

VON NEUMANN ALGEBRAS AND WAVELETS

D.R. LARSON
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368

Abstract. Orthonormal wavelets can be regarded as complete wandering vectors for a system of bilateral shifts acting on a separable infinite dimensional Hilbert space. The local (or "point") commutant of a system at a vector ψ is the set of all bounded linear operators which commute with each element of the system locally at ψ . In the theory we shall develop, we will show that in the standard one-dimensional dyadic orthonormal wavelet theory the local commutant at certain (perhaps all) wavelets ψ contains non-commutative von Neumann algebras. The unitary group of such a locally-commuting von Neumann algebra parameterizes in a natural way a connected family of orthonormal wavelets. We will outline, as the simplest nontrivial special case, how Meyer's classical class of dyadic orthonormal wavelets with compactly supported Fourier transform can be derived in this way beginning with two wavelets (an interpolation pair) of a much more elementary nature. From this pair one computes an interpolation von Neumann algebra. Wavelets in Meyer's class then correspond to elements of its unitary group. Extensions of these results and ideas are also discussed.

1. Introduction

A dyadic orthonormal (also called orthogonal) wavelet in one dimension is simply a unit vector ψ in the complex Hilbert space $L^2(\mathbb{R}, \mu)$, with μ Lebesgue measure, with the property that the set

$$\{2^{\frac{n}{2}}\psi(2^n t - \ell): n, \ell \in \mathbf{Z}\} \quad (1)$$

is an orthonormal basis for $L^2(\mathbb{R})$. Most of the work we will discuss will be for this type of wavelet. Some results we (and others) have obtained for

other types of wavelets will be discussed in context. Many results for dyadic orthonormal wavelets generalize, some considerably.

The above definition (1) is the one given in, e.g., Chui's book ([5], p. 4), and Hernandez and Weiss' recent book [31], and it is referred to in Meyer's book ([41], p. 28) as the Franklin-Stromberg definition. The simplest function satisfying this is the Haar wavelet

$$\psi_H = \chi_{[0, \frac{1}{2})} - \chi_{[\frac{1}{2}, 1)}. \quad (2)$$

Wavelets having this property (see, e.g., [14]) include those of Stromberg, Meyer, Battle-Lemarié and Daubechies. These wavelets also satisfy strong auxiliary regularity (differential and moment) properties and time-frequency localization properties which make them useful in applications. Dilation factors other than 2 (the dyadic case) have also been studied. Generalizations to \mathbf{R}^n with matrix dilations (see below), and $L^p(\mathbf{R}, \mu)$ for other $1 \leq p < \infty$, are also frequently studied. Representative articles are contained in the collections [2,6,15,44].

The term "mother wavelet" is also used in the literature for a function ψ satisfying the above definition of orthogonal wavelet. In this case the functions

$$\psi_{n,\ell} := 2^{\frac{n}{2}} \psi(2^n t - \ell) \quad (3)$$

are called elements of the wavelet basis generated by the "mother". The functions $\psi_{n,\ell}$ will not themselves be mother wavelets unless $n = 0$.

The mathematics we will describe in this article belongs to a program that has been under development over the past four years. Much of it is joint work with Xingde Dai which is due to appear in an AMS Memoir [9]. It began in June, 1992, when we made two simple discoveries that caused us to think that, as operator algebraists, we *should* be interested in wavelet theory. Just before this, the point of view had been taken in two preprints by the approximation theorists Goodman, Lee and Tang [19, 20] that orthonormal dyadic wavelets in $L^2(\mathbf{R})$ and multiwavelets in $L^2(\mathbf{R}^n)$ were simply unit vectors or tuples of unit vectors in $L^2(\mathbf{R}^n)$ that could be "pushed around" by an appropriate set of unitary operators, like (11) below, to yield orthonormal bases.

Going beyond this, our *first* discovery was that the set of *all* such wavelets could be parameterized in a natural way by a fixed wavelet together with the set of all unitary operators in an associated linear subspace (12) of $\mathcal{B}(\mathcal{H})$ which we first, informally, called the "point" commutant in a number of talks we gave, and later we called "local" commutant.

Our *second* discovery was that the joint *commutant* of the dilation and translation unitary operators was nontrivial and could be characterized simply in terms of the Fourier-Plancherel transform. Using the first discovery, this gave an *algorithm* (22) for obtaining "new wavelets from old" by

changing their phase. To our surprise our colleague and resident wavelet expert at Texas A&M, Charles Chui, told us that he thought that this algorithm was new. He qualified this response by telling us that, while it was interesting mathematically, it probably would not be significant to applications-oriented wavelet theory. Because of this, and also because of the sheer simplicity of our proofs, we made a decision not to publish until we had reached a “certain level of depth” and/or made some positive impact on “genuine” applications-oriented wavelet theory. However, it encouraged us that we could obtain new results concerning “concrete” wavelets using simple functional-analytic techniques that either had escaped notice (over 2000 wavelet papers had appeared by then!), or had not been easily obtained, by wavelet theorists using “traditional” approaches.

A *third* discovery we made that same summer of 1992 without which we could not have proceeded further, was the realization that the Littlewood-Paley (33) wavelet (also called Shannon’s wavelet) could be easily “perturbed” to yield a large class of wavelets that have sufficiently elementary structure to permit “hand-computation” experimentation and testing of our ideas. These are described in chapter 4, where the examples we abstracted from [9] were obtained over approximately a two-year period. Here is where there has been some interaction with the group of Guido Weiss and his students and colleagues. Basically, over roughly the same period of time, we both discovered what I shall informally call here the same class of computationally elementary wavelets. We obtained some of the same characterizations of them, and we posed some of the same problems to our students, although many of our results, techniques and terminology, were still quite different. We only discovered this connection in late spring (May-June) 1995 when our colleagues Charles Chui, Bill Johnson, and John McCarthy saw talks each of us gave separately, and alerted both of us.

The initial computationally elementary examples led, in turn, to our methods of *operator-theoretic interpolation*, which are described in chapter 5. This took nearly two years to work out, with the derivation of Meyer’s family taking the first year, and it is still under further development. *Finally* (actually “piecemeal” along the way) we realized that many of the results that we, and others, have obtained for orthonormal wavelet theory have natural generalizations to what we called *abstract unitary systems*, and some proofs are actually more transparent in the abstract setting than in the “concrete” wavelet setting. In our writing of [9] (and herein) this was presented first, not last. Some of this is done in chapter 2. Finding our stopping-point took two years, resulting in our memoir.

The paper [9] was the beginning of a program of investigation of wavelet theory from a functional analysis point of view. In [9] we posed six basic open questions, labeled *A* to *F*. In the past year, problems *B*, *C*, *D*, the

“finite” case of E , and F have been *solved*, and we will discuss these solutions in context. This reveals something about how rapidly the theory is evolving. Problem A was the *connectedness* problem (described after Example 2.4). It is still open, although much progress has been made. This problem was *also* raised *independently* in [29, 30], where it belongs just as much to Guido Weiss’ beautifully penetrating and fruitful program of building a unified approach to wavelet theory via the Fourier transform.

