HOUSTON JOURNAL OF MATHEMATICS
Volume 17, No. 4, 1991

SEMITRIANGULAR OPERATORS

DOMINGO A. HERRERO
DAVID R. LARSON
WARREN R. WOGEN

Introduction. An operator T in the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear transformations on a complex separable Hilbert space is called *triangular* if $\mathcal{H}$ has an orthonormal basis $\{e_n\}$ with the property that $Te_n \in$ span $\{e_1, \dots, e_n\}$ for each n. This class of operators has a rich structure, and the reader is referred to the recent survey article of the first author [H3] for properties of triangular operators and an extensive reference list. An interesting subclass is the class of *bitriangular* operators studied in [**DH**]. An operator T is called bitriangular if both T and T^* are triangular, perhaps with respect to different orthonormal bases. In this article we introduce a natural generalization of triangularity which we call *semitriangularity* which was motivated by the construction of some counterexamples to problems in single operator theory and operator algebras by the third author in [**W2**].

Let Δ denote the class of all triangular operators, and let $(B\Delta)$ denote the class of bitriangular operators. If $S \in \mathcal{B}(\mathcal{H})$, let us call a vector $x \in \mathcal{H}$ an *algebraic vector* for S if there is a nonzero polynomial $p(t)$ for which $p(S)x = 0$. Then x is algebraic for S if and only if the cyclic subspace for S determined by x is finite dimensional. Let $\mathcal{E}_S$ denote the *set of algebraic vectors* for S. Then $\mathcal{E}_S$ is a linear space: if $x_1, x_2 \in \mathcal{E}_S$ there are nonzero polynomials $p_1(t), p_2(t)$ such that $p_i(S)x_i = 0$, and so setting $p = p_1p_2$ we have $p(S)(x_1 + x_2) = 0$, showing that $x_1 + x_2 \in \mathcal{E}_S$. We note that in our terminology, a well-known theorem of Kaplansky [K] states that S is algebraic (i.e., satisfies a nontrivial polynomial identity) if and only if $\mathcal{E}_S = \mathcal{H}$. If $S \in \Delta$ it is clear that $\mathcal{E}_S$ is dense in $\mathcal{H}$. Conversely, if $\mathcal{E}_S$ is dense then $\mathcal{H}$ is the closed span of finite-dimensional invariant subspaces

This research was partially supported by grants from the NSF.

for S so an elementary argument shows that S is triangular. This gives a coordinate-free description of Δ. In [DH] a useful stronger coordinate-free description of Δ was exploited: $S \in \Delta$ if and only if

$$\text{span}\,\{\ker(S-\lambda)^k : \lambda \in \mathbb{C}, k = 1, 2, \dots\}$$

is dense in $\mathcal{H}$.

We will define an operator T to be *semitriangular* $(S\Delta)$ if $[\mathcal{E}_T]$ has finite codimension in $\mathcal{H}$. (Here $[\cdot]$ denotes closed linear span.) Equivalently, T is semitriangular if T is an extension of a triangular operator by a finite rank operator. We call the codimension of $[\mathcal{E}_T]$ the *index of semitriangularity* of T and denote it $i_{S\Delta}(T)$. It is equal to the minimum possible dimension of the Hilbert space on which T_{22} acts in a representation of T as

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

with T_{11} triangular.

We will see that semitriangular operators may display properties much different from those of triangular operators. With one exception the counterexamples constructed in [W2] are semitriangular. While some (but not all) of the questions that were settled in [W2] can be settled with triangular counterexamples, (see [LW1]), others remain open for the class Δ. In this article, along with the development of structural properties of the class $(S\Delta)$, we give some new results for the classes Δ and $(B\Delta)$. We also discuss some open questions.

This paper is organized as follows. In Section 1 we give some elementary results, examples, and exposition to illustrate some fundamental differences between properties of the classes of triangular and semi-triangular operators. Our main section is Section 2 where we consider several extension properties. We describe the triangular operators which arise as the restriction of some $T \in (S\Delta)$ to the invariant subspace $[\mathcal{E}_T]$, we consider the structure of restrictions of T to invariant subspaces of finite codimension, and we define the class of bi-semitriangular operators $(BS\Delta)$ and derive some of its elementary properties. In Section 3 we show that every bitriangular operator T has a cyclic commutant. It follows that $\mathcal{W}(T)$, the weakly closed algebra generated by T, has a separating vector. (A *separating* vector for a linear space of operators S is a vector x for which the map $S \to Sx, S \in \mathcal{S}$, is injective.) We give an example to show that the commutant of a triangular operator need not be cyclic. We conclude with a discussion of some open questions for the classes $\Delta, (B\Delta), (S\Delta)$ and $(BS\Delta)$.

1. Some Comparisons. We will comment on several results and examples to illustrate that semi-triangular operators can behave differently from triangular operators (see also [**LW2**]). We begin with a simple (but apparently new) result.

Proposition 1.1. *If $T \in \Delta$, then*

$$\{T\}' \cap \text{ Alg Lat } T = \{T\}'' \cap \text{ Alg Lat } T.$$

Proof: First note that $\mathcal{W}(A) = \{A\}' \cap$ Alg Lat $A = \{A\}'' \cap$ Alg Lat$A = \{A\}''$ if A is algebraic (cf. [B, p. 74]). Then fix $\lambda \in \mathbb{C}, n > 0$, and let $\mathcal{M} = \ker(T-\lambda)^n$. Then $\mathcal{M}$ is hyperinvariant for T, and $T|_{\mathcal{M}}$ is algebraic. Thus if $S \in \{T\}' \cap$ Alg Lat T, then

$$S|_{\mathcal{M}} \in \{T|_{\mathcal{M}}\}' \cap \text{ Alg Lat } (T|_{\mathcal{M}}).$$

Hence $S|_{\mathcal{M}} = p(T|_{\mathcal{M}}) = p(T)|_{\mathcal{M}}$ for some polynomial p. If $R \in \{T\}'$, then $R|_{\mathcal{M}}$ and $S|_{\mathcal{M}}$ commute. So since $\mathcal{H}$ is the closed span of subspaces of the form $\ker(T-\lambda)^n$ the operators R and S commute. So $S \in \{T\}''$. Thus $\{T\}' \cap$ Alg Lat $T \subset \{T\}'' \cap$ Alg Lat T. But the reverse inclusion always holds, since $\{T\}'' \subset \{T\}'$. □

The above proposition *fails* for semitriangular operators of index ≥ 2, as shown in [**W2**]. It remains an interesting open question whether it holds if the index is *one*. (See [**LW2**].)

We now consider a density property that the predual $\mathcal{A}_*$ of a dual algebra $\mathcal{A}$ can have. Let $C_1(\mathcal{H})$ denote the ideal of trace class operators on $\mathcal{H}$ and identify $\mathcal{B}(\mathcal{H})$ with $(C_1(\mathcal{H}))^*$ via the pairing $\langle A, f\rangle := Tr(Af)$. A ***dual algebra*** is a unital w^*-closed subalgebra of $\mathcal{B}(\mathcal{H})$. It can be identified with the dual space of $\mathcal{A}_* := C_1(\mathcal{H})/\mathcal{A}_\perp$ in the standard fashion, where $\mathcal{A}_\perp$ denotes the preannihilator of $\mathcal{A}$ in $C_1(\mathcal{H})$. The algebra (or more generally, w^*-closed subspace) $\mathcal{A}$ is called *elementary* [A] if each coset in $\mathcal{A}_*$ has the form $f + \mathcal{A}_\perp$ with f of rank 1 or 0. We will say that $\mathcal{A}$ is ***approximately elementary*** if the set of such "rank-one" elements of $\mathcal{A}_*$ (those of the form $f + \mathcal{A}_\perp$ with f of rank 1 or 0) is dense in $\mathcal{A}_*$ in the quotient topology. For many operators T, $\mathcal{W}(T)$ is elementary. In [HN] an operator was constructed for which $\mathcal{W}(T)$ is not elementary. It is not hard to see that it is approximately elementary. In [W2] the first example was given of an operator T such that $\mathcal{W}(T)$ is not even approximately elementary. This answered negatively a question of the second author. The example in [W2] can be taken to be semi-triangular of index 2. If T is an operator for which $\mathcal{W}(T)$ is not approximately elementary, then T cannot be triangular, and $\mathcal{W}(T)$ cannot have a separating vector. (In fact, it is an open question [LW2] whether $T \in \Delta$ implies that $\mathcal{W}(T)$ has a separating vector.)

