REFLEXIVITY, ALGEBRAIC REFLEXIVITY AND LINEAR INTERPOLATION

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Dedicated to the memory of Constantin Apostol

Introduction. This paper was motivated by a discovery that "most" finite dimensional subalgebras of $\mathcal{B}(H)$, for H an infinite dimensional Hilbert space, are reflexive, with the only obstructions to reflexivity being finite-rank considerations. Proofs (section 2) do not depend on multiplicative structure, nor on topology, so extend to linear subspaces of transformations in an abstract setting. Abstract reflexivity has been studied in [3, 4, 7], primarily for singly generated algebras. Reflexivity properties can be interpreted as linear interpolation properties, and we shall adopt this point of view.

The results of section 2 extend, in section 3, to algebraic reflexivity counterparts for countably generated (algebraically) linear subspaces of bounded linear transformations acting on a Banach space. As consequences, in section 4 we obtain generalizations of two single operator results. One is a multivariate version of the result, due to Kaplansky [10, Theorem 15], that a bounded locally algebraic operator acting on a Banach space is algebraic. The other is a multivariate version of the result, due to Douglas and Foias [5] for Hilbert space and to Hadwin [7] for a general Banach space, extending work of Fillmore [6], that a bounded non-algebraic operator acting on a Banach space is (topologically) algebraically reflexive. In section 5 we give an application, kindly suggested by D. Hadwin, concerning joint strong similarity orbits of n-tupules of operators.

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dence, concerning aspects of this work. The initial steps were accomplished during a visit to Leeds University which was partially supported by the SERC and by the NSF. Some of the ideas involved were developed during a visit to the University of Georgia.

1. Preliminaries. Reflexivity was introduced by Halmos for lattices of closed subspaces of a Hilbert space H, and for subalgebras of $\mathcal{B}(H)$. A lattice $\mathcal{L}$ is *reflexive* if $\mathcal{L} = \text{lat alg}(\mathcal{L})$, where for any set of closed subspaces $\mathcal{P}$, alg $\mathcal{P}$ denotes the algebra of all operators in $\mathcal{B}(H)$ leaving every member of $\mathcal{P}$ invariant, and for any set of operators $\mathcal{E} \subseteq \mathcal{B}(H)$, $\text{lat}(\mathcal{E})$ is the lattice of closed subspaces left invariant by $\mathcal{E}$. An algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ is reflexive if $\mathcal{A} = \text{alg lat } \mathcal{A}$. Loginov and Sulman [15] have extended these terms to include linear subspaces $\mathcal{S}$ of $\mathcal{B}(H)$ which are not necessarily algebras. We write $\text{ref}(\mathcal{S}) = \{T \in \mathcal{B}(H): Tx \in [\mathcal{S}x], x \in H\}$ where $[\cdot]$ denotes closure. If $\mathcal{A}$ is a unital algebra $\text{lat}(\mathcal{A})$ is determined by the closed cyclic subspaces of $\mathcal{A}$, so $\text{ref}(\mathcal{A}) = \text{alg lat } \mathcal{A}$. A linear subspace $\mathcal{S}$ is *reflexive* if $\mathcal{S} = \text{ref}(\mathcal{S})$. Reflexive subspaces have been useful in the analysis of operator algebras [1, 11, 12, 13, 15]. These definitions make sense with H replaced by an arbitrary topological vector space.

The notion of algebraic reflexivity has appeared in different contexts. The term was first used by Hadwin [7]. Let V be a vector space over a field $\mathcal{F}$, and let $\mathcal{L}(V)$ denote the algebra of all $\mathcal{F}$-linear transformations from V into itself. If V is a topological vector space (over $\mathbf{R}$ or $\mathbf{C}$) we write $\mathcal{B}(V)$ for the algebra of continuous elements of $\mathcal{L}(V)$. The definitions of algebraic "alg," "lat" and "ref" are analogous to those above with "closed subspace" replaced with "subspace" and with $[\mathcal{S}x]$ replaced with $\mathcal{S}x$. We write $\text{ref}_a(\mathcal{S}) = \{T \in \mathcal{L}(V): Tx \in \mathcal{S}x, x \in V\}$, and if V is topological, for $\mathcal{S} \subseteq \mathcal{B}(V)$ we write $\text{ref}_{at}(\mathcal{S}) = \{T \in \mathcal{B}(V): Tx \in \mathcal{S}x, x \in V\}$. So $T \in \text{ref}_a(\mathcal{S})$ iff for each $x \in V$ there exists $S \in \mathcal{S}$, depending on x, such that $Sx = Tx$. That is, T *interpolates* $\mathcal{S}$. We may say that T is *locally* in $\mathcal{S}$. If $\mathcal{S}$ is a finite dimensional subspace of $\mathcal{B}(V)$, then it is easily verified that $\text{ref}(\mathcal{S}) = \text{ref}_a(\mathcal{S}) = \text{ref}_{at}(\mathcal{S})$. If $\mathcal{S} \subseteq \mathcal{B}(V)$ is infinite dimensional then (see Example 3.3) $\text{ref}_a(\mathcal{S})$ may contain discontinuous operators. A subspace of linear transformations $\mathcal{S}$ is *algebraically reflexive* if $\mathcal{S} = \text{ref}_a(\mathcal{S})$, and (*topologically*) *algebraically reflexive* if $\mathcal{S} = \text{ref}_{at}(\mathcal{S})$. A linear transformation T is defined to possess one of these properties if the algebra $\mathcal{P}(T)$ of all polynomials in T with coefficients in $\mathcal{F}$ has the property. T is *algebraic* if $P(T) = 0$ for some nontrivial polynomial $P(t)$, and is *locally algebraic* if for each $x \in V$ there is a nontrivial polynomial $P(t)$, depending on x, such that $P(T)x =$

0. A *separating vector* for a subspace $\mathcal{S} \subseteq \mathcal{L}(V)$ is a vector $x \in V$ such that $S \to Sx$, $S \in \mathcal{S}$, is an injective map. We write $\mathcal{L}_F(V)$ for the space of all finite-rank (finite dimensional range) transformations in $\mathcal{L}(V)$, and $\mathcal{B}_F(V)$ for the space of all finite-rank transformations in $\mathcal{B}(V)$.

In section 2 we consider finite dimensional subspaces in an abstract setting. Since here the notions of reflexivity coincide, all results (and proofs) remain valid if topology is assumed and "ref_a" is replaced with "ref_{at}" or "ref." Assumption of topology would not aid in proofs, except in Hilbert space, where an advantage could be gained. (We thank K. Davidson for pointing this out, and for a useful idea in simplification.)

In section 3 we give our results for countably generated (algebraically) subspaces of $\mathcal{B}(X)$ for X a Banach space. In the absence of completeness of the underlying space, extensions of the results of section 2 to even singly generated subalgebras are not always possible; thus we restrict our attention.