I want to take the opportunity to thank the organizers of the NATO Advanced Study Institute on Operator Algebras and Applications, August 19-28, 1996, of which this article is to be part of the proceedings, especially Aristides Katavolos, for running a splendid conference in an ideal environment on the beautiful island of Samos, Greece, and for inviting me to be a participant.

A considerable amount of the material presented in my talks and included herein was previously presented, in various preliminary stages, in 22 talks over the past four years, beginning with an AMS Special Session in October, 1992, and including hour talks at GPOTS-1994 and SEAM-1995. Current wavelet work is also discussed, especially relating to some interaction between the work of my group of colleagues and students, and the group of Guido Weiss based at Washington University.

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2. Wavelets and Unitary Systems

If S is a set of operators we will let $U(S)$ denote the set of unitary operators in S .

Let T and D be the translation (by 1) and dilation (by 2) operators in $B(L^2(\mathbf{R}))$ given by

$$\begin{aligned} (Tf)(t) &= f(t-1) \quad \text{and} \\ (Df)(t) &= \sqrt{2} f(2t) \end{aligned} \tag{4}$$

for $f \in L^2(\mathbf{R})$. Then T and D are unitary operators. They are in fact bilateral shifts of infinite multiplicity, with complete wandering subspaces

$$L^2([0,1]) \quad \text{and} \quad L^2([-2,-1] \cup [1,2]),$$

respectively, considered as subspaces of $L^2(\mathbf{R})$. We have

$$TD = DT^2. \tag{5}$$

Indeed, we have

$$\begin{aligned} ((TD)f)(t) &= T(\sqrt{2}f(2t)) = \sqrt{2}f(2(t-1)) = D(f(t-2)) \\ &= (DT^2f)(t), f \in L^2(\mathbf{R}) \end{aligned}$$

It is clear that

$$\{2^{\frac{n}{2}}\psi(2^n t - \ell): n, \ell \in \mathbf{Z}\} = \{D^n T^\ell \psi: n, \ell \in \mathbf{Z}\}.$$

We define a *unitary system* to be simply a set of unitary operators \mathcal{U} acting on a Hilbert space \mathcal{H} which contains the identity operator. In analogy with the notion of a wandering vector for the bilateral or unilateral shift, we say that a vector $\psi \in \mathcal{H}$ is *wandering* for \mathcal{U} if the set

$$\mathcal{U}\psi := \{U\psi: U \in \mathcal{U}\} \quad (6)$$

is an orthonormal set, and we call ψ *complete* if $\mathcal{U}\psi$ spans \mathcal{H} . More generally, a linear subspace $E \subseteq \mathcal{H}$ is called *wandering* if the subspaces $\{UE: U \in \mathcal{U}\}$ are pairwise orthogonal, and *complete* if $\mathcal{H} = \bigoplus_{U \in \mathcal{U}} UE$. Following our notation in [9] we write $\mathcal{W}(\mathcal{U})$ for the set of complete wandering vectors for \mathcal{U} .

Let $\mathcal{U}_{D,T}$ be the unitary system defined by

$$\mathcal{U}_{D,T} := \{D^n T^\ell: n, \ell \in \mathbf{Z}\}. \quad (7)$$

Then ψ is a dyadic orthonormal wavelet if and only if ψ is a complete wandering vector for the unitary system $\mathcal{U}_{D,T}$. We will use the abbreviation

$$\mathcal{W}(D, T) := \mathcal{W}(\mathcal{U}_{D,T}) \quad (8)$$

to denote the set of all dyadic orthonormal wavelets.

More generally, let $1 \leq n < \infty$, and let A be an $n \times n$ real matrix which is *expansive* as a transformation from $\mathbf{R}^n \rightarrow \mathbf{R}^n$. This means that $\|A^m x - x\| \rightarrow \infty, x \neq 0$. This is equivalent to the condition that all eigenvalues have modulus > 1 (cf. [47]). Another equivalence is that A is similar in $M_n(\mathbf{C})$ to a strict dilation – the inverse of a strict contraction. By a dilation- A orthonormal wavelet we mean a function $\psi \in L^2(\mathbf{R}^n)$ such that

$$\{|\det A|^{\frac{m}{2}}\psi(A^m t - (\ell_1, \ell_2, \dots, \ell_n)^t): m, \ell \in \mathbf{Z}\} \quad (9)$$

where $t = (t_1, \dots, t_n)^t$, is an orthonormal basis for $L^2(\mathbf{R}^n)$. (Here “ t ” means transpose.) We introduce dilation and translation unitary operators, as in the 1-dimensional dyadic case. Define D_A by

$$(D_A f)(t) = |\det A|^{\frac{1}{2}} f(At), \quad (10)$$

$f \in L^2(\mathbb{R}^n)$. For $1 \leq i < n$ let T_i be the unitary operator determined by translation by 1 in the i^{th} coordinate direction. The set (9) is then

$$\{D_A^k T_1^{\ell_1} \dots T_n^{\ell_n} \psi: k, \ell_i \in \mathbf{Z}\}.$$

Write

$$\mathcal{U}_{D_A, T_1, \dots, T_n} = \{D_A^k T_1^{\ell_1} \dots T_n^{\ell_n}: k, \ell_i \in \mathbf{Z}\}. \quad (11)$$

Then, as in the 1-dimensional dyadic case, dilation- A orthonormal wavelets are precisely the complete wandering vectors for $\mathcal{U}_{D_A, T_1, \dots, T_n}$.

The term “orthogonal wavelet” has been extended in the literature to include a “multi” notion, which is an orthonormal p -tuple (f_1, \dots, f_p) of functions in $L^2(\mathbb{R}^n)$, a *multiwavelet*, each of which separately generates an incomplete orthonormal set under the system of unitaries, and which together form an orthonormal basis.

Until very recently, the *only* orthonormal wavelets known in this multivariate setting were such *multi-wavelets* for very special expansive matrices associated with n -dimensional multiresolution analyses. In [10] we showed the existence of *single function wavelets*, associated with n -dimensional *wavelet sets*, for any arbitrary $n \times n$ real expansive matrix.

According to certain reports we have been given, wavelet folklore had indicated that single-function wavelets could not exist for dimension greater than 1. So our results in [10] have evidently been a bit of surprise to some wavelet theorists. Since we announced our results in talks, and at the end of [9], and in Speegle’s work [46], a number of concrete examples of mother wavelets in the plane have been worked out by several mathematicians including Guido Weiss’ students, Xingde Dai and his student, and my students at A&M. In particular, we refer to the interesting paper [45]. The proof in [10] was constructive, but a direct application of the constructive techniques in that paper does not directly yield examples of elegance. In [10] we showed that *many* wavelet sets exist: for every expansive dilation matrix factor in \mathbb{R}^n there are sufficiently many Borel wavelet sets to generate the Borel structure of \mathbb{R}^n .