The following two observations have been known for some time. For perspective, and completeness, we include them here.

Proposition 1.2. *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be a w^*-closed linear subspace. If $\mathcal{S}$ has a separating vector x, then $\mathcal{S}$ is approximately elementary.*

Proof: Let $\mathcal{G} := \{x \otimes y^* + \mathcal{S}_\perp : y \in \mathcal{H}\}$. (Here $x \otimes y^*$ denotes the operator $w \to \langle w, y\rangle x$.) Then $\mathcal{G}$ is a set of "rank-1" elements of $\mathcal{S}_*$ that is clearly a linear space. We claim that $\mathcal{G}$ is dense in $\mathcal{S}_*$. Indeed, if not then there is a nonzero element of $(\mathcal{S}_*)^* = \mathcal{S}$ which annihilates $\mathcal{G}$. Call this S. Then $Tr(S(x \otimes y^*)) = 0$ for all $y \in H$. That is, $\langle Sx, y\rangle = 0$ for all $y \in H$. But this implies $Sx = 0$. So $S = 0$, a contradiction, since x separates $\mathcal{S}$. □

Proposition 1.3. *If $T \in \Delta$ then $\mathcal{W}(T)$ is approximately elementary.*

Proof: Suppose that $\{e_n\}_1^\infty$ is an orthonormal basis which triangularizes T. For each $n \geq 1$, let P_n be the projection of $\mathcal{H}$ onto $\mathcal{M}_n := [e_1, \ldots e_n]$. Let $g \in C_1(\mathcal{H})$. Since $P_n g P_n \to g$ in trace class norm, it will suffice to show that $P_n g P_n$ decomposes

$$P_n g P_n = F_n + H_n$$

where rank $F_n \leq 1$ and $H_n \in (\mathcal{W}(T))_\perp$. Let $g_n = P_n g|_{\mathcal{M}_n}$. Since $T|_{\mathcal{M}_n}$ is an operator on a finite dimensional Hilbert space, $\mathcal{W}(T|_{\mathcal{M}_n})$ is elementary. Hence $g_n = f_n + h_n$ for some $f_n \in \mathcal{B}(\mathcal{M}_n)$ of rank ≤ 1 and some $h_n \in \mathcal{W}(T|_{\mathcal{M}_n})_\perp$. Let $F_n = f_n \oplus 0$ and $H_n = h_n \oplus 0$ in $\mathcal{M}_n \oplus \mathcal{M}_n^\perp$. Then $P_n g P_n = F_n + H_n$ and rank $F_n \leq 1$. If $A \in \mathcal{W}(T)$ then since $\mathcal{M}_n \in$ Lat T we have $P_n A|_{\mathcal{M}_n} \in \mathcal{W}(T|_{\mathcal{M}_n})$, so

$$\begin{aligned} Tr(AH_n) &= Tr(AP_nH_nP_n) = Tr(P_nAP_nH_nP_n) \\ &= Tr(P_nA|_{\mathcal{M}_n} \cdot h_n) = 0 \end{aligned}$$

showing that $H_n \in (\mathcal{W}(T))_\perp$ as required. □

We now briefly discuss a strong notion of cyclicity. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called ***strictly cyclic***, with strictly cyclic vector x, if $\mathcal{W}(T)x = \mathcal{H}$. The main result of [H1] is that *no triangular* operator on an infinite dimensional Hilbert space can be strictly cyclic. (There are lower triangular strictly cyclic operators; namely, certain weighted shifts.) However, as also noted in [H1] there *are* strictly cyclic operators T which are finite dimensional extensions of triangular operators. So these are semi-triangular in our present terminology. In fact we can choose T so that $i_{S\Delta}(T) = 1$ and $T|_{[\mathcal{E}_T]}$ is diagonal, hence bitriangular.

Example 1.4. Let $x = (1, 1/2, 1/3, \dots) \in \ell^2$ and let D be the diagonal operator on ℓ^2 with x as its diagonal sequence. Then

$$T = \begin{pmatrix} D & x \\ 0 & 0 \end{pmatrix}$$

in $\mathcal{B}(\ell^2 \oplus \mathbf{C})$ is strictly cyclic. In fact the vector $e_\infty = 0 \oplus 1$ in $\ell^2 \oplus \mathbf{C}$ is a strictly cyclic vector. To see this, first observe that

$$T^{n+1}e_\infty = (D^n x) \oplus 0$$

for $n \geq 0$, and that span $\{D^n x : n = 0, 1, \dots\}$ is dense in $\mathcal{H}$. This shows that e_∞ is (topologically) cyclic for T. Then verify that

$$\|p(T)\| \leq (1 + \sqrt{2})\|p(T)e_\infty\|$$

for all polynomials $p(t)$, using the special form of D and x.

For this, write $p(t) = a + q(t)$ with $q(0) = 0$, and compute

$$p(T) = \begin{pmatrix} p(D) & w \\ 0 & a \end{pmatrix}$$

where $w = (q(1), q(1/2), \dots) \in \ell^2$. Then $p(T)e_\infty = w \oplus a$, so $\|p(T)\| \leq \|p(D)\| + \|p(T)e_\infty\|$. Also, $\|p(D)\| \leq |a| + \|q(D)\| \leq |a| + \|w\| \leq \sqrt{2} \cdot \sqrt{|a|^2 + \|w\|^2} = \sqrt{2} \cdot \|p(T)e_\infty\|$. Thus $\|p(T)\| \leq (1 + \sqrt{2}) \cdot \|p(T)e_\infty\|$. This implies that the continuous map $\Theta : A \mapsto Ae_\infty$ from the norm closure of the polynomials $\mathcal{P}(T)$ in T into $\mathcal{H}$ is bounded below. So since it has dense range, it must be surjective. Thus x is strictly cyclic.

2. Extensions. In this section we consider finite and infinite dimensional extensions of triangular and semitriangular operators. If T is an algebraic operator then any finite dimensional extension of T is algebraic, hence in Δ. We establish a strong converse implication: if T is triangular and *not* algebraic then there is a one dimensional extension of T which is not triangular. (Hence it is semitriangular of index 1.) The methods we use in proving this generalize with no real difficulty to extensions of arbitrary dimension. Theorem 2.7 (and Corollary 2.8) show that semitriangular operators of arbitrary index are plentiful. Some finer extension and restriction results are then proven. Finally, the class of *bi-semitriangular* operators is considered and analogous results are proven.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and let*

$$S = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

in $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. Let $x \in \mathcal{H}$ and $y \in \mathcal{K}$. Then

(i) $x \oplus 0 \in \mathcal{E}_S$ *if and only if* $x \in \mathcal{E}_A$.

(ii) *If* $y \neq 0$, *then* $x \oplus y \in \mathcal{E}_S$ *if and only if* $Ax + By \in \mathcal{E}_A$.

