c_i = a_i^2$, $i = 1, \dots, k$ since each $b_i = \lambda$, and $a_1^2 \leq \lambda$ implies the remaining inequalities. \square

2. INFINITE DIMENSIONS

The main result, Theorem 6, of this section (or more particularly Corollary 12) is a generalization of Theorem 5.4 from [2], which we list here as Corollary 8. We will use Proposition 7, the operator-theoretic formulation of Corollary 8, in the proof of Theorem 6.

Corollary 12 is also a generalization of Theorem 2 from [6]. This theorem states that, given B a positive operator with $\|B\|_{ess} > 1$, there exists an ellipsoidal tight frame for B with frame bound 1. This is equivalent to the existence of a frame of unit vectors with frame operator B^{-2} , which is the special case of Corollary 12 with $\{c_i\}$ the constant sequence in which each $c_i = 1$.

Theorem 6. *Let B be a positive non-compact operator in $\mathcal{B}(\mathcal{H})$ for \mathcal{H} a real or complex Hilbert space with infinite dimension. If $\{c_i\}_{i=1}^\infty$ is a bounded sequence of positive numbers with $\sup_i c_i < \|B\|_{ess}$ and $\sum_i c_i = \infty$, then there is a sequence of rank-one projections $\{P_i\}_1^\infty \subset \mathcal{B}(\mathcal{H})$ such that*

$$B = \sum_{i=1}^{\infty} c_i P_i$$

where convergence is in the SOT.

Example 3. The condition that $\sup_i \{c_i\}$ be strictly less than $\|B\|_{ess}$ cannot be dropped in general. For instance, if $B = I$ the identity, and

$$\{c_i\}_{i=1}^\infty = \left\{ \frac{2}{3}, 1, 1, 1, \dots \right\}$$

then there is no series $\{P_i\}$ of projections with $\sum_i c_i P_i = I$. Equivalently, there is no sequence of vectors forming a Parseval frame in which the norms of the vectors $\{x_i\}$ are each $\|x_i\| = \sqrt{c_i}$. If, in fact, such a frame did exist, the vectors $\{x_2, x_2, \dots\}$ would necessarily form an orthonormal set. The assumption that the frame is Parseval then implies that x_1 is a unit vector, which is a contradiction.

We will prove Theorem 6 in a series of steps, using the finite dimensional Theorem 2. The first step is Proposition 7, which, as we previously stated, is a reformulation of Theorem 5.4 from [2]. The result is theirs, but our operator-theoretic proof is different from the one given in [2].

Proposition 7. *Let \mathcal{H} be an infinite-dimensional separable Hilbert space. Let $\{c_i\}_{i=1}^\infty$ be a sequence of numbers with $0 < c_i \leq 1$, and suppose $\sum_i c_i = \infty$ and $\sum_i (1 - c_i) = \infty$. Then there is a sequence of rank-one projections $\{P_i\}_{i=1}^\infty$ such that*

$$I = \sum_{i=1}^{\infty} c_i P_i$$

with the sum converging in the strong operator topology.

An immediate corollary is the precise statement of Theorem 5.4 from [2], which gives the existence of a Parseval frame for \mathcal{H} with norms of the frame elements coming from the sequence.

Corollary 8. [2] *If \mathcal{H} is an infinite-dimensional real or complex Hilbert space, and if $\{a_i\}$ is a sequence of real numbers with $0 < a_i \leq 1$ such that $\sum_i a_i^2 = \infty$ and $\sum_i (1 - a_i^2) = \infty$, then there is a Parseval frame $\{x_i\}$ for \mathcal{H} such that $\|x_i\| = a_i$ for all i .*