In a recent article [32] with Eugen Ionascu and Carl Pearcy, we showed that the unitary systems $\mathcal{U}_{D_A, T_1, \dots, T_n}$ in (11), for arbitrary expansive matrices A , were unitarily *inequivalent*. This proved surprisingly difficult, even for the case $n = 1$. (For the case $n = 1$ there is a very nice class of wavelets for other dilation factors than 2. See example 4.5 (x) in this article.) Thus the corresponding wavelet theories for two different (expansive) dilation factors are not unitary equivalent, even if one dilation matrix is simply a scalar multiple of the other. We are interpreting this to mean that general wavelet theory is mathematically “rich”.

Even if an $n \times n$ real invertible matrix is not expansive the corresponding dilation operator D_A given by (10) is unitary, and one can ask whether

wavelets (either single function or multi) might exist for the corresponding unitary system (11). An obvious necessary condition is that D_A must be a bilateral shift of infinite multiplicity. In [32] we proved that this condition is satisfied if and only if the $n \times n$ real invertible matrix A is *not* similar to a $n \times n$ complex *unitary* matrix via a complex similarity transformation. So in particular, the matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

both induce bilateral shifts. We left open the question in [32] of whether these (or indeed, any other such “nonstandard” unitary systems) had complete wandering vectors (which could be called “nonstandard wavelets”). If so, there may be applications.

Problem 2.A. If \mathcal{U} is a unitary system of the form (11), and if $\mathcal{W}(\mathcal{U})$ is nonempty, must either A or A^{-1} be expansive?

In this article we will discuss higher dimension wavelets and dilation factors in one dimension only in passing. For simplicity we will focus primarily on the one-dimensional dyadic theory.

It is not the purpose of this article to discuss applications. But a few words to this effect are in order. In a setting where signals of some type are represented by vectors in a Hilbert space \mathcal{H} , a prescribed orthonormal basis $\{e_n\}$ for \mathcal{H} can be used as a signal-processing “black box” in (at least) two natural ways: Firstly “noise” can be filtered out by “prescribing” a fixed N and discarding all Fourier coefficients $\langle x, e_n \rangle$ for $n > N$; then

$$\tilde{x} := \sum_{j=1}^N \langle x, e_j \rangle e_j$$

will be the “processed” signal, and, secondly, a signal can be “compressed” by replacing x with the finite sequence of numbers $\{\langle x, e_j \rangle\}_{j=1}^N$; then

$$\sum_{j=N+1}^{\infty} \langle x, e_j \rangle e_j$$

is the error term, and the compression is “effective” in a case if the error is negligible. It turns out that bases given by orthonormal (and Riesz) wavelets (and frames (cf. [14, 31])) are particularly suitable to use in this fashion for certain types of signal processing problems.

We introduce operator theory into the wavelet framework in an elementary way. Fix a (one-dimensional dyadic) wavelet ψ and consider the set of

all bounded linear operators S which commute with the “action” of dilation and translation on ψ . That is, we require

$$(S\psi)(2^n t - \ell) = S(\psi(2^n t - \ell)),$$

or equivalently

$$D^n T^\ell S\psi = S D^n T^\ell \psi,$$

for all $n, \ell \in \mathbf{Z}$. A motivating example is that if η is any *other* wavelet, let S be the unitary determined by mapping the orthonormal basis $\psi_{n\ell} := D^n T^\ell \psi$ onto $\eta_{n\ell} := D^n T^\ell \eta$. That is, $S\psi_{n\ell} = \eta_{n\ell}$, $n, \ell \in \mathbf{Z}$. Then $\eta = S\psi$, so

$$S D^n T^\ell \psi = D^n T^\ell S\psi,$$

as required. This simple-minded idea is reversible, so for every unitary V in this “point commutant” of $\mathcal{U}_{D,T}$ at ψ , the vector $V\psi \in \mathcal{W}(D, T)$. Indeed

$$D^n T^\ell (V\psi) = V(D^n T^\ell \psi).$$

Moreover, this simple correspondence between unitaries in this point commutant and wavelets is 1-1. This turns out to be useful because it leads to some new formulas relating to decomposition and factorization results for wavelets, making use of the linear and multiplicative properties of the point commutant.

We capture the above notion formally.

Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be a set of vectors, and let $x \in \mathcal{H}$ be a nonzero vector. Define the *local* (or “point”) commutant of \mathcal{S} at x by

$$\mathcal{C}_x(\mathcal{S}) := \{A \in \mathcal{B}(\mathcal{H}) : (AS - SA)x = 0, \quad s \in \mathcal{S}\}. \quad (12)$$

It is clear that this is a linear subspace of $\mathcal{B}(\mathcal{H})$ which is closed in the strong operator topology and the weak operator topology, and it contains the commutant \mathcal{S}' of \mathcal{S} . If x is cyclic for \mathcal{S} in the sense that $\text{span}(\mathcal{S}x)$ is dense, then

$$x \text{ separates } \mathcal{C}_x(\mathcal{S}).$$

Indeed, if $A \in \mathcal{C}_x(\mathcal{S})$ and if $Ax = 0$, then for any $S \in \mathcal{S}$ we have $ASx = SAx = 0$, so $ASx = 0$, and hence $A = 0$. If in addition \mathcal{S} is a multiplicative semigroup, then in fact $\mathcal{C}_x(\mathcal{S}) = \mathcal{S}'$, so in this case the local commutant is not a new structure. To see this, suppose $A \in \mathcal{C}_x(\mathcal{S})$. Then for each $S, T \in \mathcal{S}$ we have $ST \in \mathcal{S}$, and so

$$AS(Tx) = (ST)Ax = S(ATx) = (SA)Tx.$$

So since $T \in \mathcal{S}$ was arbitrary and $[\mathcal{S}x] = \mathcal{H}$, it follows that $AS = SA$.

In the wavelet case $\mathcal{U}_{D,T}$, if $\psi \in \mathcal{W}(D,T)$ then it turns out that $\mathcal{C}_\psi(\mathcal{U}_{D,T})$ is in fact *much larger* than $(\mathcal{U}_{D,T})' = \{D,T\}'$, underscoring the fact that $\mathcal{U}_{D,T}$ is *not* a group. In particular, $\{D,T\}'$ is abelian while $\mathcal{C}_\psi(\mathcal{U}_{D,T})$ is nonabelian for every wavelet ψ . (See the remark after Proposition 5.3.) To see that $\mathcal{U}_{D,T}$ is not a group, note that

$$TD^{-1} = D^{-1}T^{\frac{1}{2}} \notin \mathcal{U}_{D,T}. \quad (13)$$

In fact, it is not hard to show that the group generated by D and T is

$$\{D^n T_\beta : \beta \text{ a dyadic rational}\}, \quad (14)$$

where for $r \in \mathbb{R}$ T_r is the unitary operator corresponding to translation by r .

We note that *most* unitary systems \mathcal{U} do not have complete wandering vectors. For $\mathcal{W}(\mathcal{U})$ to be nonempty, the set \mathcal{U} must be very special. It must be countable if it acts separably, and it must be discrete in the strong operator topology (pointwise convergence) because if $U, V \in \mathcal{U}$ and if x is a wandering vector for \mathcal{U} then

$$\|U - V\| \geq \|Ux - Vx\| = \sqrt{2}. \quad (15)$$

Certain other properties are forced on \mathcal{U} by the presence of a wandering vector. One purpose of our work is to study such properties. Indeed, it was a matter of some surprise to us to discover that such a theory is viable even in some considerable generality. The more immediate purpose, however, is to study structural properties of $\mathcal{W}(\mathcal{U})$ for special systems \mathcal{U} which are relevant to *wavelet* theory.