2. Finite dimensional subspaces of $\mathcal{L}(V)$. Let V be a vector space over a field $\mathfrak{F}$. If $\mathcal{S}$ is a subspace of $\mathcal{L}(V)$ we write $\mathcal{S}_F = \mathcal{S} \cap \mathcal{L}_F(V)$. We will show that if $\mathcal{S}$ is finite dimensional, than the reflexivity properties of $\mathcal{S}$ are determined by those of $\mathcal{S}_F$. Theorem 2.6 states that $\text{ref}_a(\mathcal{S}) = \mathcal{S} + \text{ref}_a(\mathcal{S}_F)$. A consequence is that $\mathcal{S}$ is reflexive if and only if $\mathcal{S}_F$ is reflexive. The proof is accomplished in steps.

LEMMA 2.1. *Let $\mathcal{S}$ be a finite dimensional linear subspace of $\mathcal{L}$(V). If W_1, W_2 are linear subspaces of V of finite codimension, then $\{x \in W_1 : \mathcal{S}x \subseteq W_2\}$ has finite codimension in V.*

Proof. Let $\{S_1, \ldots, S_n\}$ be a basis for $\mathcal{S}$, and let P be any projection (obtained via a Hamel basis) from V onto W_2. Then $(I - P)\mathcal{S}$ is a set of finite rank linear transformations. Let K be the intersection of the kernels of $\{(I - P)S_1, \ldots, (I - P)S_n\}$. Then K has finite codimension, thus so does $K \cap W_1$. We have $(I - P)\mathcal{S}(K \cap W_1) = 0$, so $\mathcal{S}(K \cap W_1) \subseteq W_2$.

If $\mathcal{S}_F = \{0\}$ then, intuitively, $\mathcal{S}$ should have "infinite multiplicity" and hence a separating vector. We next prove this. A somewhat stronger form is necessary.

LEMMA 2.2. *Let $\mathcal{S}$ be a finite dimensional linear subspace of $\mathcal{L}$(V) with $\mathcal{S}_F = \{0\}$. Then if W_1, W_2 are linear subspaces of V of finite codimension, there exists a vector $x \in W_1$ which is separating for $\mathcal{S}$ such that $\mathcal{S}x \subseteq W_2$.*

Proof. First, if $\dim(\mathcal{S}) = 1$, then $\mathcal{S}$ consists of scalar multiples of a single linear transformation S_1 of infinite rank. By Lemma 2.1, the subspace $\{u \in W_1 : S_1 u \in W_2\}$ has finite codimension in V, so contains a vector x with $Sx \neq 0$. This satisfies our requirements.

Inductively, assume the conclusion holds for all subspaces of dimension $\leq n$. Suppose $\dim(\mathcal{S}) = n + 1$, and write $\mathcal{S} = \operatorname{span}\{S_1, \ldots, S_{n+1}\}$ where $\{S_1, \ldots, S_{n+1}\}$ are linearly independent. Let $\tilde{W}_1 = \{u \in W_1 : \mathcal{S}u \in W_2\}$, and let $\tilde{\mathcal{S}} = \operatorname{span}\{S_1, \ldots, S_n\}$. By Lemma 2.1 $\tilde{W}_1$ has finite codimension in V, so by the induction hypothesis there exists $x \in \tilde{W}_1$ which is separating for $\tilde{\mathcal{S}}$. By definition of $\tilde{W}_1$ we have $\mathcal{S}x \subseteq W_2$, so if x separates $\mathcal{S}$ we are finished. If not, there exists a nonzero element $S \in \mathcal{S}$ with $Sx = 0$. Then $\mathcal{S} = \operatorname{span}\{S_1, \ldots, S_n, S\}$.

Now let M denote any vector space complement of $\mathcal{S}x$ in V, and let $\hat{W}_2 = W_2 \cap M$. Let $\hat{W}_1 = \{u \in \tilde{W}_1 : \mathcal{S}u \in \hat{W}_2\}$. Then $\hat{W}_1$ has finite codimension in V, so since S has infinite rank there exists $y \in \hat{W}_1$ with $Sy \neq 0$. We will show that $x + y$ is separating for $\mathcal{S}$. Since $x, y \in W_1$ and $\mathcal{S}x \subseteq W_2$, $\mathcal{S}y \subseteq W_2$, the proof will then be complete.

If $T \in \mathcal{S}$, $T \neq 0$, then $T = \alpha S + \alpha_1 S_1 + \cdots + \alpha_n S_n$ for some $\alpha, \alpha_i \in \mathfrak{F}$. Since $Tx \in \mathcal{S}x$ and $Ty \in M$, to show that $T(x + y) \neq 0$ it will suffice to show that either $Tx \neq 0$ or $Ty \neq 0$. Since $Sx = 0$ we have $Tx = \alpha_1 S_1 x + \cdots + \alpha_n S_n x$, and since x separates $\operatorname{span}\{S_1, \ldots, S_n\}$, if $Tx = 0$ then necessarily $\alpha_i = 0$, $1 \leq i \leq n$. But then $T = \alpha S$ with $\alpha \neq 0$. So $Ty \neq 0$.

We next strengthen Lemma 2.2, and show that for finite dimensional subspaces nonexistance of a separating vector is a finite-rank obstruction.

PROPOSITION 2.3. *Let $\mathcal{S}$ be a finite dimensional linear subspace of $\mathcal{L}(V)$. Then $\mathcal{S}$ has a separating vector if and only if $\mathcal{S}_F$ has a separating vector.*

Proof. Only one direction requires proof. Write $\mathcal{S} = \operatorname{span}\{S_1, \ldots, S_k, S_{k+1}, \ldots, S_n\}$, where we may assume $\mathcal{S}_F = \operatorname{span}\{S_1, \ldots, S_k\}$, and $\operatorname{span}\{S_{k+1}, \ldots, S_n\} \cap \mathcal{L}_F(V) = \{0\}$. Let u be a separating vector for $\mathcal{S}_F$, and let M be a complement of $\mathcal{S}u$ in V. Let K be the intersection of the kernels of $S_1, \ldots, S_k$. Since K and M have finite codimension in V, by Lemma 2.2 there is a vector $v \in K$ which is separating for $\operatorname{span}\{S_{k+1}, \ldots, S_n\}$ such that $\{S_{k+1}v, \ldots, S_n v\} \subseteq M$. Let $x = u + v$. If $T \in \mathcal{S}$, write $T = T_1 + T_2$ with $T_1 \in \mathcal{S}_F$, $T_2 \in \operatorname{span}\{S_{k+1}, \ldots, S_n\}$. We have $Tu \in \mathcal{S}u$, and $Tv = T_2 v \in M$, so if $Tx = 0$ then $Tu = 0$ and $Tv = 0$. But then $T_2 v = 0$ so $T_2 = 0$, hence $T = T_1$. But u separates $\mathcal{S}_F$, hence $T_1 = 0$ also, and so $T = 0$. Thus x is separating for $\mathcal{S}$.

LEMMA 2.4. *Let $\mathcal{S}$ be a finite dimensional linear subspace of $\mathcal{L}(V)$ with $\mathcal{S}_F = \{0\}$. Then $\mathcal{S}$ is reflexive.*

Proof. If $T \in \mathcal{L}(V)$ is arbitrary, then $\mathcal{L}_F(V) \cap \text{span}\{T, \mathcal{S}\}$ can be at most one dimensional, so has a separating vector. Hence $\text{span}\{T, \mathcal{S}\}$ has a separating vector x, by Proposition 2.3. So if $T \notin \mathcal{S}$ then $Tx \notin \mathcal{S}x$. So $T \notin \text{ref}_a(\mathcal{S})$.