Proof of Proposition 7. For each $n \in \mathbb{N}$, let $s(n)$ denote the smallest integer such that

$$c_1 + c_2 + \dots + c_{s(n)} > n$$

(Such a value exists for every n since $\sum_i c_i = \infty$.) Denote what we will call the *residual* by

$$r(n) = c_1 + c_2 + \cdots + c_{s(n)} - n$$

and denote the *integer gap* by

$$g(n) = s(n) - n$$

Then $0 < r(n) \leq 1$, and clearly $g(n) \leq g(n+1)$ for all $n \in \mathbb{N}$. Denote $\delta_i = 1 - c_i$, and let $d(n) = \delta_1 + \delta_2 + \cdots + \delta_n$. We have

$$\begin{aligned} g(n) = s(n) - n &> s(n) - (c_1 + \cdots + c_{s(n)}) \\ &= d(s(n)) \\ &\geq d(n) \end{aligned}$$

By hypothesis, $\sum_i \delta_i = \infty$, so $d(n) \rightarrow \infty$ and thus $g(n) \rightarrow \infty$. We can therefore inductively choose an increasing sequence $\{n_i\}_{i=1}^\infty$ of natural numbers such that for all $i \geq 2$,

$$(6) \quad g(n_i) \geq g(n_{i-1}) + 1$$

and

$$(7) \quad d(s(n_{i-1}) + n_i - n_{i-1} + 1) - d(s(n_{i-1})) > 2$$

Note that condition 6 implies that $s(n_i) - s(n_{i-1}) \geq n_i - n_{i-1} + 1$, and condition 7 implies that:

$$(8) \quad c_{(s(n_{i-1}+1))} + \cdots + c_{(s(n_{i-1})+n_i-n_{i-1}+1)} < n_i - n_{i-1} - 1$$

Let $\{e_i\}_1^\infty$ be an orthonormal basis for \mathcal{H} , and let $E_i = e_i \otimes e_i$ for all $i \in \mathbb{N}$. Let

$$B_1 = E_1 + E_2 + \cdots + E_{n_1} + r(n_1)E_{n_1+1}$$

and for $i \geq 2$, let

$$B_i = ((1 - r(n_{i-1}))E_{n_{i-1}+1} + [E_{n_{i-1}+2} + \cdots + E_{n_i}]) + r(n_i)E_{n_i+1}$$

This gives

$$\text{trace}(B_1) = c_1 + \cdots + c_{s(n_1)}$$

and for $i \geq 2$,

$$\begin{aligned} \text{trace}(B_i) &= [1 - r(n_{i-1})] + [n_i - n_{i-1} - 1] + r(n_i) \\ &= [r(n_i) + n_i] - [r(n_{i-1}) + n_{i-1}] \\ &= \sum_{j=1}^{s(n_i)} c_j - \sum_{j=1}^{s(n_{i-1})} c_j \\ &= \sum_{j=s(n_{i-1})+1}^{s(n_i)} c_j \end{aligned}$$

We have $\text{rank}(B_1) = n_1+1$, and for $i \geq 2$, we have $\text{rank}(B_i) \leq n_i - n_{i-1} + 1$. When $i \geq 2$, at least the first $n_i - n_{i-1} - 1$ (in decreasing order) of the

eigenvalues of B_i are 1. (That is, all but the last two eigenvalues of B_i are 1.) This, together with the condition (8), implies that B_i and the sequence $\{c_{s(n_{i-1})+1}, \dots, c_{s(n_i)}\}$ satisfy the hypotheses of Theorem 2. To see this, for $1 \leq p \leq n_i - n_{i-1} - 1$, the inequalities (2) in Theorem 2 are trivially satisfied. For $p = n_i - n_{i-1}$, the inequality (8) implies that the sum of the first $n_i - n_{i-1}$ of the terms in $\{c_{s(n_{i-1})+1}, \dots, c_{s(n_i)}\}$ is less than $n_i - n_{i-1} - 1$, which is less than or equal to the sum of the first $n_i - n_{i-1}$ eigenvalues of B_i .