In operator theory, wandering vectors have been studied for groups of unitaries and semigroups of isometries, particularly those that are singly generated. Wavelet theory entails the study of wandering vectors for unitary systems which are not even semigroups.

It is useful for perspective to note that the *reversed* set

$$\mathcal{U}_{T,D} = \{T^n D^\ell : n, \ell \in \mathbb{Z}\}$$

fails to have a wandering vector. To see this choose dyadic $\beta_k \rightarrow 0$, $\beta_k \neq 0$, as above. Write $\beta_{\ell_k} = p_k/2^{q_k}$. A straightforward computation (see [9, Lemma 3.2]) shows that

$$D^{-q_k} T_{\beta_k} = T^{p_k} D^{-q_k}.$$

If $\psi \in \mathcal{H}$ then,

$$\|T^{p_k} D^{-q_k} \psi - D^{-q_k} \psi\| = \|D^{-q_k} T_{\beta_k} \psi - D^{-q_k} \psi\| = \|T_{\beta_k} \psi - \psi\| \rightarrow 0.$$

So if $\psi \in \mathcal{W}(\mathcal{U}_{T,D})$, then orthonormality of $\mathcal{U}_{T,D}\psi$ implies $T^{pk} = I$ for all but finitely many k , contradicting the assumption that $\beta_k \neq 0$.

The following is a simple result which is more useful than one might expect. In fact, it is the key to our approach.

Proposition 2.1. *Let \mathcal{U} be a unitary system in $B(\mathcal{H})$. Suppose $\psi \in \mathcal{W}(\mathcal{U})$. Then*

$$\mathcal{W}(\mathcal{U}) = \{V\psi: V \in \mathcal{U}(\mathcal{C}_\psi(\mathcal{U}))\}.$$

Moreover, the correspondence

$$V \rightarrow V\psi, \quad \mathcal{U}(\mathcal{C}_\psi(\mathcal{U})) \rightarrow \mathcal{W}(\mathcal{U}),$$

is one-to-one.

Proof. Let $V \in \mathcal{U}(\mathcal{C}_\psi(\mathcal{U}))$. Let $\eta = V\psi$. For $U \in \mathcal{U}$ we have $U\eta = UV\psi = VU\psi$ since U commutes with V at ψ . Thus

$$U\eta = VU\psi,$$

and so $U\eta$ is an orthonormal basis for \mathcal{H} since V is unitary. So $\eta \in \mathcal{W}(\mathcal{U})$. Conversely, let $\eta \in \mathcal{W}(\mathcal{U})$ be arbitrary. Since $\mathcal{U}\psi$ and $\mathcal{U}\eta$ are orthonormal bases, there is a unique unitary operator V with $VU\psi = U\eta$, $U \in \mathcal{U}$. Then $V\psi = \eta$ since $I \in \mathcal{U}$. So

$$VU\psi = UV\psi$$

for all $U \in \mathcal{U}$. Thus $V \in \mathcal{C}_\psi(\mathcal{U})$. So since ψ separates points of $\mathcal{C}_\psi(\mathcal{U})$, the map $V \rightarrow V\psi$ is one-to-one. ■

Proposition 2.1 shows that if \mathcal{U} is a unitary system with $\mathcal{W}(\mathcal{U})$ nonempty, then given any $\psi \in \mathcal{W}(\mathcal{U})$ the entire set $\mathcal{W}(\mathcal{U})$ can be parameterized in a natural way by the set of unitary operators in the local commutant of \mathcal{U} at ψ .

A *Riesz basis* for \mathcal{H} is a bounded unconditional basis. Equivalently, (since \mathcal{H} is a Hilbert space) a Riesz basis is the image under a bounded invertible operator of an orthonormal basis. A (one-dimensional) Riesz dyadic orthonormal wavelet is a vector ψ with the property that $\{D^n T^\ell \psi: n, \ell \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$. In some applications, Riesz wavelets are more useful (and are more widely studied) than orthogonal wavelets. Proposition 2.1 easily generalizes to generators of Riesz bases:

Proposition 2.1'. *Let \mathcal{U} be a unitary system in $B(\mathcal{H})$, and suppose $\psi \in \mathcal{H}$ is a vector for which $\{U\psi: U \in \mathcal{U}\}$ is a Riesz basis for \mathcal{H} . If $\eta \in \mathcal{H}$, then $\{U\eta: U \in \mathcal{U}\}$ is a Riesz basis for \mathcal{H} if and only if $\eta = S\psi$ for some invertible operator S in $\mathcal{C}_\psi(\mathcal{U})$.*

We follow with a very elementary application of this “operator” approach to aspects of wavelet theory, which in fact appears to be new to wavelet theory. It says that, in particular, “most” convex combinations of orthonormal wavelets are Riesz wavelets.

Proposition 2.2. *Let ψ and η be orthonormal wavelets. Let*

$$\xi = \lambda\psi + (1 - \lambda)\eta$$

for some scalar $\lambda \in \mathbb{C}$ with $|\lambda| \neq |\lambda - 1|$. Then ξ is a Riesz wavelet.

Proof. Let $V = V_\psi^\eta$ be the (unique) unitary in $\mathcal{C}_\psi(\mathcal{U}_{D,T})$ such that $V\psi = \eta$. Then $\xi = S\psi$, where

$$S = \lambda I + (1 - \lambda)V.$$

We may assume $\lambda \notin \{0, 1\}$. Since V is unitary $\sigma(V) \subseteq \{z \in \mathbb{C}: |z| = 1\}$, so if $|\lambda| \neq |\lambda - 1|$ then

$$S = (1 - \lambda)[V - \lambda(\lambda - 1)^{-1}I]$$

is invertible, and hence ξ is a Riesz wavelet by Proposition 2.1'. ■

Next we give a second elementary application. It shows that in certain cases new wandering vectors can be obtained by "interpolating" between a given pair. It happens that in wavelet theory, pairs (ψ, η) as above with

$$(V_\psi^\eta)^2 = I$$

are not uncommon. Proposition 2.3 can be viewed as the "prototype" of our operator-theoretic interpolation results we will discuss later, in which after conjugating with the Fourier-Plancherel transform, the scalar α in (16) can be replaced by a real-valued function in a certain class. See, for instance, the "form" of (56).

Proposition 2.3. *Let \mathcal{U} be a unitary system, let $\psi, \eta \in \mathcal{W}(\mathcal{U})$, and let V be the unique unitary operator in $\mathcal{C}_\psi(\mathcal{U})$ with $V\psi = \eta$. Suppose $V^2 = I$. Then*

$$\cos \alpha \cdot \psi + i \sin \alpha \cdot \eta \tag{16}$$

is in $\mathcal{W}(\mathcal{U})$ for all $0 \leq \alpha \leq 2\pi$.