LEMMA 2.5. *Let $\mathcal{S}$ be a finite dimensional linear subspace of $\mathcal{L}(V)$. Then* $\text{ref}_a(\mathcal{S}) \subseteq \mathcal{S} + \mathcal{L}_F(V)$.

Proof. Suppose $T \in \text{ref}_a(\mathcal{S})$. Let $M = (\mathcal{S} \cap \mathcal{F})V$, a finite dimensional subspace of V, and let Q be any projection from V onto M. Then $((I - Q)\mathcal{S}) \cap \mathcal{L}_F(V) = 0$, so $(I - Q)\mathcal{S}$ is reflexive by Lemma 2.4. We have $(I - Q)T \in \text{ref}_a((I - Q)\mathcal{S}) = (I - Q)\mathcal{S}$, so $(I - Q)T = (I - Q)S$ for some $S \in \mathcal{S}$. So $T = S + Q(T - S)$, and thus $T \in \mathcal{S} + \mathcal{L}_F(V)$.

THEOREM 2.6. *Let $\mathcal{S}$ be a finite dimensional linear subspace of $\mathcal{L}(V)$. Then* $\text{ref}_a(\mathcal{S}) = \mathcal{S} + \text{ref}_a(\mathcal{S}_F)$.

Proof. By Lemma 2.5 it will suffice to show that any finite rank linear transformation in $\text{ref}_a(\mathcal{S})$ is in $\text{ref}_a(\mathcal{S}_F)$. So let $T \in \mathcal{L}_F(V) \cap \text{ref}_a(\mathcal{S})$. We will show that $Tu \in (\mathcal{S}_F)u$ for such $u \in V$.

Let $u \in V$ be arbitrary. Write $\mathcal{S} = \text{span}\{S_1, \ldots, S_k, S_{k+1}, \ldots, S_n\}$ where $\{S_1, \ldots, S_n\}$ is a basis for $\mathcal{S}$, and where $\mathcal{S}_F = \text{span}\{S_1, \ldots, S_k\}$. Let $\mathcal{S}_I = \text{span}\{S_{k+1}, \ldots, S_n\}$. Then $\mathcal{L}_F(V) \cap \mathcal{S}_I = \{0\}$. Let N be any vector space complement of $\mathcal{S}u$ in V. Let K be the intersection of the kernels of $\{T, S_1, \ldots, S_k\}$. Then N and K have finite codimension in V. By Lemma 2.2 there is a separating vector y for $\mathcal{S}_I$ with $y \in K$ and with $\mathcal{S}_I y \subseteq N$. Let $x = u + y$. Since $T \in \text{ref}_a(\mathcal{S})$, there exist scalars $\{\alpha_1, \ldots, \alpha_n\}$ in $\mathcal{F}$, depending on x, such that $Tx = (\alpha_1 S_1 + \cdots + \alpha_n S_n)x$. Since $y \in \ker(S_i)$ for $i \geq k + 1$, and $y \in \ker(T)$, we have

$$Tu = Tx = T(u + y) = \sum_{i=1}^{n} \alpha_i S_i(u + y)$$

$$= \sum_{i=1}^{n} \alpha_i S_i u + \sum_{i=k+1}^{n} \alpha_i S_i y.$$

So since $Tu \in \mathcal{S}u$ and $\mathcal{S}_I y \subseteq N$, we have $\Sigma_{i=k+1}^{n} \alpha_i S_i y = 0$. But y separates $\mathcal{S}_I$ and $\{S_{k+1}, \ldots, S_n\}$ are linearly independent, so $\alpha_i = 0, k + 1 \leq i \leq n$. Thus $Tu \in (\mathcal{S}_F)u$.

We have shown that $Tu \in (\mathcal{S}_F)u$ for each $u \in V$. So $T \in \text{ref}_a(\mathcal{S}_F)$, as desired.

We need an observation.

LEMMA 2.7. *Let $\mathcal{S}$ be a finite dimensional linear subspace of $\mathcal{L}(V)$ consisting of finite rank operators. Then* $\text{ref}_a(\mathcal{S})$ *is finite dimensional and consists of finite rank operators.*

Proof. Then there are finite rank projections P, Q with $\mathcal{S} = P\mathcal{S}Q$. It is easily shown that $PL(V)Q$ is algebraically reflexive so must contain $\text{ref}_a(\mathcal{S})$. Since $PL(V)Q$ is finite dimensional and is contained in $\mathcal{L}_F(V)$, the conclusion follows.

COROLLARY 2.8. *Let $\mathcal{S}$ be a finite dimensional subspace of $\mathcal{L}(V)$. Then $\mathcal{S}$ is reflexive if and only if $\mathcal{S}_F$ is reflexive.*

Proof. If $\text{ref}_a(\mathcal{S}_F) = \mathcal{S}_F$, then by Theorem 2.6 we have $\text{ref}_a(\mathcal{S}) = \mathcal{S} + \text{ref}_a(\mathcal{S}_F) = \mathcal{S}$, so $\mathcal{S}$ is reflexive. Conversely, suppose $\mathcal{S}$ is reflexive. Then $\mathcal{S} = \text{ref}_a(\mathcal{S}) = \mathcal{S} + \text{ref}_a(\mathcal{S}_F)$, so $\text{ref}_a(\mathcal{S}) \subseteq \mathcal{S}$. But by Lemma 2.7 $\text{ref}_a(\mathcal{S}_F) \subseteq \mathcal{L}_F(V)$. Hence $\text{ref}_a(\mathcal{S}_F) = \mathcal{S}_F$.

COROLLARY 2.9. *Let $\mathcal{S}$ be a finite dimensional linear subspace of $\mathcal{L}(V)$. Then* $\text{ref}_a(\mathcal{S})$ *is also finite dimensional.*

Proof. This is immediate from Theorem 2.6 and Lemma 2.7.

3. Countably generated subspaces of $\mathcal{B}(X)$. We extend the results of section 2 to counterparts for linear subspaces of $\mathcal{B}(X)$ with denumerable Hamel basis, where X is a real or a complex Banach space. The main results of this section are Theorems 3.2, 3.5 and 3.7.

We first require a generalization of Lemma 2.2. An exact generalization is not possible because if W_2 is a proper subspace of X which is closed, it may be that $\mathcal{S}$ is topologically transitive so no separating vector x for $\mathcal{S}$ will satisfy $\mathcal{S}x \subseteq W_2$. However, if E is a subspace of X with denumerable Hamel basis there *is* enough control to insure that if $\mathcal{S}_F = \{0\}$, where $\mathcal{S}_F = \mathcal{S} \cap \mathcal{B}_F(X)$, then there exists a separating vector x for $\mathcal{S}$ such that $(\mathcal{S}x) \cap E = \{0\}$, so that $\mathcal{S}x$ will be contained in *some* complement of E, although not necessarily a closed complement even if E is finite dimensional. This will be enough.