Proof: Item (i) is clear. For (ii), suppose that p is a polynomial and write $p(t) = a + tq(t)$, where q is a polynomial. Then

$$p(S) = \begin{pmatrix} aI + q(A)A & q(A)B \\ 0 & aI \end{pmatrix}.$$

If $y \neq 0$ and $p(S)(x \oplus y) = 0$, then $a = 0$ and $q(A)Ax + q(A)By = 0$, so $Ax + By \in \mathcal{E}_A$. Conversely, if $Ax + By \in \mathcal{E}_A$, there is a polynomial q so that $q(A)(Ax + By) = 0$. Hence $Sq(S)(x \oplus y) = 0$ and $x \oplus y \in \mathcal{E}_S$. □

Lemma 2.1 has the following immediate corollary.

Corollary 2.2. *With the notation of Lemma 2.1, $\mathcal{E}_S \subseteq \mathcal{H} \oplus 0$ if and only if B is injective and* $(\text{ran } B) \cap (\mathcal{E}_A + \text{ ran } A) = \{0\}$.

Corollary 2.3. *With the notation of Lemma 2.1, suppose also that $A \in \Delta$. Then $S \in \Delta$ if and only if there is a dense subspace $\mathcal{K}_0$ of $\mathcal{K}$ with*

$B\mathcal{K}_0 \subseteq \mathcal{E}_A +$ ran A. In particular, if $\mathcal{K}$ is finite dimensional, then $S \in \Delta$ if and only if ran $B \subseteq \mathcal{E}_A +$ ran A.

Proof: Since $A \in \Delta$, we have $[\mathcal{E}_A] = \mathcal{H}$ and $\mathcal{H} \oplus 0 \subset [\mathcal{E}_S]$. Thus $S \in \Delta$ if and only if the projection of $\mathcal{E}_S$ onto $0 \oplus \mathcal{K}$ is dense in $0 \oplus \mathcal{K}$. Now apply Lemma 1 (ii) to complete the proof. □

Lemma 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(i) $\mathcal{E}_T +$ ran $T = \cup_{k=1}^{\infty} \ker(T^k) +$ ran T.

(ii) *If $\mathcal{E}_T +$ ran T has finite codimension in $\mathcal{H}$, it is closed. If $\mathcal{E}_T +$ ran T does not have finite codimension, it is contained in an operator range of infinite algebraic codimension in $\mathcal{H}$.*

Proof: We will show that

$$\cup_1^{\infty} \ker(T^k) \subset \mathcal{E}_T \subset \cup_1^{\infty} \ker(T^k) + \text{ran}\, T$$

from which (i) follows. The inclusion $\cup_1^{\infty} \ker(T^k) \subset \mathcal{E}_T$ is clear. For the second inclusion, it is enough to show that if $x \in \ker(T-\lambda)^k$ for some $k \geq 1$ and some $\lambda \neq 0$, then $x \in$ ran T. But if $(T-\lambda)^k x = 0$, then (e.g., using the Binomial Theorem) $x = Tq(T)x$ for some polynomial q. So $x \in$ ran T.

To prove (ii), let P_k be the orthogonal projection onto $\ker(T^k)$. Then

$$\mathcal{E}_T + \text{ ran } T = \cup_1^{\infty}(\text{ ran } P_k + \text{ ran } T).$$

By [FW],

$$\text{ran } P_k + \text{ ran } T = \text{ ran}\sqrt{TT^* + P_k} \quad .$$

(More generally, Theorem 2.2 in [FW] states that if $A, B \in \mathcal{B}(\mathcal{H})$, then ran $A +$ ran $B =$ ran$\sqrt{AA^* + BB^*}$.) If $\mathcal{E}_T +$ ran T is proper, then for all $k \geq 1$, ran$\sqrt{TT^* + P_k}$ is proper, so that $0 \in \sigma(TT^* + P_k)$.

There are two cases to consider. If $0 \notin \sigma_e(TT^* + P_k)$ for some k, then

$$\text{ran } \sqrt{TT^* + P_k} = \text{ ran } T + \text{ ran } P_k$$

has finite codimension in $\mathcal{H}$. Hence there is a k_0 so that

$$\text{ran } T + \text{ ran } P_k = \text{ ran } T + \text{ ran } P_{k_0} \text{for all } k \geq k_0.$$

Thus $\mathcal{E}_T + \text{ ran } T = \text{ ran } P_{k_0} + \text{ ran } T$ is an operator range of finite codimension. It follows easily (see [FW]) that $\mathcal{E}_T + \text{ ran } T$ is closed.

In the remaining case, $0 \in \sigma_e(TT^* + P_k)$ for all $k \geq 1$. Let $B_m = \sum_{k=1}^m 2^{-k} P_k$ for each $m \geq 1$ and let $B = \sum_1^\infty 2^{-k} P_k$. Then

$$\text{ran } T + \text{ ran } B_m = \text{ ran } T + \text{ ran } P_m$$

has infinite codimension, so that $0 \in \sigma_e(TT^* + B_m^2)$ for all $m \geq 1$. But $\{TT^* + B_m^2\}$ converges in norm to $TT^* + B^2$, so by upper semicontinuity of σ_e, $0 \in \sigma_e(TT^* + B^2)$. Let $\mathcal{M} = \text{ ran } T + \text{ ran } B$. Then $\mathcal{M}$ is an operator range of infinite codimension, and $\text{ran } P_m \subset \text{ ran } B$ for all $m \geq 1$, so $\mathcal{E}_T + \text{ ran } T \subset \mathcal{M}$. □

For $\lambda \in \mathbf{C}$ and $T \in \mathcal{B}(\mathcal{H})$ note that $\mathcal{E}_{T-\lambda} = \mathcal{E}_T$. We next characterize points in $\partial\sigma(T)$ for which $\mathcal{E}_T + \text{ ran } (T - \lambda) = \mathcal{H}$.

Lemma 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose $\lambda \in \partial\sigma(T)$. Then $\mathcal{E}_T +$ ran $(T - \lambda) = \mathcal{H}$ if and only if λ is an isolated point of $\sigma(T)$ for which $(T-\lambda)P_\lambda$ is nilpotent, where P_λ is the Riesz idempotent for T corresponding to $\{\lambda\}$.*

Proof: Suppose that λ satisfies the right hand side of the equivalence. (If in addition P_λ is of finite rank, then λ is called a *normal* eigenvalue. See [H4, p. 5].) Then $P_\lambda \mathcal{H} \subset \mathcal{E}_T$ and $(I - P_\lambda)\mathcal{H} \subset \text{ ran } (T - \lambda)$, so $\mathcal{H} = \mathcal{E}_T + \text{ran } (T - \lambda)$.

For the converse, assume $\lambda \in \partial\sigma(T)$ and that $\mathcal{E}_T + \text{ ran } (T - \lambda) = \mathcal{H}$. It will suffice to find $\mathcal{M} \in \text{ Lat } T, \mathcal{M} \neq 0$, so that the matrix of $T - \lambda$ relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ has the form

$$\begin{pmatrix} N & R \\ O & S \end{pmatrix}, \qquad (*)$$

with N nilpotent and S invertible. First note that since λ is in the *boundary* of $\sigma(T)$, $T - \lambda$ is not surjective. From this, Lemma 2.4(i) implies that $\ker(T - \lambda) \neq 0$. Again, by Lemma 2.4(i),

$$\mathcal{H} = \cup_1^\infty \left(\ker(T - \lambda)^k + \text{ ran } (T - \lambda)\right).$$

Each subspace

$$\ker\,(T - \lambda)^k + \text{ ran } (T - \lambda)$$

is an operator range and hence an F_σ set. So an application of the Baire Category Theorem shows that

$$\mathcal{H} = \ker(T-\lambda)^m + \operatorname{ran}(T-\lambda)$$

for some $m \geq 1$.

Let $\mathcal{M} = \ker(T-\lambda)^m$ and write $T-\lambda$ in the form $(*)$. Then N is nilpotent as required, and we claim that S is invertible. It is easy to see that S is surjective. If S were singular, then (since $\sigma(T-\lambda) = \sigma(N)\cup\sigma(S)$) we would have $\sigma(S) = \sigma(T-\lambda)$. But then $0 \in \partial\sigma(S)$, contradicting the surjectivity of S. □

It T is algebraic, then $\mathcal{E}_T = \mathcal{H}$ so of course $\mathcal{H} = \mathcal{E}_T +$ ran $(T-\lambda)$ for all λ. We now prove a strong form of the converse.