Proof. Let $P = \frac{1}{2}(V + I)$. Then P is a projection which is contained in $\mathcal{C}_\psi(\mathcal{U})$. Let $\omega = \cos \alpha + i \sin \alpha$, and let

$$W = \omega P + \bar{\omega}(I - P).$$

Then W is a unitary contained in $\mathcal{C}_\psi(\mathcal{U})$ so $W\psi \in \mathcal{W}(\mathcal{U})$. We have $W\psi = \omega P\psi + \bar{\omega}(I - P)\psi$, and $P = \frac{1}{2}(V + I)$, so $P\psi = \frac{1}{2}(\eta + \psi)$ and $(I - P)\psi = \frac{1}{2}(\psi - \eta)$. Thus

$$\begin{aligned} W\psi &= \frac{1}{2}\omega(\psi + \eta) + \frac{1}{2}\bar{\omega}(\psi - \eta) = \\ &= \cos \alpha \cdot \psi + i \sin \alpha \cdot \eta. \quad \blacksquare \end{aligned}$$

Example 2.4. Let $\{e_n\}_{-\infty}^{\infty}$ be an orthonormal basis for a separable Hilbert space \mathcal{H} , and let $Se_n = e_{n+1}$ be the bilateral shift of multiplicity one. Let $\mathcal{U} = \{S^n: n \in \mathbf{Z}\}$ be the group generated by S . This is the simplest unitary system with a complete wandering vector. Each e_n is in $\mathcal{W}(\mathcal{U})$. Since \mathcal{U} is a group $\mathcal{C}_\psi(\mathcal{U}) = \mathcal{U}'$. So by Proposition 2.1

$$\mathcal{W}(\mathcal{U}) = \{Ve_0: V \text{ is a unitary in } \{S\}'\}.$$

Here $\{S\}'$ coincides with the set of Laurent operators. Let \mathbf{T} be the unit circle. If we represent S on $L^2(\mathbf{T})$ in the usual way by identifying it with the multiplication operator M_z , then $\mathbf{U}\{S\}'$ is identified with (multiplication by) the set of unimodular functions on \mathbf{T} , and e_0 is identified with the constant function 1. Then Proposition 2.1 just recovers the well-known fact that the set of complete wandering vectors for the shift coincides (under this representation) with the set of unimodular functions on \mathbf{T} . In this case $\mathcal{W}(\mathcal{U})$ is clearly a closed, connected subset of the unit ball of \mathcal{H} in the norm topology with dense linear span.

In the program we outlined in [9] we mentioned that the most *basic* problems from a functional analysis point of view that could be raised for wandering vectors, or wavelets, were the *topological* ones: Is $\mathcal{W}(\mathcal{U})$ a closed subset of the unit ball of \mathcal{H} ? Is it arcwise connected? Does $\mathcal{W}(\mathcal{U})$ have dense linear span? The last question is nontrivial because, while $sp(\mathcal{U}\psi)$ is certainly dense for $\psi \in \mathcal{W}(\mathcal{U})$, in most cases unless, say, \mathcal{U} is a group, the individual basis vectors $U\psi$ will *not* be in $\mathcal{W}(\mathcal{U})$.

For the wavelet case, we first showed in [9] (see Example 4.5 (ii) in the present article) that $\mathcal{W}(D, T)$ is not closed. Fang and Wang [17] also showed this independently using, in fact, the same simple example derived independently. We also proved that $\text{span } \mathcal{W}(D, T)$ is dense in $L^2(\mathbf{R})$ [9, Corollary 3.17]. Partial results on the connectedness problem were obtained in [9]. The connectedness problem was also raised independently in [17, 29, 30] and partial results, some similar to ours and some quite different, were obtained. Recently, it has been shown [11, 28] that the set of all MRA-wavelets is connected.

In [17, 29, 30] Fang, Wang, Hernandez and Weiss proved that the set of all MRA-wavelets (see (23)) are relatively closed in $\mathcal{W}(D, T)$, and techniques of "smoothing" were developed. The connectedness problem for the entire set $\mathcal{W}(D, T)$ is still open. However, Speegle [46] has shown that the set of all s -elementary (equiv. MSF) wavelets (see chapter 4) are connected. This set contains non-MRA wavelets (see Example 4.5 (i)).

Example 2.5. Let \mathcal{G} be a countable group, let $\mathcal{H} = \ell^2(\mathcal{G})$, and let π_L be the left regular representation of \mathcal{G} on \mathcal{H} . That is, for $h \in \mathcal{G}$ and

$\{\lambda_g\}_{g \in \mathcal{G}} \in \ell^2(\mathcal{G})$, define $\pi_L(h)\{\lambda_g\} = \{\lambda_{h^{-1}g}\}$; so writing $\lambda(g) \equiv \{\lambda_g\}_{g \in \mathcal{G}}$, we have $(\pi_L(h_1)\pi_L(h_2)\lambda)(g) = (\pi_L(h_1)\lambda(h_2^{-1}g))(g) = (\lambda(h_2^{-1}g))(h_1^{-1}g) = \lambda(h_2^{-1}h_1^{-1}g) = \lambda((h_1h_2)^{-1}g) = (\pi_L(h_1h_2)\lambda)(g)$. The standard basis for \mathcal{H} is $\{e_g: g \in \mathcal{G}\}$, where $e_g = \chi_{\{g\}} \equiv \{\delta_{g,k}\}_{k \in \mathcal{G}}$. Then

$$\pi_L(h)e_g = \{\delta_{g,h^{-1}k}\}_{k \in \mathcal{G}} = \{\delta_{hg,k}\}_{k \in \mathcal{G}} = e_{hg}.$$

The vectors e_g are clearly in $\mathcal{W}(\pi_L(\mathcal{G}))$. Since $\pi_L(\mathcal{G})$ is a group the local commutant of $\pi_L(\mathcal{G})$ at e_I is just the commutant, (where I denotes the identity element of \mathcal{G}). Since $\{\pi_L(\mathcal{G})\}'$ is a von Neumann algebra, its group of unitaries is connected in the norm topology. Since the map $V \rightarrow Ve_I$ is continuous, Proposition 2.1 implies that $\mathcal{W}(\pi_L(\mathcal{G}))$ is a connected subset of the unit ball of \mathcal{H} . Since $e_g \in \mathcal{W}(\pi_L(\mathcal{G}))$, $g \in \mathcal{G}$, the set $\mathcal{W}(\pi_L(\mathcal{G}))$ has dense span.

The above example of a complete wandering vector for a group is generic. The following "result" essentially proves *itself*.

Proposition 2.6. *If π is a representation of a countable group G on a Hilbert space \mathcal{H} , and if $\pi(G)$ has a complete wandering vector ψ , then π is unitarily equivalent to the left regular representation of G on $\ell^2(G)$ under a unitary transformation which takes ψ to e_I .*

Again, let \mathcal{G} be a countable group, and let $\mathcal{G}_0 \subseteq \mathcal{G}$ be a subset containing the identity element I . Sometimes it is possible to obtain a faithful unitary representation π of \mathcal{G} on $\ell^2(\mathcal{G}_0)$ satisfying the requirement that if $h \in \mathcal{G}$ and $g \in \mathcal{G}_0$ are such that $h^{-1}g \in \mathcal{G}_0$, then $\pi(h)e_g = e_{hg}$. Then $e_I \in \mathcal{W}(\pi(\mathcal{G}_0))$ trivially.