LEMMA 3.1. *Let X be a Banach space, and let $\mathcal{S}$ be a linear subspace of $\mathcal{B}(X)$ with denumerable Hamel basis. Let W be a closed linear subspace* *of X of finite codimension in X, and let E be a linear subspace of X of denumerable Hamel basis. If $\mathcal{S}_F = \{0\}$, then W contains a vector x which is separating for $\mathcal{S}$ such that* $(\mathcal{S}x) \cap E = \{0\}$.

Proof. Write $\mathcal{S} = \text{span}\{S_1, S_2, \ldots\}$, and $E = \text{span}\{v_1, v_2, \ldots\}$, where the operators S_i and the vectors v_i are not necessarily linearly independent and may be repeated. For each n, let $\mathcal{S}_n = \text{span}\{S_1, \ldots, S_n\}$ and $E_n = \text{span}\{v_1, \ldots, v_n\}$. Let $\{\epsilon_i\}_{i=1}^{\infty}$ be any sequence of positive numbers with $\Sigma\epsilon_i < \infty$. We will inductively construct a sequence of vectors $\{x_i\}$, with $||x_i|| \leq \epsilon_i$, and a sequence $\{L_i\}$ of closed subspaces of X, which together satisfy properties that insure that $x = \Sigma_{i=1}^{\infty} x_i$ meets our requirements. (The convergence of Σx_i is where completion of X is required.)

The properties to be satisfied are:

(1) Each L_i is closed, has finite codimension in X, and $L_i \cap E_i = \{0\}$.
(2) $L_i \supseteq L_{i+1}$, $i \geq 1$.
(3) x_i is a vector in W which separates $\mathcal{S}_i$ and satisfies $||x_i|| \leq \epsilon_i$.
(4) $\mathcal{S}x_i \subseteq L_i$ for each i.
(5) $L_{i+1} \cap (E_i + \mathcal{S}_{i+1}F_i) = \{0\}$, $i \geq 1$, where we define $F_i = \text{span}\{x_1, \ldots, x_i\}$.

Begin by letting L_1 be any closed complement of E_1. Since L_1 and W have finite codimensions, Lemma 2.2 yields $x_1 \in W$, separating for $\mathcal{S}_1$, with $\mathcal{S}_1 x_1 \subseteq L_1$. Multiply by a scalar, if necessary, so that $||x_1|| \leq \epsilon_1$.

Suppose that $\{x_1, \ldots, x_n\}$ and $\{L_1, \ldots, L_n\}$ have been constructed satisfying properties (1) $\rightarrow$ (5). Let M be a closed complement of $(E_{n+1} + \mathcal{S}_{n+1}F_n)$, and let $L_{n+1} = M \cap L_n$. Then L_{n+1} satisfies (1), (2), (5). Apply Lemma 2.2, obtaining $x_{n+1} \in W$, separating for $\mathcal{S}_{n+1}$, with $\mathcal{S}_{n+1}x_{n+1} \subseteq L_{n+1}$. Scale if necessary, so that $||x_{n+1}|| \leq \epsilon_{n+1}$. Then $\{L_1, \ldots, L_{n+1}\}$ and $\{x_1, \ldots, x_{n+1}\}$ satisfy (1) $\rightarrow$ (5).

Now let $x = \Sigma_{i=1}^{\infty} x_i$. Since each $x_i \in W$ and W is closed, $x \in W$. We will show that if $S \in \mathcal{S}$, and if $Sx \in E$, then $S = 0$.

Suppose $S \in \mathcal{S}$ with $Sx \in E$. Then $Sx \in E_i$ for some i, and $S \in \mathcal{S}_j$ for some j, so if $n = \max\{i,j\}$ then $S \in \mathcal{S}_n$ and $Sx \in E_n$. We may assume $n \geq 2$. Let $y = \Sigma_{i=1}^{n-1} x_i$ and $z = \Sigma_{i=n}^{\infty}$. So $x = y + z$ and $Sy + Sz \in E_n$. For $i \geq n$ we have $Sx_i \in L_i \subseteq L_n$, so $Sz \in L_n$ since L_n is closed and S is continuous. Also, $Sy \in \mathcal{S}_n F_{n-1}$ and $Sx \in E_n$, so $Sz = Sx - Sy \in E_n + \mathcal{S}_n F_{n-1}$. But by property (5) we have $L_n \cap (E_n + \mathcal{S}_n F_{n-1}) = \{0\}$, hence $Sz = 0$. That is, $\Sigma_{i=n}^{\infty} Sx_i = 0$. But also, $Sx \in E_{n+1}$ and $S \in \mathcal{S}_{n+1}$. Thus the same argument yields that $\Sigma_{i=n+1}^{\infty} Sx_i = 0$. Hence $Sx_n = 0$. Since x_n is separating for $\mathcal{S}_n$ and $S \in \mathcal{S}_n$ we conclude that $S = 0$.

The above paragraph shows that $(\mathcal{S}x) \cap E = (0)$. It also shows that x is separating for $\mathcal{S}$, for if $S \in \mathcal{S}$ with $Sx = 0$ then $Sx \in E$ and hence $S = 0$. The proof is complete.

We obtain a partial generalization of Proposition 2.3. If $\mathcal{S} \subseteq \mathcal{B}(X)$ has a separating vector, then that vector separates $\mathcal{S}_F$. The partial converse we obtain is that if $\mathcal{S}$ has denumerable Hamel basis, if the intersection of the kernels of the elements of $\mathcal{S}_F$ has finite codimension in X, and if $\mathcal{S}_F$ has a separating vector, then $\mathcal{S}$ has a separating vector. It is not known whether the "finite dimensional support" condition on $\mathcal{S}_F$ can be dropped.

THEOREM 3.2. *Let X be a Banach space, and let $\mathcal{S}$ be a linear subspace of $\mathcal{B}(X)$ with denumerable Hamel basis such that $\{v \in X : \mathcal{S}_F v = \{0\}\}$ has finite codimension in X. Then $\mathcal{S}$ has a separating vector if and only if* $\mathcal{S}_F$ has a separating vector.

Proof. Only one direction requires proof. Let $\mathcal{S}_I$ be any vector space complement (not necessarily closed) of $\mathcal{S}_F$ in $\mathcal{S}$. Let u be a separating vector for $\mathcal{S}_F$, and let $E = \mathcal{S}u$. Let W be the intersection of the kernels of the elements of $\mathcal{S}_F$. Then $\mathcal{S}_1$, W, E satisfy the hypotheses of Lemma 3.1 so there exists $v \in W$, separating for $\mathcal{S}_I$, such that $(\mathcal{S}_I v) \cap E = (0)$. Let $x = u + v$. The proof now proceeds as in Proposition 2.3. If $S \in \mathcal{S}$, then $S = S_1 + S_2$ with $S_1 \in \mathcal{S}_F$, $S_2 \in \mathcal{S}_I$. We have $Su \in E$, $Sv = S_2 v \in \mathcal{S}_I v$. If $Sx = 0$ then $Sv = -Su \in E \cap (\mathcal{S}_I v) = (0)$, hence $Sv = Su = 0$. But then $S_2 v = 0$ so $S_2 = 0$ since v separates $\mathcal{S}_I$, hence $S = S_1$. Then $S_1 u = 0$, hence $S_1 = 0$ since u separates $\mathcal{S}_F$. Thus $S = 0$. We have shown that x separates $\mathcal{S}$.