Corollary 2.6. *If $T \in \mathcal{B}(\mathcal{H})$ and if $\mathcal{H} = \mathcal{E}_T +$ ran $(T-\lambda)$ for all $\lambda \in \partial\sigma_e(T)$, then T is algebraic.*

Proof: First note that if $\lambda \in \partial\sigma(T)$ but $\lambda \notin \partial\sigma_e(T)$, then $\mathcal{H} = \mathcal{E}_T +$ ran $(T-\lambda)$ by Lemma 2.5. Thus our hypothesis implies that in fact $\mathcal{H} = \mathcal{E}_T +$ ran $(T-\lambda)$ for all $\lambda \in \partial\sigma(T)$. By Lemma 2.5, each $\lambda \in \partial\sigma(T)$ is isolated in $\sigma(T)$, so $\sigma(T)$ is finite. Again by Lemma 2.5, each restriction $(T-\lambda)|_{P_\lambda\mathcal{H}}$ is nilpotent, so T is algebraic. □

We consider the existence of semitriangular extensions of triangular operators.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be triangular. The following statements are equivalent.*

(i) *T is not algebraic.*

(ii) *For some $\lambda \in \partial\sigma_e(T)$, there is a $Z \in \mathcal{B}(\mathbf{C},\mathcal{H})$ so that the operator*

$$A = \begin{pmatrix} T & Z \\ 0 & \lambda \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H}\oplus\mathbf{C})$ satisfies $i_{S\Delta}(A) = 1$.

(iii) *For some $\lambda \in \partial\sigma_e(T)$ and for each $p > 0$, there is a $Z_p \in \mathcal{B}(\mathbf{C}^p,\mathcal{H})$ so that the operator*

$$A = \begin{pmatrix} T & Z_p \\ 0 & \lambda \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H} \oplus \mathbb{C}^p)$ *satisfies* $i_{S\Delta}(A) = p$.

(iv) *For some* $\lambda \in \partial\sigma_e(T)$, *there is a* $Z \in \mathcal{B}(\mathcal{H})$ *so that the operator*

$$A = \begin{pmatrix} T & Z \\ 0 & \lambda \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ *is not semitriangular.*

Proof: It is clear that each of the conditions (ii), (iii), and (iv) imply that T is not algebraic. Conversely, suppose (i) holds. Then by Corollary 2.6, there is a $\lambda \in \partial\sigma_e(T)$ with $\mathcal{E}_T +$ ran $(T-\lambda) \neq \mathcal{H}$. The set $\mathcal{E}_T +$ ran $(T-\lambda)$ is dense in $\mathcal{H}$, so it follows from Lemma 2.4(ii) that $\mathcal{E}_T +$ ran $(T-\lambda)$ has infinite algebraic codimension in $\mathcal{H}$. Thus for each $p \geq 1$ we may choose $Z_p \in \mathcal{B}(\mathbb{C}^p, \mathcal{H})$ so that Z_p is injective and

$$(\text{ ran } Z_p) \cap (\mathcal{E}_T + \text{ ran } (T-\lambda)) = \{0\}.$$

Let

$$S := \begin{pmatrix} T-\lambda & Z_p \\ 0 & 0 \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H} \oplus \mathbb{C}^p)$. By Corollary 2.2, $i_{S\Delta}(S) = p$, so $i_{S\Delta}(S+\lambda) = p$, proving (ii) and (iii). To prove (iv), we apply Lemma 2.4(ii). There is a dense operator range $\mathcal{M}$ of infinite algebraic codimension with $\mathcal{E}_T +$ ran $(T-\lambda) \subset \mathcal{M}$. By [FW] there is a unitary operator U so that $\mathcal{M} \cap U\mathcal{M} = \{0\}$. Choose $Z \in \mathcal{B}(\mathcal{H})$ so that Z is injective and ran $Z = U\mathcal{M}$. Corollary 2.2 shows that

$$\begin{pmatrix} T-\lambda & Z \\ 0 & 0 \end{pmatrix}$$

is not semitriangular. □

Finally, it is useful for perspective to express the results of Theorem 2.7 in an alternate way.

Corollary 2.8. *Let* $T \in \mathcal{B}(\mathcal{H})$. *The following statements are equivalent.*

(i) *T is algebraic.*
(ii) *Every finite dimensional extension of T is triangular.*
(iii) *Every finite dimensional extension of T is algebraic.*

(iv) *Every extension of T by a triangular operator is triangular. (That is, whenever*

$$A = \begin{pmatrix} T & Z \\ 0 & S \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ and S is triangular, then A is triangular.)

Proof: The only part of the proof that is not immediate from Theorem 2.7 is that (i) implies (iv). But if T is algebraic, then $\mathcal{H} \oplus 0 \subset \mathcal{E}_A$. Thus clearly $\mathcal{E}_A = \mathcal{H} \oplus \mathcal{E}_S$, and so A is triangular. □

It is now a simple matter to extend Theorem 2.7 as follows.

Corollary 2.9. *If $T \in \mathcal{B}(\mathcal{H})$ is semitriangular with $i_{S\Delta}(T) = n$, then T has a one-dimensional extension A with $i_{S\Delta}(A) = n + 1$.*

Proof: If $\mathcal{H}_0 = [\mathcal{E}_T]$, then $\mathcal{H}_0^{\perp}$ has dimension n. Relative to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$ we have

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Also, $\mathcal{E}_T = \mathcal{E}_A$. Choose $\mu \in \sigma(C) = \sigma_p(C)$ and choose $e \in \mathcal{H}_0^{\perp}, e \neq 0$, so that $Ce = \mu e$. If $T_0 = T|_{\mathcal{H}_0 \oplus [e]}$, then T_0 has the form

$$T_0 = \begin{pmatrix} A & y \\ 0 & \mu \end{pmatrix},$$

and $i_{S\Delta}(T_0) = 1$. By Lemma 2.1, $y \notin \mathcal{E}_A +$ ran $(A - \mu)$. Since A is triangular $\mathcal{E}_A$ is dense. From these two facts Lemma 2.4(ii) implies that $\mathcal{E}_A +$ ran $(A - \mu)$ is contained in an operator range $\mathcal{M}$ of infinite algebraic codimension in $\mathcal{H}_0$. Thus $\mathcal{E}_T +$ ran $(T - \mu)$ has infinite algebraic codimension in $\mathcal{H}$. So choose $Z \in \mathcal{B}(\mathbb{C}, \mathcal{H})$ with

$$\text{ran } Z \cap (\mathcal{E}_T + \text{ ran } (T - \mu)) = \{0\}.$$

Let

$$R = \begin{pmatrix} T & Z \\ 0 & \mu \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H} \oplus \mathbf{C})$. By Lemma 2.1, $\mathcal{E}_R = \mathcal{E}_T = \mathcal{E}_A$, so $i_{S\Delta}(R) = n+1$. □

We now consider a converse to Theorem 2.7. If T is semitriangular, what can be said about the *restrictions* of T? Since *every* strict contraction is the restriction of a triangular operator (namely, the backward unilateral shift of infinite multiplicity) we must impose some conditions on the restrictions of T. We need the following lemma.