This contrived "example" with respect to a distinguished subset of a group is generic. If \mathcal{U} is any unitary system on a Hilbert space \mathcal{H} with a complete wandering vector ψ , let $\tilde{\mathcal{U}}$ be the group generated by \mathcal{U} in $B(\mathcal{H})$. Let $\mathcal{G} = \tilde{\mathcal{U}}$ as an abstract group, and let $\mathcal{G}_0 = \mathcal{U}$, a subset of \mathcal{G} . Let $\mathcal{K} = \ell^2(\mathcal{G}_0)$. Define a unitary operator

$$W: \mathcal{H} \rightarrow \mathcal{K} \quad \text{by} \quad Wg\psi = e_g, g \in \mathcal{U},$$

making use of the fact that $\mathcal{U}\psi$ is an orthonormal basis for \mathcal{H} . Define

$$\pi: B(\mathcal{H}) \rightarrow B(\mathcal{K}) \quad \text{by} \quad \pi(A) = WAW^*,$$

and restrict to $\mathcal{G} \equiv \tilde{\mathcal{U}}$. Then π satisfies the property of the above paragraph, and is unitarily equivalent to the identity representation of $\mathcal{G} \equiv \tilde{\mathcal{U}}$ on \mathcal{H} .

Let us call a unitary representation π of a group \mathcal{G} relative to a unital subset \mathcal{G}_0 a *wandering vector representation* of the pair $(\mathcal{G}, \mathcal{G}_0)$ if it is faithful on \mathcal{G} and if $\pi(\mathcal{G}_0)$ has a complete wandering vector.

Problem 2.B. Given a group \mathcal{G} , what are the unital subsets \mathcal{G}_0 which are *allowable* in the sense that $(\mathcal{G}, \mathcal{G}_0)$ has a wandering vector representation?

In particular, if \mathcal{G} is generated as a group by an ordered pair of elements $\{g_1, g_2\}$, and if $\mathcal{G}_i = \text{Group}\{g_i\}$, is the set

$$\{\mathcal{G}_1\mathcal{G}_2 = h_1h_2: h_i \in \mathcal{G}_i\}$$

an allowable subset of \mathcal{G} ? Work here may aid in understanding wavelet systems, in particular. The question is obviously nontrivial in view of wavelet theory. This question generalizes to ordered n -tuples of generators. As earlier, we have the three “basic” problems: when does $\mathcal{W}(\pi(\mathcal{G}_0))$ have dense span, when is it closed, and when is it connected?

For the interested reader, other examples of unitary systems with complete wandering vectors are given in Chapter 1 of [9]. There is a way of taking “twisted tensor products” of known examples to get new nontrivial (i.e. where the two factors do not commute) examples, showing that a wealth of such examples exist, even in finite dimensions.

It is likely impossible to obtain a complete classification of all unitary systems which have complete wandering vectors. However, interesting “clean” structural results are possible for certain systems, including those most pertinent to wavelet theory. We next present some of these, for perspective and for independent interest. Moreover, it is a “two way street” in the sense that some known wavelet results have natural abstract generalizations, and results obtained abstractly sometimes have concrete wavelet implications.

Let \mathcal{U} be a unitary system in $\mathcal{B}(\mathcal{H})$, and suppose \mathcal{U} contains a subset \mathcal{U}_0 which is a group such that $\mathcal{U}\mathcal{U}_0 = \mathcal{U}$. This is the situation for the wavelet theory case

$$\mathcal{U} = \{D^n T^\ell: n, \ell \in \mathbf{Z}\},$$

where $\mathcal{U}_0 = \{T^\ell: \ell \in \mathbf{Z}\}$. Suppose $\psi \in \mathcal{W}(\mathcal{U})$. Then $\mathcal{U}_0\psi \subseteq \mathcal{W}(\mathcal{U})$ clearly. Usually \mathcal{U}_0 will not be contained in $\mathcal{C}_\psi(\mathcal{U})$. For each $U \in \mathcal{U}_0$, let V_U be the unique unitary in $\mathcal{C}_\psi(\mathcal{U})$ with $V_U\psi = U\psi$ given by Proposition 2.1. Let

$$\kappa_\psi: \mathcal{U}_0 \rightarrow \mathbf{U}(\mathcal{C}_\psi(\mathcal{U})) \quad (17)$$

denote the map $\kappa_\psi(U) = V_U, U \in \mathcal{U}_0$. The following result is easily obtained by using the definition of local commutant. In the case where $\mathcal{U} = \mathcal{U}_0$ (the group case) it is just the usual antihomomorphism of the image of the left regular representation of a group into its commutant.

Theorem 2.7. *With the above notation, $\kappa_\psi(\mathcal{U}_0)$ is a group and κ_ψ is a group anti-isomorphism. The set $\mathcal{U}_0\psi$ is contained in a connected subset of $\mathcal{W}(\mathcal{U})$.*

Corollary 2.8. *With the notation of Theorem 2.7, the set $\mathcal{U}_0\psi$ is contained in a connected subset of $\mathcal{W}(\mathcal{U})$.*

Proof. Then $\kappa_\psi(\mathcal{U}_0)$ is a group of unitaries contained in $\mathcal{C}_\psi(\mathcal{U})$, so since the latter is a weakly closed linear space it contains $\overline{\text{sp}}^{\text{WOT}}(\kappa_\psi(\mathcal{U}_0)) = w^*(\kappa_\psi(\mathcal{U}_0))$. The unitary group of the von Neumann algebra $w^*(\kappa_\psi(\mathcal{U}_0))$ is norm-connected. Thus $\mathbf{U}(w^*(\kappa_\psi(\mathcal{U}_0)))\psi$ is a norm-connected subset of $\mathcal{W}(\mathcal{U})$. It contains $\mathcal{U}_0\psi$ since for each $U \in \mathcal{U}_0$ we have $U\psi = \kappa_\psi(U)\psi$. ■

An alternate way of obtaining Corollary 2.8 is to directly show that if $V \in w^*(\mathcal{U}_0)$ then V is a wavelet multiplier (see Chapter 3). The proof of Proposition 3.3 easily generalizes.

In [9, Chapter 2] a number of abstract structural results are obtained starting with Theorem 2.7. It is shown that in the case where \mathcal{U}_0 is abelian, which is the case relevant to multi-variate wavelet theory, the map κ_ψ of Theorem 2.7 extends to a $*$ -homomorphism of $w^*(\mathcal{U}_0)$ into $\mathcal{C}_\psi(\mathcal{U})$. Then W. Li, J. McCarthy and D. Timotin [38] and independently my students D. Han and V. Kamat [26] extended this to the case where \mathcal{U}_0 is nonabelian. Both groups then used this to answer affirmatively Problem B in [9], which asked whether if $\eta \in \mathcal{W}(\mathcal{U})$ and $\eta \in E_\psi := [\mathcal{U}_0\psi]$, is there always a unitary operator U in $w^*(\mathcal{U}_0)$ such that $\eta = U\psi$? It was known for the case where \mathcal{U}_0 was abelian.