The following example points out the distinction between (topological) algebraic reflexivity and algebraic reflexivity for infinite dimensional subspaces.

Example 3.3. Let H be an infinite dimensional Hilbert space. For u, $v \in H$ let $u \otimes v^*$ denote the operator in $\mathcal{B}(H)$ which maps w to $\langle w, v\rangle u$, $w \in H$. For fixed $u \neq 0$ the subspace $\mathcal{S}_u = \{u \otimes v^* : v \in H\}$ is reflexive, so is (topologically) algebraically reflexive, but fails to be algebraically reflexive. If ϕ is any discontinuous linear functional on H, then the discontinuous operator T defined by $Tw = \phi(w)u$, $w \in H$, will interpolate $\mathcal{S}_u$ on H yet not be in $\mathcal{S}_u$.

LEMMA 3.4. *Let X be a Banach space, and let $\mathcal{S}$ be a linear subspace* *of* $\mathcal{B}(X)$ *with denumerable Hamel basis such that* $\mathcal{S}_F = \{0\}$. *Then* $\mathcal{S}$ *is (topologically) algebraically reflexive.*

Proof. Let $T \in \mathcal{B}(X)$ be arbitrary and let $\tilde{\mathcal{S}} = \text{span}\{T, \mathcal{S}\}$. Then $(\tilde{\mathcal{S}})_F$ is at most one dimensional, so has a separating vector and satisfies the support condition of Theorem 3.2, thus $\tilde{\mathcal{S}}$ has a separating vector x. So if $T \notin \mathcal{S}$ then $Tx \notin \mathcal{S}x$, so $T \notin \text{ref}_{at}(\mathcal{S})$.

THEOREM 3.5. *Let* X *be a Banach space, and let* $\mathcal{S}$ *be a linear subspace of* $\mathcal{B}(X)$ *with denumerable Hamel basis. Then* $\text{ref}_{at}(\mathcal{S}) \subseteq \mathcal{S} + \mathcal{B}_F(X)$.

Proof. We use a Baire Category argument. Write $\mathcal{S} = \text{span}\{S_1, S_2, \ldots\}$, and let $\mathcal{S}_n = \text{span}\{S_1, \ldots, S_2\}$, $n \geq 1$, so that $\mathcal{S} = \cup_n \mathcal{S}_n$. Suppose $T \in \text{ref}_{at}(\mathcal{S})$. For each n, $k \geq 1$ let $\mathcal{E}_n{}^k = \{x \in X$: there exists $S \in \mathcal{S}_n$ with $||S|| \leq k$ such that $Tx = Sx\}$. Then $X = \cup \{\mathcal{E}_n{}^k : n, k \geq 1\}$.

Suppose $x = \lim x_i$, $x_i \in \mathcal{E}_n{}^k$. For each i there exists $S_i \in \mathcal{S}_n$ with $||S_i|| \leq k$ such that $Tx_i = S_i x_i$. Since $\mathcal{S}_n$ is finite dimensional $\{S \in \mathcal{S}_i : ||S|| \leq k\}$ is compact, so $\{S_i\}$ has a convergent subsequence $\{S_{i_j}\}$. Let $\hat{S} = \lim S_{i_j}$. Then $x_{i_j} \rightarrow x$, so $Tx = \lim Tx_{i_j} = \lim S_{i_j} x_{i_j} = \hat{S}x$. We have $||\hat{S}|| \leq k$. So $x \in \mathcal{E}_n{}^k$. Hence $\mathcal{E}_n{}^k$ is closed.

By the Baire Category Theorem some $\mathcal{E}_n{}^k$ has nonempty interior in X. So for some $z \in X$ and $\epsilon > 0$, and for some n and k, the set $\mathcal{E}_n{}^k$ contains $B_\epsilon(z) = \{v \in X : ||v - z|| < \epsilon\}$. Then for each $x \in X$ with $||x|| < \epsilon$ there exists $S_x \in \mathcal{S}_n$, depending on x, with $T(x + z) = S_x(x + z)$. [This will *not* imply that $T \in \text{ref}(\mathcal{S}_n)$. See Example 3.6.]

Let $F_1 = \mathcal{S}_n z$, and let $F_2 = (\mathcal{S}_n)_F X$. Then F_1, F_2 are finite dimensional subspaces of X. Let $F = F_1 \vee F_2 = \text{span}\{F_1, F_2\}$, and let P be any continuous projection from X onto F. We have $(I - P)(\mathcal{S}_n)_F = \{0\}$, and so since P has finite rank, $((I - P)\mathcal{S}_n)_F = \{0\}$, so $(I - P)\mathcal{S}_n$ is reflexive. Note that $Tz = S_0 z$ for some $S_0 \in \mathcal{S}_n$, so $Tz \in F$, so $(I - P)Tz = 0$. Thus if $x \in X$ with $||x|| < \epsilon$, then $(I - P)Tx = (I - P)T(x + z) = (I - P)S_x(x + z) = (I - P)S_x x$ since $(I - P)\mathcal{S}_n z = 0$. This shows that $(I - P)T \in \text{ref}((I - P)\mathcal{S}_n) = (I - P)\mathcal{S}_n$. Thus for some $S \in \mathcal{S}_n$ we have $(I - P)T = (I - P)S$. So for this S we have $T = S + P(T - S)$. Since $S \in \mathcal{S}$ and $P(T - S) \in \mathcal{B}_F(X)$, the proof is complete.

Example 3.6. A linear transformation can interpolate a linear subspace $\mathcal{S}$ on a *ball* with center different from 0 and yet not be in $\text{ref}_a(\mathcal{S})$. Let

$$\mathcal{S} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbf{C} \right\}$$

represent the algebra of upper triangular operators with respect to an o.n. basis $\{e_1, e_2\}$ for a 2-dimensional Hilbert space H. Then $\mathcal{S}$ is reflexive. Let $B_1(e_2) = \{x \in H: ||x - e_2|| < 1\}$. If $x \in B_1(e_2)$ then $\mathcal{S}x = H$. It follows that

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

interpolates $\mathcal{S}$ on $B_1(e_2)$.

We obtain a partial generalization of Theorem 2.6. In view of Example 3.9, this may be the best general result possible.