Therefore, for $i \geq 2$, there exist rank-one projections $\{P_{s(n_{i-1})+1}, \dots, P_{s(n_i)}\}$ in the range of B_i such that

$$B_i = c_{s(n_{i-1})+1}P_{s(n_{i-1})+1} + \dots + c_{s(n_i)}P_{s(n_i)}$$

For the special case B_1 , since there is at most one eigenvalue different from 1, the hypotheses of Theorem 2 are clearly satisfied, and so there exist rank-one projections $P_1, \dots, P_{s(n_1)}$ such that

$$B_1 = \sum_{j=1}^{s(n_1)} c_j P_j$$

For all $i \in \mathbb{N}$,

$$B_1 + \dots + B_i = E_1 + \dots + E_{n_i} + r(n_i)E_{n_i+1}$$

Since $\{r(n_i)\}$ is bounded, the sum $\sum_{i=1}^{\infty} B_i$ converges strongly to the identity I . It follows that $\sum_{i=1}^{\infty} c_i P_i$ converges strongly to I . \square

Corollary 9. *Let E be an infinite rank projection in $\mathcal{B}(\mathcal{H})$, and let $\{c_i\}$ be a sequence of real numbers with $0 < c_i \leq 1$ such that $\sum_i c_i = \infty$ and $\sum_i (1 - c_i) = \infty$. Then E has a rank-one decomposition corresponding to $\{c_i\}$:*

$$E = \sum_{i=1}^{\infty} c_i P_i$$

Proof. Apply Proposition 7 to the subspace $E\mathcal{H}$ and the result follows immediately. \square

In [6], it was proven (Theorem 2) that if B is a positive operator with essential norm strictly greater than 1, then B has a projection decomposition. That is, $B = \sum_i P_i$, where $\{P_i\}$ are self-adjoint projections and convergence is in the strong operator topology. Clearly $\{P_i\}$ can be taken to be rank-one. We need here a stronger result, that each of the projections can be required to have infinite rank.

Proposition 10. *Let B be a positive operator in $\mathcal{B}(\mathcal{H})$ for \mathcal{H} a real or complex separable Hilbert space with infinite dimension, and suppose $\|B\|_{ess} > 1$. Then B has an infinite-rank projection decomposition.*

Proof. The hypothesis that $\|B\|_{ess} > 1$ implies there exists a sequence $\{Q_i\}_{i=1}^{\infty}$ of mutually orthogonal infinite-rank projections commuting with

B such that $\|B|_{Q_i\mathcal{H}}\|_{ess} > 1$ for all i , and with $\sum_i Q_i = I$. Let $A_i = B|_{Q_i\mathcal{H}}$. By Theorem 2 in [6], each A_i has a projection decomposition:

$$A_i = \sum_{j=1}^{\infty} E_{ij}$$

Here we can assume without loss of generality that for each i , $\{E_{i,j}\}_{j=1}^{\infty}$ is an infinite sequence of nonzero projections because A_i is not finite rank. Note that for each j , the projections $\{E_{1,j}, E_{2,j}, \dots\}$ are mutually orthogonal. Let

$$E_j = E_{1,j} + E_{2,j} + \dots$$

with the sum converging strongly. Then E_j is an infinite-rank projection, and we have $\sum_{j=1}^{\infty} E_j = B$, as required. \square

Remark 3. If \mathcal{E} is a countably infinite set of positive numbers, then it is well-known that $\sum_{\lambda \in \mathcal{E}} \lambda$ makes sense as an extended real number, independent of enumeration. It is also clear that if $\sum_{\lambda \in \mathcal{E}} \lambda = \infty$, then there exists a partition $\{\mathcal{E}_i\}_{i=1}^{\infty}$ of \mathcal{E} into infinitely many subsets such that for each i , $\sum_{\lambda \in \mathcal{E}_i} \lambda = \infty$. Similarly, if $\{c_i\}_{i=1}^{\infty}$ is a sequence of positive numbers such that $\sum_i c_i = \infty$, there exists a partition of the sequence into infinite subsequences $\Lambda_i = \{c_{i,j}\}_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty} a_{i,j} = \infty$ for each i . We will use this partition in the proof of the next theorem.