Theorem 2.9. ([26, 38]). *In the terminology of Theorem 2.7, the map κ_ψ extends to a $*$ -antihomomorphism of $w^*(\mathcal{U}_0)$ into $\mathcal{C}_\psi(\mathcal{U})$.*

The above result generalizes the standard conjugate-linear isomorphism between the von Neumann algebra generated by the left regular representation of a group G on $\ell^2(G)$ and its commutant. In the group case the commutant is the von Neumann algebra generated by the right regular representation on $\ell^2(G)$. In the abstract unitary system case the situation is analogous but more complicated, with the role of the commutant replaced by the local commutant. Moreover, the conjugate linear map $\kappa_\psi: w^*(\mathcal{U}_0) \rightarrow \mathcal{C}_\psi(\mathcal{U})$ need not be onto and need not be 1-1. If (as above) we set $E_\psi = [\mathcal{U}_0\psi]$, and if we define $\mathcal{C}_\psi^P(\mathcal{U})$ to be the linear subspace of $\mathcal{C}_\psi(\mathcal{U})$ consisting of the operators in $\mathcal{C}_\psi(\mathcal{U})$ which are reduced by E_ψ , it turns out that $\kappa_\psi(w^*(\mathcal{U}_0)) = \mathcal{C}_\psi^P(\mathcal{U})$. Moreover, $\mathcal{C}_\psi^P(\mathcal{U})$ is $*$ -antiisomorphic to $w^*(\mathcal{U}_0)|_{E_\psi}$. In [9] this was done for the case where \mathcal{U}_0 is abelian and \mathcal{U} has the special form $\mathcal{U} = \mathcal{U}_1\mathcal{U}_0$ where \mathcal{U}_1 is also a group with trivial intersection with \mathcal{U}_0 . In [26, 38] it was done for the general case.

We conclude with a result from [9] which we find rather intriguing, although in practice we have found it less useful than might be expected. It is not difficult to prove.

Proposition 2.10. Let \mathcal{U}_1 and \mathcal{U}_0 be unitary groups in $\mathcal{B}(\mathcal{H})$, and let \mathcal{U} be the unitary system $\mathcal{U} = \mathcal{U}_1\mathcal{U}_0 = \{UV: U \in \mathcal{U}_1, V \in \mathcal{U}_0\}$. If $\psi \in \mathcal{W}(\mathcal{U})$, then

$$\mathcal{C}_\psi(\mathcal{U}) = \mathcal{U}'_1 \cap \{\mathcal{U}'_0 + \mathcal{B}(\mathcal{H})P_\psi^\perp\}.$$

The above result gives a simple *structural description* of the local commutant for an important special case. All unitary systems relevant to wavelet theory fit this case. However, it does not seem to aid in the solution of open problems such as: when is $\mathcal{C}_\psi(\mathcal{U})$ the *SOT*-closed span of the invertible operators it contains? It does point out, at least in principle, the way in which $\mathcal{C}_\psi(\mathcal{U})$ can be much larger in some cases (especially *wavelet* cases) than the commutant $\mathcal{U}' = \mathcal{U}'_1 \cap \mathcal{U}'_0$.

3. One dimensional dyadic orthonormal wavelets

Let D, T be the dilation by 2 and translation by 1 operators (4) discussed in section 2. The following results are simple and are useful.

Lemma 3.1. Let $\psi \in \mathcal{W}(D, T)$ be arbitrary. Then

- (i) $T \notin \mathcal{C}_\psi(D, T)$ and $D \notin \mathcal{C}_\psi(D, T)$.
- (ii) $\mathcal{C}_\psi(D, T) \subseteq \{D\}'$.
- (iii) If $\eta \in \mathcal{W}(D, T)$, let $V \in \mathcal{C}_\psi(D, T)$ with $V\psi = \eta$. Then $\mathcal{C}_\eta(D, T) = \mathcal{C}_\psi(D, T)V^*$.

Proof. (i) If either $T \in \mathcal{C}_\psi(D, T)$ or $D \in \mathcal{C}_\psi(D, T)$ we would have $TD\psi = DT\psi$. But we have

$$TD = DT^2,$$

and DT and DT^2 are distinct elements of $\mathcal{U}_{D, T}$. Hence $TD\psi$ and $DT\psi$ are orthogonal unit vectors.

(ii) Let $A \in \mathcal{C}_\psi(D, T)$. Then for all $n, \ell \in \mathbf{Z}$ we have $(AD - DA)\psi_{n\ell} = (AD - DA)D^n T^\ell \psi = A(D^{n+1} T^\ell)\psi - D(AD^n T^\ell)\psi = D^{n+1} T^\ell A\psi - DD^n T^\ell A\psi = 0$.

So since $\{\psi_{n\ell}\}_{n, \ell \in \mathbf{Z}}$ spans $L^2(\mathbf{R})$, it follows that $AD - DA = 0$.

(iii) We have $A \in \mathcal{C}_\eta(D, T) \Leftrightarrow AD^n T^\ell \eta = D^n T^\ell A\eta \forall n, \ell \in \mathbf{Z} \Leftrightarrow AD^n T^\ell V\psi = D^n T^\ell AV\psi \forall n, \ell \in \mathbf{Z} \Leftrightarrow (AV)D^n T^\ell \psi = D^n T^\ell (AV)\psi \forall n, \ell \in \mathbf{Z} \Leftrightarrow AV \in \mathcal{C}_\psi(D, T)$. ■

If $a \in \mathbf{R}$, let T_α denote the unitary corresponding to translation by α on $L^2(\mathbf{R})$. A computation shows

$$DT_\alpha D^{-1} = T_{\frac{\alpha}{2}}.$$

This implies that Group (D, T) contains the abelian subgroup $\{T_\alpha: \alpha \text{ is a dyadic rational}\}$. It is easy to see that $\chi_{[0,1]}$ is a cyclic vector for the linear span of these dyadic translations. It follows that the closure of this linear

span in the strong operator topology is a maximal abelian von Neumann subalgebra of $L^2(\mathbf{R})$. Denote this by \mathcal{A}_T . Then the commutant $\{D, T\}'$ is contained in \mathcal{A}_T . This proves that $\{D, T\}'$ is abelian. A better way to see this is via the Fourier transform.