THEOREM 3.7. *Let X be a Banach space and let $\mathcal{S}$ be a linear subspace of $\mathcal{B}(X)$ with denumerable Hamel basis such that $\{v \in X: \mathcal{S}_F v = \{0\}\}$ has finite codimension in X. Then*

(i) $\operatorname{ref}_{at}(\mathcal{S}) = \mathcal{S} + \operatorname{ref}_{at}(\mathcal{S}_F)$, *and*
(ii) *$\mathcal{S}$ is (topologically) algebraically reflexive if and only if $\mathcal{S}_F$ is (topologically) algebraically reflexive.*

Proof. (i) By Theorem 3.5, it will suffice to show that if $T \in \operatorname{ref}_{at}(\mathcal{S})$ and if T has finite rank, then $T \in \operatorname{ref}_{at}(\mathcal{S}_F)$. Let $\mathcal{S}_I$ be any complement (not necessarily closed) of $\mathcal{S}_F$ in $\mathcal{S}$. Let $u \in X$ be arbitrary. We will show that $Tu \in \mathcal{S}_F u$. Let $E = \mathcal{S}u$, let $K = \text{kernel}(T)$, and let $W = K \cap \{v \in X: \mathcal{S}_F v = \{0\}\}$. Then W is closed and has finite codimension in X, so by Lemma 3.1, W contains a vector y which is separating for $\mathcal{S}_I$ such that $(\mathcal{S}_I y) \cap E = (0)$. Then $Ty = 0$, so $Tu = T(u + y)$. Since $T \in \operatorname{ref}_{at}(\mathcal{S})$ there exists $S \in \mathcal{S}$ with $T(u + y) = S(u + y)$. Decompose $S = S_1 + S_2$ with $S_1 \in \mathcal{S}_F$, $S_2 \in \mathcal{S}_I$. Noting that $S_1 y = 0$, we have $Tu = S(u + y) = S_1 u + S_2 u + S_2 y$, and since $Tu \in \mathcal{S}u$ it follows that $S_2 y \in \mathcal{S}u$. But $(\mathcal{S}_I y) \cap (\mathcal{S}u) = \{0\}$, and so $S_2 y = 0$, and thus $S_2 = 0$ since y separates $\mathcal{S}_I$. Thus $S = S_1 \in \mathcal{S}_F$.

We have shown that for each $u \in X$ there is an $S \in \mathcal{S}_F$ such that $Tu = Su$. So $T \in \operatorname{ref}_{at}(\mathcal{S}_F)$, completing the proof of (i).

Assertion (ii) follows readily from (i). If $\mathcal{S}_F$ is (topologically) algebraically reflexive then $\operatorname{ref}_{at}(\mathcal{S}) = \mathcal{S} + \operatorname{ref}_{at}(\mathcal{S}) = \mathcal{S} + \mathcal{S}_F = \mathcal{S}$. Conversely, if $\mathcal{S}$ is reflexive then $\operatorname{ref}_{at}(\mathcal{S}_F) \subseteq \mathcal{S}$. Let $M = \{v \in X: \mathcal{S}_F v = \{0\}\}$. M has finite codimension by hypothesis, and is closed, so there is a bounded projection P from X onto M. Then $S(I - P) = S$, $S \in \mathcal{S}_F$, and $\mathcal{B}(X)(I - P)$ is a reflexive subspace of $\mathcal{B}(X)$, hence $\text{ref}_{at}(\mathcal{S}_F) \subseteq \mathcal{B}(X)(I - P) \subseteq \mathcal{B}_F(X)$. Thus $\text{ref}_{at}(\mathcal{S}_F) \subseteq \mathcal{S} \cap \mathcal{B}_F(X) = \mathcal{S}_F$, as required. The proof is complete.

If $\mathcal{S} \subseteq \mathcal{B}_F(X)$ is a subspace with denumerable Hamel basis then $\text{ref}_{at}(\mathcal{S})$ need *not* have denumerable Hamel basis, even if $\mathcal{S}$ is the union of an increasing chain, $\mathcal{S}_n \subseteq \mathcal{S}_{n+1}$, of finite dimensional reflexive subspaces. Thus Corollary 2.9 does not generalize completely. Also, even if a subspace $\mathcal{S}$ with denumerable Hamel basis is such that $\mathcal{S}_F$ has codimension *one* in $\mathcal{S}$, it may happen that $\text{ref}_{at}(\mathcal{S}) \neq \mathcal{S} + \text{ref}_{at}(\mathcal{S}_F)$ if the "finite dimensional support" condition on $\mathcal{S}_F$ in Theorem 3.7 is not satisfied. In addition, if $\mathcal{S} \subseteq \mathcal{B}(H)$, with H a Hilbert space, it can happen that $\mathcal{S}$ is (topologically) algebraically reflexive while $\mathcal{S}^* = \{S^* : S \in \mathcal{S}\}$ is not, in contrast to the situation for reflexivity. Two examples illustrate these pathologies.

Example 3.8. Let $\{e_i : i = 1, 2, \ldots\}$ be an orthonormal basis for infinite dimensional separable Hilbert space H. For each i, j let E_{ij} denote the corresponding matrix unit: the operator assigning to each vector x the vector $\langle x, e_j \rangle e_i$. Let $\mathcal{S}$ be the span of $\{E_{1n} : n = 1, \ldots\}$. Then $\mathcal{S} \subseteq \mathcal{B}_F(H)$, and $\mathcal{S} = \bigcup_n \mathcal{S}_n$ where $\mathcal{S}_n = \text{span}\{E_{11}, E_{12}, \ldots, E_{1n}\}$. Each $\mathcal{S}_n$ has the form $P\mathcal{B}(H)Q$ with P, Q projections, so is reflexive. If $w \neq 0$ is arbitrary, the operator $(e_1 \otimes w^*)$, mapping x to $\langle x, w \rangle e_1$, is in $\text{ref}_{at}(\mathcal{S})$. In fact $\text{ref}_{at}(\mathcal{S}) = \{e_1 \otimes w^* : w \in H\}$. This has uncountable Hamel basis. Now note that while $\mathcal{S}$ does not satisfy the finite dimensional support condition, the adjoint $\mathcal{S}^*$ does. In fact $\mathcal{S}^*$ is (topologically) algebraically reflexive: If T interpolates $\mathcal{S}^* = \text{span}\{E_{n1} : n = 1, \ldots\}$, then if $x \in H$ with $x \perp e_1$ we have $\mathcal{S}^* x = 0$, so $Tx = 0$; thus $T = (w \otimes e_1^*)$ for some $w \in H$, and since $Te_1 = S^* e_1$ for some $S \in \mathcal{S}$, w must be a finite linear combination of the basis vectors. Hence $T \in \mathcal{S}^*$, as desired.

Example 3.9. Let terms be as in Example 3.8. Let $\mathcal{A}_0 = \text{span}\{E_{ij} : i < j\}$, a strictly upper triangular algebra, and let $\mathcal{A} = \mathcal{A}_0 + \mathbf{C}I$. Then $\mathcal{A}_F = \mathcal{A}_0$. Let $T = E_{11}$. Then $T \notin \mathcal{A}$, but T interpolates $\mathcal{A}$. Indeed, let $x \in H$ be arbitrary. If $\langle x, e_1 \rangle = 0$ then $Tx = 0$, so $Tx \in \mathcal{A}x$. If $\langle x, e_1 \rangle = \alpha \neq 0$, and if x is not a scalar multiple of e_1, then $\langle x, e_j \rangle = \beta \neq 0$ for some $j > 1$, so $Tx = \alpha e_1 = (\alpha/\beta) E_{1j} x \in \mathcal{A}x$. Finally, $Te_1 = Ie_1 \in \mathcal{A}e_1$. Thus $Tx \in \mathcal{A}x$, $x \in H$, so $T \in \text{ref}_{at}(\mathcal{A})$. Since $Te_1 = e_1$ and $\mathcal{A}_0 e_1 = \{0\}$, T fails to interpolate $\mathcal{A}_0$. This shows that $\text{ref}_{at}(\mathcal{A}) \neq \mathcal{A} + \text{ref}_{at}(\mathcal{A}_F)$.