Lemma 2.10. *If $\mathcal{E}$ is a linear manifold in $\mathcal{H}$ with the property that $[\mathcal{E}]$ has finite codimension n in $\mathcal{H}$, and if $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ of finite codimension m in $\mathcal{H}$, then*

$$n - m \leq \dim\,(\mathcal{M} \ominus [\mathcal{E} \cap \mathcal{M}]) \leq n.$$

Proof: The key to the proof is that $\mathcal{M} \cap [\mathcal{E}] = [\mathcal{M} \cap \mathcal{E}]$. To show this, let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}^\perp$. Then $P\mathcal{E}$ has dimension $k \leq m$. Thus there is a subspace $\mathcal{U} = [u_1, u_2, \dots, u_k] \subset \mathcal{E}$ so that $P\mathcal{U} = \mathcal{P}\mathcal{E}$. For each $x \in \mathcal{E}$, we can find $u \in \mathcal{U}$ with $Px = Pu$. Thus $x - u \in \mathcal{E} \cap \mathcal{M}$ and $\mathcal{E} = (\mathcal{E} \cap \mathcal{M}) + \mathcal{U}$. Hence $[\mathcal{E}] = [\mathcal{E} \cap \mathcal{M}] + \mathcal{U}$. The intersection $\mathcal{U} \cap \mathcal{M} = \{0\}$, so

$$[\mathcal{E}] \cap \mathcal{M} \;=\; ([\mathcal{E} \cap \mathcal{M}] + \mathcal{U}) \cap \mathcal{M} \;=\; [\mathcal{E} \cap \mathcal{M}].$$

Now

$$\begin{aligned} \mathcal{M} \ominus [\mathcal{E} \cap \mathcal{M}] &= \mathcal{M} \ominus ([\mathcal{E}] \cap \mathcal{M}) \\ &= ([\mathcal{E}] \cap \mathcal{M})^\perp \ominus \mathcal{M}^\perp \\ &= (\mathcal{E}^\perp + \mathcal{M}^\perp) \ominus \mathcal{M}^\perp. \end{aligned}$$

Since

$$n = \dim\, \mathcal{E}^\perp \leq \dim\,(\mathcal{E}^\perp + \mathcal{M}^\perp) \leq m + n,$$

we see that $n - m \leq \dim\,(\mathcal{M} \ominus [\mathcal{M} \cap \mathcal{E}]) \leq n$. □

Corollary 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ be semitriangular with $i_{S\Delta}(T) = n$, and suppose $\mathcal{M}$ is an invariant subspace for T of finite codimension m in $\mathcal{H}$. Then $T|_{\mathcal{M}}$ is semitriangular and*

$$\max\{0, n-m\} \;\leq\; i_{S\Delta}(T|_{\mathcal{M}}) \leq n.$$

Proof: First observe that $\mathcal{E}_{T|_{\mathcal{M}}} = \mathcal{E}_T \cap \mathcal{M}$. Now apply Lemma 2.10, with $\mathcal{E} = \mathcal{E}_T$. □

A special case of Corollary 2.11 states that the restriction of a triangular operator to an invariant subspace of finite codimension is triangular. Also, it is well-known (cf. [H3]) that the compression of a triangular operator to the *orthogonal complement* of an *arbitrary* (no dimension restriction) invariant subspace is triangular. The same is true for a semitriangular operator. The reason for this is the following.

Lemma 2.12. *Let $T \in \mathcal{B}(\mathcal{H})$ be arbitrary, and let $P \in$ Lat T be arbitrary. Then*

$$P^{\perp}\mathcal{E}_T \subseteq \mathcal{E}_{P^{\perp}T|P^{\perp}H}.$$

Proof: Let $x \in \mathcal{E}_T$ and let $q(t)$ be a nonzero polynomial with $q(T)x = 0$. Then $P^{\perp}q(T)x = 0$. So since $P^{\perp}q(T) = P^{\perp}q(P^{\perp}TP^{\perp})P^{\perp}$, it follows that $P^{\perp}x \in \mathcal{E}_{P^{\perp}T|P^{\perp}H}$. □

Corollary 2.13. *Let $T \in (S\Delta)$ and let $P \in$ Lat T. Then $P^{\perp}T|P^{\perp}H \in (S\Delta)$ and $i_{S\Delta}(P^{\perp}T|P^{\perp}H) \leq i_{S\Delta}(T)$.*

Proof: If the codimension of $[\mathcal{E}_T]$ in $\mathcal{H}$ is n then the codimension of $[P^{\perp}\mathcal{E}_T]$ in $P^{\perp}\mathcal{H}$ is $\leq n$. □

Let us call an operator T *bi-semitriangular* $(BS\Delta)$ if both T and T^* are semitriangular. This is closely related to bitriangularity. First note that if T is semitriangular and $T|_{[\mathcal{E}_T]}$ is bitriangular, then T^* is triangular. To see this fact, simply write

$$T = \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}$$

in $\mathcal{B}([\mathcal{E}_T] \oplus \mathcal{N})$ with $\dim \mathcal{N} < \infty$ and A bitriangular. Then

$$T^* = \begin{pmatrix} B^* & Z^* \\ 0 & A^* \end{pmatrix}$$

in $\mathcal{B}(\mathcal{N} \oplus [\mathcal{E}_T])$, where B^* is finite-dimensional and A^* is triangular. So by Corollary 2.8, T^* is triangular. Next note that if T_1 and T_2 are arbitrary operators then $\mathcal{E}_{T_1 \oplus T_2} = \mathcal{E}_{T_1} \oplus \mathcal{E}_{T_2}$. Indeed, for each polynomial $p(t)$ and for each vector $x_1 \oplus x_2$ we have $p(T_1 \oplus T_2)(x_1 \oplus x_2) = p(T_1)x_1 \oplus p(T_2)x_2$, yielding "$\subseteq$", and conversely, if $x_i \in \mathcal{E}_{T_i}$ choose nonzero polynomials $q_i(t)$ with $q_i(T_i)x_i = 0$ and note that if $q = q_1q_2$ then $q(T_1 \oplus T_2)(x_1 \oplus x_2) = 0$, yielding "$\supseteq$". A consequence of this second observation is:

Lemma 2.14. *If* T_1 *and* T_2 *are semitriangular operators then* $i_{S\Delta}(T_1 \oplus T_2) = i_{S\Delta}(T_1) + i_{S\Delta}(T_2)$.

Proof: Note that $[\mathcal{E}_{T_1 \oplus T_2}] = [\mathcal{E}_{T_1}] \oplus [\mathcal{E}_{T_2}]$. □

From Lemma 2.14 and the fact that *adjoints* of finite-dimensional extensions of bitriangular operators are triangular, it follows that bi-semitriangular operators will *not* in general be simply finite-dimensional extensions of bitriangular operators. For an example let A_1 and A_2 be nonalgebraic bitriangular operators, let n_1 and n_2 be prescribed positive integers, let T_1 and T_2 be finite-dimensional extensions of A_1 and A_2 with $i_{S\Delta}(T_i) = n_i$, and let $T = T_1 \oplus T_2^*$. Then $i_{S\Delta}(T) = n_1$ and $i_{S\Delta}(T^*) = n_2$.

The operator $T = T_1 \oplus T_2^*$ above is a *dilation* of the bitriangular operator $A = A_1 \oplus A_2^*$. In general, operators in $(BS\Delta)$ are finite-dimensional dilations of operators in $(B\Delta)$. The following proposition captures this. We note that item (ii) below yields the following matrix form for $T \in (BS\Delta)$:

$$(**) \qquad T = \begin{pmatrix} B_1 & * & * \\ 0 & T_0 & * \\ 0 & 0 & B_2 \end{pmatrix}$$

where T_0 is bitriangular and B_1, B_2 act on Hilbert spaces of dimension $i_{S\Delta}(T^*)$ and $i_{S\Delta}(T)$, respectively, giving the correct dilation form.

Proposition 2.15. *The following statements are equivalent for* $T \in \mathcal{B}(\mathcal{H})$:

(i) $T \in (BS\Delta)$ *with* $i_{S\Delta}(T) = m, i_{S\Delta}(T^*) = n$.