The reader will note that Theorem 11 is an alternate restatement of Theorem 6. We include it because it has a simpler form and has independent interest. We prove it first and then obtain Theorem 6 by a simple scaling argument.

Theorem 11. *Let B be a positive operator in $\mathcal{B}(\mathcal{H})$ for \mathcal{H} with $\|B\|_{ess} > 1$. Let $\{c_i\}_{i=1}^{\infty}$ be any sequence of numbers with $0 < c_i \leq 1$ such that $\sum_i c_i = \infty$. Then there exists a sequence of rank-one projections $\{P_i\}_{i=1}^{\infty}$ such that*

$$B = \sum_{i=1}^{\infty} c_i P_i$$

Proof. Choose α , $0 < \alpha < 1$ such that $\|\alpha B\|_{ess} > 1$. Using Proposition 10, write $\alpha B = \sum_n Q_n$, a projection decomposition with each Q_n having infinite rank. Next, let $\gamma_i = \alpha c_i$ for all i , and partition the sequence $\{\gamma_i\}$ into subsequences Λ_n as described in Remark 3. Note that for each n , $\sup_j \{\gamma_{n,j}\} < 1$. Apply Corollary 9 to each operator Q_n with the subsequence Λ_n , obtaining rank-one projections $\{F_{n,j}\}_{j=1}^{\infty}$ with

$$Q_n = \sum_{j=1}^{\infty} \gamma_{n,j} F_{n,j}$$

Scaling by α , we have the rank-one decomposition of B :

$$B = \frac{1}{\alpha} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \gamma_{n,j} F_{n,j} = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} c_{n,j} F_{n,j}$$

□

Proof of Theorem 6. We apply Theorem 11. Given B non-compact, we have $m := \|B\|_{ess} > 0$ and $\sup_i c_i < m$. Take $\alpha = \frac{1}{m-\epsilon}$ for some $\epsilon > 0$, so that we have $\|\alpha B\|_{ess} > 1$, and let $\gamma_i = \alpha c_i$, which gives $\sup_i \gamma_i < 1$. By Theorem 11, there is a rank-one decomposition for αB corresponding to the sequence $\{\gamma_i\}$:

$$\alpha B = \sum_{i=1}^{\infty} \gamma_i P_i$$

Scaling the above by $m - \epsilon$ gives the final result. □

The following is the frame-theoretic version of Theorem 6. It is now clear that Theorem 5.4 from [2] is the special case of Corollary 12 in which B is the identity operator, and Theorem 2 from [6] is the case where $\{c_i\}$ is a constant sequence.

Corollary 12. *Let \mathcal{H} be a real or complex separable Hilbert space of infinite dimension, and let B be any bounded positive invertible operator on \mathcal{H} . Let $\{a_i\}$ be an arbitrary sequence of positive numbers with infinite sum and with $\sup_i \{a_i^2\} < \|B\|_{ess}$. Then there is a frame $\{z_i\}$ for \mathcal{H} with frame operator B such that $\|z_i\| = a_i$ for all i .*

Proof. Let $c_i = a_i^2$, and let $\alpha = \sup_i \{a_i^2\} = \sup_i \{c_i\}$. Then $\alpha < \|B\|_{ess}$, so by Theorem 6, B has a rank-one decomposition corresponding to $\{c_i\}$. Let $\{x_i\}$ be unit vectors in the range of the rank-one operators of this decomposition, and let $z_i = \sqrt{c_i} x_i$ for all i .

$$B = \sum_i c_i (x_i \otimes x_i) = \sum_i z_i \otimes z_i$$

By the expression (1) of the frame operator, $\{z_i\}$ is the desired frame. □

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TEXAS A&M UNIVERSITY
E-mail address: `keri@math.tamu.edu`

TEXAS A&M UNIVERSITY
E-mail address: `larson@math.tamu.edu`