4. Two multivariate results. Nonalgebraic but locally algebraic bounded linear operators exist on incomplete normed linear spaces. An example is given by the backward unilateral shift restricted to the dense subspace of Hilbert space consisting of all finite linear combinations of the corresponding basis vectors. Theorem 15 in [10] says that if X is a real or complex Banach space, and if $S \in \mathcal{B}(X)$ is locally algebraic, then S is algebraic. We obtain a generalization of that result, using section 3.

THEOREM 4.1. *Let X be a Banach space, and let $\{S_1, \ldots, S_n\} \subset \mathcal{B}(X)$. Suppose for each $x \in X$ there is a nonzero polynomial $P_x(t_1, \ldots, t_n)$ in n (noncommuting) variables such that $P_x(S_1, \ldots, S_n)x = 0$. Then there exists a nonzero polynomial $P(t_1, \ldots, t_n)$ such that $P(S_1, \ldots, S_n) = 0$.*

Proof. Let $\mathcal{P}$ denote the vector space of all polynomials in n noncommuting variables. [We assume real (complex) coefficients if X is real (complex).] If $P_x(S_1, \ldots, S_n) = 0$ for some x the proof is complete. If each operator $P_x(S_1, \ldots, S_n)$ is nonzero, then $\mathcal{S} = \{P(S_1, \ldots, S_n): P \in \mathcal{P}\}$ fails to have a separating vector. Since $\mathcal{S}$ is a linear subspace of $\mathcal{B}(X)$ with denumerable Hamel basis, Theorem 3.2 (or Lemma 3.1) implies that $\mathcal{S}_F \neq \{0\}$. So for some $Q \in \mathcal{P}$, $Q(S_1, \ldots, S_n)$ is a nonzero finite rank operator, hence algebraic. Thus there is a nonzero polynomial $F(t)$ such that $F(Q(T_1, \ldots, T_n)) = 0$. Let $P = F \circ Q$.

Remarks. An inspection of the above proof shows that Theorem 4.1 can be adapted to situations in which some or all of the operators $\{S_1, \ldots, S_n\}$ commute by replacing $\mathcal{P}$ in the proof with a subspace of, or a quotient of, $\mathcal{P}$. Also, we may have $1 \leq n \leq \aleph_0$. That is, if $\{S_1, S_2, \ldots\} \subset \mathcal{B}(X)$ is any sequence, and $\mathcal{P}$ is the set of polynomials in denumerably many variables (where each $P \in \mathcal{P}$ has finitely many nonzero coefficients), the statement and proof of the theorem remain the same.

One of the results obtained in [5] is that if $T \in \mathcal{B}(H)$ is not algebraic then the set of operators in $\mathcal{B}(H)$ which leave invariant every linear submanifold of H left invariant by T coincides with $\mathcal{P}(T)$, the set of all polynomials in T. This extended work initiated in [6]. In [7] a more algebraic proof was obtained which extended this result to arbitrary Banach spaces, and aspects to abstract vector spaces. It was proven that if $\mathfrak{F}$ is infinite then non locally algebraic operators have this property. So an application of Kaplansky's theorem yielded the Banach space result. Hadwin used the term "algebraically reflexive" to denote operators T, and unital algebras of operators, which are thus completely determined by invariant linear submanifolds of the underlying vector space. For unital algebras this is equivalent to the subspace definition we use. We make the distinction between algebraically reflexive and (topologically) algebraically reflexive for reasons given earlier. Using section 3 we obtain a generalization of the result that a nonalgebraic bounded linear operator acting on a Banach space is (topologically) algebraically reflexive.

THEOREM 4.2. *Let X be a Banach space, and let $\{S_1, \ldots, S_n\} \subset \mathcal{B}(X)$. If $P(S_1, \ldots, S_n) \neq 0$ for each nonzero polynomial $P(t_1, \ldots, t_n)$ in n (noncommuting) variables, then the algebra $\mathcal{P}(S_1, \ldots, S_n)$ of all polynomials in the operator variables $S_1, \ldots, S_n$ is (topologically) algebraically reflexive.*

Proof. The proof is similar to that of Theorem 4.1, with Theorem 3.7 (or Lemma 3.4) used instead of Theorem 3.2. The algebra $\mathcal{P}(S_1, \ldots, S_n)$ is a linear subspace of $\mathcal{B}(X)$ with denumerable Hamel basis, so if it is not (topologically) algebraically reflexive then some $P(S_1, \ldots, S_n)$ is a nonzero finite-rank operator. But then $(Q \circ P)(S_1, \ldots, S_n) = 0$ for some nonzero $Q(t)$, a contradiction.

Remarks. As in Theorem 4.1, the above theorem adapts to commuting variables, and we may have $1 \leq n \leq \aleph_0$. Also, even if $\{S_1, \ldots, S_n\}$ are algebraically dependent, $\mathcal{P}(S_1, \ldots, S_n)$ will be (topologically) algebraically reflexive if no polynomial in $\{S_1, \ldots, S_n\}$ is a nonzero finite rank operator, or more generally, if the finite-rank subalgebra of $\mathcal{P}(S_1, \ldots, S_n)$ is (topologically) algebraically reflexive and satisfies the "finite dimensional support" condition of Theorem 3.7.

5. An application of Hadwin. We thank Donald Hadwin for obtaining, and kindly suggesting inclusion of, the following application of Lemma 2.2 which generalizes Theorem 1 in [9].

Let X be an infinite dimensional Banach space, and $1 \leq n \leq \infty$. Equip $\mathcal{B}(X)$ with the strong operator topology (pointwise convergence), and let τ denote the corresponding product topology on $\mathcal{B}(X) \times \cdots \times \mathcal{B}(X)$, ($n$-times). Given $\{T_1, \ldots, T_n\} \subseteq \mathcal{B}(X)$, denote the joint similarity orbit by

$$\mathcal{S}(T_1, \ldots, T_n) = \{(S^{-1}T_1S, \ldots, S^{-1}T_nS) : S \in \mathcal{B}(X),\ S \text{ invertible}\}.$$

THEOREM 5.1. *$\mathcal{S}(T_1, \ldots, T_n)$ is τ-dense in $\mathcal{B}(X) \times \cdots \times \mathcal{B}(X)$ if* *and only if* $\{I, T_1, \ldots, T_n\}$ *is linearly independent modulo* $\mathcal{B}_F(X)$, *where* I *denotes the identity operator on* X.