(ii) $\mathcal{H}$ *has a decomposition* $\mathcal{H} = \mathcal{N} \oplus \mathcal{H}_0 \oplus \mathcal{M}$, *with* $\dim \mathcal{N} = n, \dim \mathcal{M} = m$, *and* $[\mathcal{E}_T] = \mathcal{N} \oplus \mathcal{H}_0$, $[\mathcal{E}_{T^*}] = \mathcal{H}_0 \oplus \mathcal{M}$. *In this case the compression of* T *to* $\mathcal{H}_0$ *is bitriangular.*

Proof: Only the implication (i) → (ii) needs proof. Let $T \in (BS\Delta), i_{S\Delta}(T) = m, i_{S\Delta}(T^*) = n$, and let $\mathcal{M} := [\mathcal{E}_T]^\perp$ and $\mathcal{N} := [\mathcal{E}_{T^*}]^\perp$. Then $\dim \mathcal{M} = m$ and $\dim \mathcal{N} = n$. Since $\mathcal{N}$ is a finite dimensional invariant subspace for T we have $\mathcal{N} \subseteq \mathcal{E}_T$. Similarly, $\mathcal{M} \subseteq \mathcal{E}_{T^*}$. Hence $\mathcal{N} \perp \mathcal{M}$. Let $\mathcal{H}_0 := [\mathcal{E}_T] \ominus \mathcal{N} = [\mathcal{E}_{T^*}] \ominus \mathcal{M}$. Then $\mathcal{H} = \mathcal{N} \oplus \mathcal{H}_0 \oplus \mathcal{M}$ as required. Now let $P_0 :=$ proj$(\mathcal{H}_0)$ and let $T_0 := P_0 T | \mathcal{H}_0$. Let T_1 be the restriction of T to $[\mathcal{E}_T]$. Then T_1 is triangular, and T_0 is the compression of T_1 to $[\mathcal{E}_T] \ominus \mathcal{N}$. So T_0 is triangular. Since $T_0^* = P_0 T^* | \mathcal{H}_0$ a similar argument shows that $T_0^* \in \Delta$. Hence $T_0 \in (\mathcal{B}\Delta)$. □

Finally, we note the analog of Theorem 2.7 for bi-semitriangularity.

Proposition 2.16. *If $T_0 \in \mathcal{B}(\mathcal{H}_0)$ is bitriangular and not algebraic, then for every $m \geq 0$ and $n \geq 0$ there is a $T \in (BS\Delta)$ of form $(**)$ with $i_{S\Delta}(T) = m$ and $i_{S\Delta}(T^*) = n$.*

Proof: If either $m = 0$ or $n = 0$, this result is immediate from Theorem 2.7 and the fact that adjoints of finite dimensional extensions of bitriangular operators are triangular. So suppose m and n are positive. By Theorem 2.7, T_0 has an extension

$$T_1 = \begin{pmatrix} T_0 & Z \\ 0 & M \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H}_0 \oplus \mathcal{M})$ with $\dim \mathcal{M} = m$ and $i_{S\Delta}(T_1) = m$. Then $T_1^* \in \Delta$ and T_1^* is not algebraic. Again by Theorem 2.7,

$$T_1^* = \begin{pmatrix} M^* & Z^* \\ 0 & T_0^* \end{pmatrix}$$

on $\mathcal{M} \oplus \mathcal{H}_0$ has an extension

$$T^* = \begin{pmatrix} M^* & Z^* & Y^* \\ 0 & T_0^* & X^* \\ 0 & 0 & N^* \end{pmatrix}$$

in $\mathcal{B}(\mathcal{M} \oplus \mathcal{H}_0 \oplus \mathcal{N})$, where $n = \dim \mathcal{N}$ and $i_{S\Delta}(T^*) = n$.

It remains to show that $i_{S\Delta}(T) = m$. For this, note that

$$\begin{pmatrix} T_0^* & X^* \\ 0 & N^* \end{pmatrix}$$

has a triangular adjoint since it is a finite-dimensional extension of a bitriangular operator. Thus

$$T = \begin{pmatrix} N & X & Y \\ 0 & T_0 & Z \\ 0 & 0 & M \end{pmatrix}$$

on $\mathcal{N} \oplus \mathcal{H}_0 \oplus \mathcal{M}$ is an m dimensional extension of a triangular operator, and so $i_{S\Delta}(T) \leq m$. But by Corollary 2.11, $m = i_{S\Delta}(T_1) \leq i_{S\Delta}(T)$, so $i_{S\Delta}(T) = m$ as required. □

We return to a reconsideration of Lemma 2.14 and a natural generalization of it giving the behavior of $i_{S\Delta}$ under (perhaps infinite) direct sums.

Lemma 2.17. *Let $\{T_i\}$ be a uniformly bounded sequence of operators acting on Hilbert spaces $\{\mathcal{H}_i\}$, and let $T =$ diag $\{T_1, T_2, \dots\} = \oplus_i T_i$ acting on $\mathcal{H} = \oplus_i \mathcal{H}_i$. Then $[\mathcal{E}_T] = \oplus_i[\mathcal{E}_{T_i}]$.*

Proof: Let $x = \oplus_i x_i \in \mathcal{H}$ and suppose $p(T)x = 0$ for some polynomial p. Then $p(T_i)x_i = 0$ for all i. This shows "$\subset$". Now suppose $z \in \oplus_i[\mathcal{E}_{T_i}]$. Then z is a norm limit of vectors $y = \oplus_i y_i$ which have at most finitely many of the summands y_i equal to 0, and with $y_i \in \mathcal{E}_{T_i}$ for all i. Such vectors y are in $\mathcal{E}_T$. (Use the same type of argument as in Lemma 2.14.) This shows that the closures of $\mathcal{E}_T$ and $\oplus_i \mathcal{E}_{T_i}$ are the same, as required. □

Proposition 2.18. *A direct sum of semitriangular (bi-semitriangular) operators is semitriangular (bi-semitriangular) if and only if all but a finite number of the summands is triangular. In this case the index of semitriangularity of the direct sum is the sum of the indices of the summands.*

Proof: This follows immediately from Lemma 2.17. □

3. Some Related Results and Questions. First, we will show that every bitriangular operator T has a cyclic commutant. Since T is quasisimilar to a certain Jordan model J [**DH**], we first consider cyclic vectors for J.

Thus suppose that $J \in \mathcal{B}(\mathcal{H})$ is the direct sum of Jordan blocks. That is, $\mathcal{H} = \oplus_1^\infty \mathcal{H}_k$, where each $\mathcal{H}_k$ is finite dimensional and reduces J, and also $J|_{\mathcal{H}_k} = \lambda_k + J_k$, where $\lambda_k \in \mathbf{C}$ and J_k is a (cyclic) nilpotent Jordan block (a finite dimensional truncated shift of multiplicity one). For each k, let x_k be a unit vector so that $[x_k] = \mathcal{H}_k \ominus J\mathcal{H}_k$. Note that x_k is a cyclic vector for J_k, and if $y_k \in \mathcal{H}_k$ with $(y_k, x_k) \neq 0$ then y_k is also cyclic for J_k.

Lemma 3.1. *For J as above*

(i) *If $y \in \mathcal{H}$ and $(y, x_k) \neq 0$ for all $k \geq 1$, then y is cyclic for $\{J\}'$.*
(ii) *If $Y \in \mathcal{B}(\mathcal{H})$ is a quasiaffinity, then ran Y contains a cyclic vector for $\{J\}'$.*

Proof: Suppose $(y, x_k) \neq 0, k \geq 1$. Write $y = \oplus_1^\infty y_k$. Then y_k is cyclic for J_k. If $\{p_k\}$ is any sequence of polynomials such that $\{\|p_k(\lambda_k + J_k)\|\}$ is bounded, then $\oplus_{k=1}^\infty p_k(\lambda_k + J_k) \in \{J\}'$. Choosing $p_j \equiv 1$ and $p_k \equiv 0$ for $k \neq j$, we see that $y_j \in (\{J\}')y$ for all $j \geq 1$. This shows that y is cyclic for $\{J\}'$.