Proof. Suppose $\{I, T_1, \ldots, T_n\}$ is linearly independent modulo $\mathcal{B}_F(X)$. Obtain a sequence of vectors $\{u_i\}$ such that the set

$$\{u_i : 1 \leq i < \infty\} \cup \{T_j u_i : 1 \leq j \leq n, 1 \leq i < \infty\}$$

is linearly independent by applying Lemma 2.2 inductively to $\mathcal{S} = \text{span}\{I, T_1, \ldots, T_n\}$, adjusting the subspace W_2 at the i-th step so that it is a complement of

$$\mathcal{S}u_1 \vee \mathcal{S}u_2 \vee \cdots \vee \mathcal{S}u_{i-1}.$$

Let $(A_1, \ldots, A_n)$ be an arbitrary n-tupule and V a product neighborhood. Then there is a linearly independent set of vectors $\{x_1, \ldots, x_\ell\}$, and $\epsilon > 0$, with

$$V \supseteq \{(B_1, \ldots, B_n) : \|(B_i - A_i)x_j\| < \epsilon, 1 \leq i \leq n, 1 \leq j \leq \ell\}.$$

Choose vectors $\{z_{ij} : 1 \leq i \leq n, 1 \leq j \leq \ell\}$ with $\|z_{ij} - A_i x_j\| < \epsilon$ for all i and j, such that the set

$$\{x_1, \ldots, x_\ell, z_{11}, \ldots, z_{1\ell}, \ldots, z_{n1}, \ldots, z_{n\ell}\}$$

is linearly independent. Let S be any bounded invertible operator such that $Sx_j = u_j$, $1 \leq j \leq \ell$, and $Sz_{ij} = T_i u_j$, $1 \leq i \leq n$, $1 \leq j \leq \ell$. Then $z_{ij} = S^{-1}T_i S x_j$, so

$$\|S^{-1}T_i S x_j - A_i x_j\| < \epsilon$$

for all $1 \leq i \leq n$ and $1 \leq j \leq \ell$, and hence $(S^{-1}T_1 S, \ldots, S^{-1}T_n S) \in V$, as required.

For the converse, if $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is a set of scalars, not all 0, for which $\alpha_0 I + \alpha_1 T_1 + \cdots + \alpha_n T_n$ has rank $0 \leq k < \infty$, and if $\{S_\lambda\}$ is a net for which $S_\lambda^{-1} T_i S_\lambda$ converges strongly to an operator A_i, $1 \leq i \leq n$, then $\text{rank}(\Sigma\, \alpha_i A_i) \leq \text{rank}(\Sigma\, \alpha_i S_\lambda^{-1} T_i S_\lambda) = \text{rank}(\Sigma\, \alpha_i T_i) = k$, hence the n-tupule $(A_1, \ldots, A_n)$ cannot be arbitrary. The proof is complete.

Theorem 5.1 can be used to point out the nonpreservation under strong limits of virtually every nonlinear algebraic relationship between operators. For example, T can be a nilpotent of index $n+1$, and S_λ can be a net of invertible operators for which $S_\lambda^{-1}T^\ell S_\lambda \to T^{n+1-\ell}$ strongly, $1 \le \ell \le n$, actually reversing the order of powers.

6. Some concluding remarks. (1) The results of Section 2 and Section 3 extend in a straightforward way to algebraic reflexivity properties for linear subspaces of $\mathcal{L}(V, W)$ and $\mathcal{B}(X, Y)$. We have not done so here because our applications in this paper do not require the extended theory.

(2) Our original motivation for this study was the need to construct examples of finite dimensional reflexive subalgebras of $\mathcal{B}(H)$. The results of Section 2 provide a suitable characterization. Subspace theory is useful here because given a reflexive linear subspace $\mathcal{S} \subseteq \mathcal{B}(H)$, the corresponding algebra

$$\begin{bmatrix} \mathbf{C}I & \mathcal{S} \\ 0 & \mathbf{C}I \end{bmatrix} = \left\{ \begin{bmatrix} \lambda I & S \\ 0 & \mu I \end{bmatrix} : S \in \mathcal{S},\ \lambda,\ \mu \in \mathbf{C} \right\}$$

is a reflexive subalgebra of $\mathcal{B}(H \oplus H)$, and subspaces with requisite properties are often simpler to construct than algebras. This technique was utilized in [11] in the construction of a nonhyperreflexive reflexive algebra.

(3) Let H be a Hilbert space and $\mathcal{S}$ a finite dimensional linear subspace of $\mathcal{B}(H)$ which contains no nonzero finite rank operator. By section 2, $\mathcal{S}$ is reflexive and has a separating vector. Finite dimensionality and existance of a separating vector implies that $\mathcal{S}$ has *property* $\mathbf{A}_1$ [2]. (Equivalently D_σ [8], P_1 [14], elementary [1], spatially elementary [12]). Moreover, if n is any positive integer, and if M_n denotes the $n \times n$ complex matrices, then the subspace $\mathcal{S} \otimes M_n$, acting on the n-fold direct sum $H \oplus \cdots \oplus H$, has the same properties. It follows that $\mathcal{S}$ has property $\mathbf{A}_n$ (see [2]) for each positive integer n. Thus "most" finite dimensional subspaces of $\mathcal{B}(H)$ have this property.

(4) Algebraic reflexivity properties can be of interest in the study of certain subspaces of linear transformations acting on a Banach algebra $\mathcal{A}$. If $\theta \in \mathcal{B}(\mathcal{A})$ is locally a derivation, must θ be a derivation? (Is the space $\mathcal{D}$ of all bounded derivations of $\mathcal{A}$ algebraically reflexive?) If θ is a derivation which is locally an inner derivation, must θ be inner? (Is the space $\mathcal{D}_0$ of

inner derivations relatively reflexive in $\mathfrak{D}$?) The results of Section 3 do not yield answers in this generality. However, if $\{\delta_i\}$ is any sequence of bounded derivations, and if $\theta \in \mathcal{B}(\mathcal{A})$ is locally a finite linear combination of the $\{\delta_i\}$, then, by Theorem 3.5, $\theta = \theta_0 + \theta_1$ where θ_0 *is* a finite linear combination of the δ_i, and θ_1 has finite dimensional range.

(5) The emphasis of this paper has been *linear* interpolation. More generally, if $\mathcal{S}$ is a set of mappings from a set X into a set Y, we may say that a mapping θ interpolates $\mathcal{S}$ if for each $x \in X$ there is an element $S_x \in \mathcal{S}$, depending on x, with $\theta(x) = S_x(x)$. It seems appropriate to generalize our notation and write $\theta \in \text{ref}_a(\mathcal{S})$. Under suitable hypotheses we may show that θ is in $\mathcal{S}$, or at least, is "close" to being in $\mathcal{S}$. For a nonlinear example, let ϕ be a bounded automorphism of a Banach algebra $\mathcal{A}$, and let $\mathcal{S}$ be the set of integral powers of ϕ. It is easily shown using the Baire Category theorem that if $\theta \in \mathcal{B}(\mathcal{A})$, and if $\theta \in \text{ref}_a(\mathcal{S})$, then $\theta \in \mathcal{S}$. This example is nonlinear because $\mathcal{S}$ is not a linear subspace of $\mathcal{B}(\mathcal{A})$, so this cannot be obtained directly from the results of this paper.

(6) Algebraic reflexivity may have applications to systems theory. Linear systems can be represented as linear transformations acting on a vector space. A system θ interpolates a family $\mathcal{S}$ if for each input the corresponding output can be obtained by application of some member of $\mathcal{S}$, so $\theta \in \text{ref}_a(\mathcal{S})$. Under suitable hypotheses this will imply that $\theta \in \mathcal{S}$.

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