Now suppose that Y is a quasiaffinity. Then Y^* is injective, so $Y^*x_k \neq 0$ for every $k \geq 1$. Choose $y \in \mathcal{H}$ such that y is not orthogonal to any of the vectors Y^*x_k for $k = 1, 2, \ldots$. Then $(Yy, x_k) = (y, Y^*x_k) \neq 0$ for every $k \geq 1$, so Yy is cyclic for $\{J\}'$ by the first part of the Lemma. □

Theorem 3.2. *If T is bitriangular then $\{T\}'$ has a cyclic vector.*

Proof: As noted earlier, there is a Jordan model J and quasiaffinities X and Y so that $TX = XJ$ and $YT = JY$. Applying Lemma 3.1, we choose $y \in \mathcal{H}$ so that $z = Yy$ is cyclic for $\{J\}'$. We will show that y is cyclic for $\{T\}'$. First note that if $A \in \{J\}'$, then $XAY \in \{T\}'$. That is, $X\{J\}'Y \subset \{T\}'$. Now $\{J\}'z$ is dense in $\mathcal{H}$ and X has dense range, so $X\{J\}'z$ is dense in $\mathcal{H}$. But $X\{J\}'z = X\{J\}'Yy \subset \{T\}'y$, so y is a cyclic vector for $\{T\}'$. □

We note that it is an open question which has been in circulation for several years as to whether the cyclic multiplicity of the commutant of a single operator is invariant under quasisimilarity.

Now recall that if an algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ has a cyclic vector x, then x is a separating vector for $\mathcal{A}'$.

Corollary 3.3. *If T is bitriangular, then $\mathcal{W}(T)$ has a separating vector.*

Note that by Theorem 3.2 and the above remark we have a slightly stronger result: $\{T\}''$ has a separating vector.

The question of whether $\mathcal{W}(T)$ has a separating vector for every triangular operator T is still open. The following example shows that we cannot always expect to produce a separating vector for $\mathcal{W}(T)$ from a cyclic vector for $\{T\}'$.

Example 3.4. For each $n, 1 \leq n \leq \infty$, there is a triangular operator T such that $\{T\}'$ has cyclic multiplicity n.

Let $\mathcal{K}$ be a separable Hilbert space of infinite dimension and let $\mathcal{H} = \oplus_1^\infty \mathcal{K}$. Consider an operator weighted backward shift W with weight sequence $\{W_n\}_{n=0}^\infty$ on $\mathcal{H}$ defined by

$$W(x_0 \oplus x_1 \oplus \ldots) = (W_0 x_1 \oplus W_1 x_2 \oplus \ldots).$$

We have $W \in \mathcal{B}(\mathcal{H})$ provided $\{\|W_n\|\}_{n=0}^\infty$ is bounded. In [H2], an invertible weight sequence is constructed so that $\mathcal{W}(W) = \{W\}'$. (W is said to have a "tiny" commutant.) We can modify this example slightly so that W_0 is a noninvertible quasiaffinity, W_n is invertible for $n \geq 1$, and $\mathcal{W}(W) = \{W\}'$ still holds. We omit the details.

For a fixed n, $1 \leq n \leq \infty$, we also fix a Hilbert space $\mathcal{N}$ of dimension n. Choose a bounded injective operator $X : \mathcal{N} \to \ker W = \mathcal{K} \oplus 0 \oplus 0 \oplus \ldots$ such that ran $X \cap \operatorname{ran} W_0 = \{0\}$. Let

$$T = \begin{pmatrix} W & X \\ 0 & 0 \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H} \oplus \mathcal{N})$. Note that $WX = 0$ so that

$$T^k = \begin{pmatrix} W^k & 0 \\ 0 & 0 \end{pmatrix}$$

if $k > 1$. So clearly T is triangular. Suppose that

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in $\mathcal{B}(\mathcal{H} \oplus \mathcal{N})$ and $U \in \{T\}'$. Then $C = 0$, $A \in \{W\}'$, and $WB + XD = AX$. Since $\{W\}' = \mathcal{W}(W)$, the operator A has the form $A = \sum_{k=0}^{\infty} \lambda_k W^k$. (See [H2] or [Lam].) Also $W^k X = 0$ for $k \geq 1$ so that $AX = \lambda_0 X$. Hence $WB = X(\lambda_0 K - D)$. Since $(\operatorname{ran} X) \cap (\operatorname{ran} W_0) = \{0\}$ we have $(\operatorname{ran} X) \cap (\operatorname{ran} W) = \{0\}$. Thus $WB = 0$ and $D = \lambda_0 I$. This gives the following description of $\{T\}'$:

$$\{T\}' = \left\{ \begin{pmatrix} \sum_{k=0}^{\infty} \lambda_k W^k & B \\ 0 & \lambda_0 I \end{pmatrix} : \sum_{k=0}^{\infty} \lambda_k W^k \in \mathcal{W}(W), \ \operatorname{ran} B \subset \ker W \right\}.$$

We assert that $\{T\}'$ has cyclic multiplicity n. Recall (cf. [W1]) that W is a cyclic operator. Let x be a cyclic vector for W. If $\{e_k\}_{k=1}^{n}$ is a basis for $\mathcal{N}$, then one can check that $\{x \oplus e_1\} \cup \{0 \oplus e_k\}_{k=2}^{n}$ is a cyclic set for $\{T\}'$. On the other hand, if $k < n$ and $\{x_j \oplus y_j\}_{j=1}^{k}$ are vectors in $\mathcal{H} \oplus \mathcal{N}$, then the projection of $\bigvee_{j=1}^{k} \{T\}'(x_j \oplus y_j)$ onto the second summand space $\mathcal{N}$ is at most k dimensional. So $\{x_j \oplus y_j\}_{j=1}^{k}$ can not be a cyclic set for $\{T\}'$. □

We note that $\mathcal{W}(T)$ *does* have a separating vector in this example, although $\{T\}'$, which is a somewhat larger algebra, fails to have a separating vector if $n \geq 2$.

A large class of triangular operators which are cyclic has been noted in [H5, Lemma 5.2] and [H1]. For completeness, we provide a proof in Lemma 3.5 and Proposition 3.6.

Lemma 3.5. *Suppose $S, T, X \in \mathcal{B}(\mathcal{H})$ and $SX = XT$. Then if T is cyclic and if X has dense range then S is also cyclic.*

Proof: Let u be a cyclic vector for T. Then $\mathcal{P}(T)u$ is dense in $\mathcal{H}$, where $\mathcal{P}(T)$ denotes the set of all polynomials in T. So $X\mathcal{P}(T)u$ is also dense in $\mathcal{H}$ since X has dense range. Since $SX = XT, p(S)X = Xp(T)$ for all polynomials p. Hence $v := Xu$ is cyclic for S. □

Proposition 3.6. *Let T be a triangular operator whose diagonal entries with respect to some orthonormal basis for $\mathcal{H}$ are distinct. Then T is cyclic. In particular, $\mathcal{W}(T)$ has a separating vector.*

Proof: By adding a scalar to T if necessary, we may assume that the diagonal entries are all nonzero. Let $\{e_n\}_1^\infty$ be the orthonormal basis in the hypothesis. For each n let x_n be an eigenvector for T in $[e_1, \dots e_n]$ corresponding to the eigenvalue t_{nn}, where $T = (t_{ij})$. Since the numbers $t_{nn}, 1 \leq n < \infty$, are distinct it follows that $x_n \notin [1, \dots e_{n-1}]$. So $[x_1, \dots x_n] = [e_1, \dots e_n]$ for all n. Let

$$X = [x_1, x_2, \dots]$$

be the formal matrix whose column vectors are $x_1, x_2, \dots$. By multiplying the vectors x_i by scalars decreasing to 0 sufficiently fast if necessary, we may assume that X determines a bounded operator in $\mathcal{B}(\mathcal{H})$. Then $TX = XD$, where

$$D = \operatorname{diag}(t_{11}, t_{22}, \dots)$$

is the diagonal matrix with respect to $\{e_n\}_1^\infty$ with diagonal entries $t_{11}, t_{22}, \dots$. Since the t_{ii} are distinct, D is cyclic. Since $\operatorname{span}\{x_i\}_1^\infty$ is dense, X has dense range. Thus T is cyclic by Lemma 3.5. □

Next we list a few questions which were settled in the negative for semitriangular operators in [W2] but are still open for the class of triangular operators. We refer the reader to [LW2] for related questions. Let $T \in \Delta$.

Question 1. Must $\mathcal{W}(T)$ have a separating vector? The answer is yes (Corollary 3.3) if T is bitriangular. Also, the answer is yes if T has distinct diagonal entries (Proposition 3.6).

Question 2. Is there some $n \geq 1$ so that $T^{(n)}$, the direct sum of n copies of T, is reflexive? That is, must $\mathcal{W}(T^{(n)}) =$ Alg Lat $T^{(n)}$ for some $n \geq 1$? In particular, is $T^{(2)}$ reflexive?

Question 3. Must $\mathcal{W}(T) = \mathcal{A}(T)$? That is, must $\mathcal{W}(T)$ coincide with the weak $*$ closure of the polynomials in T? Actually, this question is still open for semi-triangular operators, since the counter-example for this problem in [W2] is not semitriangular.

Question 4. If T is reflexive and p is a polynomial, must $p(T)$ be reflexive? (We note that the more general question of whether each $S \in \mathcal{W}(T)$ is reflexive was settled in the negative in [LW1, Cor. 3.9].) In particular, must T^2 be reflexive?

Next, we discuss two stability properties of the index of semitriangularity.

Proposition 3.7. *If $T \in (S\Delta)$ and if S is quasisimilar to T, then $S \in (S\Delta)$ and $i_{S\Delta}(S) = i_{S\Delta}(T)$.*

Proof: Let $n = i_{S\Delta}(T)$. Suppose that X is a quasiaffinity and that $SX = XT$. Then $p(S)X = Xp(T)$ for each polynomial p. Hence if $x \in \mathcal{E}_T$ then $Xx \in \mathcal{E}_S$. So $X\mathcal{E}_T \subseteq \mathcal{E}_S$. Hence $[X\mathcal{E}_T] \subseteq [\mathcal{E}_S]$. Since $[\mathcal{E}_T]$ has codimension n in $\mathcal{H}$ and X is a quasiaffinity, $[X\mathcal{E}_T]$ has codimension n in $\mathcal{H}$. Thus $S \in (S\Delta)$ and $i_{S\Delta}(S) \leq n$. Reversing the above argument shows that $i_{S\Delta}(S) = n$. □

Lemma 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$, and let $p(t)$ be any polynomial which is not a constant function. Then $\mathcal{E}_T = \mathcal{E}_{p(T)}$.*

Proof: The inclusion "$\supseteq$" is clear. To prove "$\subseteq$", let x be an algebraic vector for T and let $q(t)$ be a nontrivial polynomial for which $q(T)x = 0$. Write $q(t) = \prod_i (t - \lambda_i)$ and let

$$r(t) = \prod_i (t - p(\lambda_i)).$$

Then q divides $r \circ p$ since $(t - \lambda_i)$ divides $(p(t) - p(\lambda_i))$ for each i. It follows that $r(p(T))x = 0$. Thus $x \in \mathcal{E}_{p(t)}$ as required. □

Proposition 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose $p(t)$ is a nonconstant polynomial. Then $T \in (S\Delta)$ if and only if $p(T) \in (S\Delta)$. In this case $i_{S\Delta}(p(T)) = i_{S\Delta}(T)$.*

Proof: This is clear from Lemma 3.8.

Let us say that a class $(\mathcal{C})$ of operators has the *polynomial property* if it satisfies the stability property of Proposition 3.9. So $(\mathcal{C})$ must be closed under nonconstant polynomial functions of its operators, and must also have the property that whenever $p(T) \in (\mathcal{C})$ for some nonconstant polynomial p then also $T \in (\mathcal{C})$. The set of *algebraic* operators has this property, but neither the set of compacts nor the set of finite rank operators has it. By Proposition 3.9 Δ and $(S\Delta)$ have the polynomial property, and by considering adjoints so do $(B\Delta)$ and $(BS\Delta)$. In fact, for each nonnegative integer n the set of semitriangular operators of index precisely n has this property, and for each pair (n, m) the set of bisemitriangular operators T with $i_{S\Delta}(T) = n$ and $i_{S\Delta}(T^*) = m$ has this property.

Next we relate the above considerations to certain nests and nest algebras. If $\{e_n\}_1^\infty$ is a basis for an infinite dimensional separable Hilbert space $\mathcal{H}$ let $\mathcal{N}_\infty$ denote the "triangular" nest $\{0, N_1, N_2, \dots\} \cup \{\mathcal{H}\}$, where $N_k = [e_1, \dots e_k]$. For $1 \leq m < \infty$ let $\mathcal{N}_m$ denote the corresponding nest on m dimensional Hilbert space. It is clear that T is semitriangular if and only if $T \in \text{Alg}\,\mathcal{N}$ for some nest $\mathcal{N}$ which is unitarily equivalent to the ordinal sum $\mathcal{N}_\infty \oplus_{\text{ord}} \mathcal{N}_m$ for some $m < \infty$. If $\mathcal{C}$ is a set of operators let us write

$$\mathcal{U}(\mathcal{C}) := \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is unitarily equivalent to an element of } \mathcal{C}\}.$$

The above discussion shows that $\mathcal{U}(\text{Alg}\,\mathcal{N})$ has the polynomial property if $\mathcal{N} = \mathcal{N}_\infty$ or if $\mathcal{N} = \mathcal{N}_\infty \oplus_{\text{ord}} \mathcal{N}_m$ for some $m < \infty$. This suggests a new type of problem:

Question 5. For which nests $\mathcal{N}$ does $\mathcal{U}(\text{Alg}\mathcal{N})$ have the polynomial property?

If $\mathcal{N}$ is the trivial nest $\{0, 1\}$ then $\text{Alg}\,\mathcal{N} = \mathcal{B}(\mathcal{N})$ and so $\text{Alg}\,\mathcal{N}$ has the property for trivial reasons. So it may be best to restrict attention to multiplicity free nests. Also, it is useful to note that if a set has the polynomial property and contains 0 then it must contain *all* algebraic operators. We note that if $\mathcal{N}$ is a *continuous* nest then $\text{Alg}\,\mathcal{N}$ fails to contain any nonzero finite rank projection. The reason is that any projection in $\text{Alg}\,\mathcal{N}$ would commute with $\mathcal{N}$, and since $\mathcal{N}$ generates a nonatomic von Neumann algebra its commutant contains no finite rank operators. Thus, if $\mathcal{N}$ is continuous, $\mathcal{U}(\text{Alg}\mathcal{N})$ does *not* have the polynomial property. Another instance of failure is the nest $\mathcal{N} = \mathcal{N}_\infty \oplus_{\text{ord}} \mathcal{N}_\infty$. Reason: There is a unicellular backward weighted shift T of multiplicity one for which $\mathcal{N}_\infty = \text{Lat}\,T$, and for this operator we have $\text{Lat}(T^2) \supseteq \mathcal{N}_\infty \oplus_{\text{ord}} \mathcal{N}_\infty$. A few additional instances of failure are known to us, but the general problem seems to be difficult.