Let \mathcal{F} be the Fourier-Plancherel transform on $\mathcal{H} = L^2(\mathbf{R})$, normalized so it is a unitary transformation. If $f, g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ then

$$(\mathcal{F}f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ist} f(t) dt := \hat{f}(s), \quad (18)$$

and

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ist} g(s) ds. \quad (19)$$

We have

$$(\mathcal{F}T_\alpha f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ist} f(t - \alpha) dt = e^{-is\alpha} (\mathcal{F}f)(s).$$

So $\mathcal{F}T_\alpha \mathcal{F}^{-1}g = e^{-is\alpha} g$. For $A \in \mathcal{B}(\mathcal{H})$ let \hat{A} denote $\mathcal{F}A\mathcal{F}^{-1}$. Thus

$$\hat{T}_\alpha = M_{e^{-is\alpha}}, \quad (20)$$

where for $h \in L^\infty$ we use M_h to denote the multiplication operator $f \rightarrow hf$. Since $\{M_{e^{-is\alpha}}: \alpha \in \mathbf{R}\}$ generates the m.a.s.a. $\mathcal{D}(\mathbf{R}) := \{M_h: h \in L^\infty(\mathbf{R})\}$ as a von Neumann algebra, we have

$$\mathcal{F}\mathcal{A}_T\mathcal{F}^{-1} = \mathcal{D}(\mathbf{R}).$$

Similarly,

$$\begin{aligned} (\mathcal{F}D^n f)(s) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ist} (\sqrt{2})^n f(2^n t) dt \\ &= (\sqrt{2})^{-n} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i2^{-n}st} f(t) dt \\ &= (\sqrt{2})^{-n} (\mathcal{F}f)(2^{2^{-n}}s) = (D^{-n} \mathcal{F}f)(s). \end{aligned}$$

So $\hat{D}^n = D^{-n} = D^{*n}$. Therefore,

$$\hat{D} = D^{-1} = D^*. \quad (21)$$

We have $\mathcal{F}\{D, T\}'\mathcal{F}^{-1} = \{\hat{D}, \hat{T}\}'$. It turns out that $\{\hat{D}, \hat{T}\}'$ has an easy characterization. As mentioned in the introduction, the following result was obtained at the beginning of this program as the "second discovery" which motivated the rest of our work. We remark that it was discovered independently by Manos Papadakis [34] in a somewhat different type of operator-theoretic study.

Theorem 3.2.

$$\{\widehat{D}, \widehat{T}\}' = \{M_h: h \in L^\infty(\mathbb{R}) \text{ and } h(s) = h(2s) \text{ a.e.}\}.$$

Proof. Since $\widehat{D} = D^*$ and D is unitary, it is clear that $M_h \in \{\widehat{D}, \widehat{T}\}'$ if and only if M_h commutes with D . So let $g \in L^2(\mathbb{R})$ be arbitrary. Then (a.e.) we have

$$\begin{aligned} (M_h Dg)(s) &= h(s)(\sqrt{2}g(2s)), \quad \text{and} \\ (DM_h g)(s) &= D(h(s)g(s)) = \sqrt{h}(2s)g(2s). \end{aligned}$$

Since these must be equal a.e. for arbitrary g , we must have $h(s) = h(2s)$ a.e. ■

Now let $E = [-2, -1) \cup [1, 2)$, and for $n \in \mathbb{Z}$ let $E_n = \{2^n x: x \in E\}$. Observe that the sets E_n are disjoint and have union $\mathbb{R} \setminus \{0\}$. So if g is any uniformly bounded function on E , then g extends uniquely (a.e.) to a function $\tilde{g} \in L^\infty(\mathbb{R})$ satisfying

$$\tilde{g}(s) = \tilde{g}(2s), \quad s \in \mathbb{R},$$

by setting

$$\tilde{g}(2^n s) = g(s), \quad s \in E, n \in \mathbb{Z},$$

and $\tilde{g}(0) = 0$. We have $\|\tilde{g}\|_\infty = \|g\|_\infty$. Conversely, if h is any function satisfying $h(s) = h(2s)$ a.e., then h is uniquely (a.e.) determined by its restriction to E . This 1-1 mapping $g \rightarrow M_{\tilde{g}}$ from $L^\infty(E)$ onto $\{\widehat{D}, \widehat{T}\}'$ is a *-isomorphism.

We will refer to a function h satisfying $h(s) = h(2s)$ a.e. as a *2-dilation periodic function*. This gives a simple algorithm for computing a large class of wavelets from a given one:

Given ψ , let $\widehat{\psi} = \mathcal{F}(\psi)$, choose a real-valued function $h \in L^\infty(E)$ arbitrarily, let $g = \exp(ih)$, extend to a 2-dilation periodic function \tilde{g} as above, and compute $\psi_{\tilde{g}} = \mathcal{F}^{-1}(\tilde{g}\widehat{\psi})$. (22)

In the description above, the set E could clearly be replaced with $[-2\pi, -\pi) \cup [\pi, 2\pi)$, or with any other "dyadic" set $[-2a, a) \cup [a, 2a)$ for some $a > 0$.

We note (cf. [31, p. 234]) that if H denotes the Hilbert transform on $L^2(\mathbb{R})$ then $\widehat{H} = M_h$, where $h(s) = -i$ for $s < 0$, $h(s) = +i$ for $s > 0$, and $h(0) = 0$. So h is a 2-dilation-periodic unimodular function of a simple type. It has long been known that the Hilbert transform maps wavelets to wavelets. So (22) can be thought of as a generalization of that fact. It is

curious that (22) is transparent from a functional analysis point of view, but was apparently unobserved in the function-theoretic wavelet theory.

Let us define a *wavelet multiplier* to be a unitary operator $V \in \mathcal{B}(L^2(\mathbf{R}))$ with the property that $V\mathcal{W}(D, T) \subseteq \mathcal{W}(D, T)$. So in particular the Hilbert transform on $L^2(\mathbf{R})$ is a wavelet multiplier. The above algorithm (22) follows from the fact that $\{D, T\}' \subseteq \mathcal{C}_\psi(D, T)$ for every $\psi \in \mathcal{W}(D, T)$, so every unitary in $\{D, T\}'$ is a wavelet multiplier. That is,

$$\{\mathcal{F}^{-1}M_h\mathcal{F}: h \text{ is a unimodular 2-dilation-periodic function}\}$$

is a *group* of wavelet multipliers. Another group of wavelet multipliers is the unitary group of $w^*(T)$.

Proposition 3.3. *Let $V \in \mathcal{U}(W^*(T))$. Then $V\mathcal{W}(D, T) \subseteq \mathcal{W}(D, T)$.*

Proof. Suppose $V \in \mathcal{U}(w^*(T))$. Let ψ be a wavelet, and let $\eta = V\psi$. Let $E_\psi := \overline{\text{span}}\{T^\ell\psi: \ell \in \mathbf{Z}\}$ denote the *translation subspace* for ψ . Then E_ψ reduces V and $\{T^\ell\eta: \ell \in \mathbf{Z}\} = V\{T^\ell\psi: \ell \in \mathbf{Z}\}$. So $\{T^\ell\eta\}_{\ell \in \mathbf{Z}}$ is an orthonormal basis for E_ψ , and since E_ψ is a complete wandering subspace for D , it follows that $\{D^n T^\ell V\psi: n, \ell \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mathbf{R})$. ■

A *multiresolution analysis* (MRA) for $L^2(\mathbf{R})$ is a nest of closed subspaces $V_n, n \in \mathbf{Z}$, of $L^2(\mathbf{R})$ satisfying

- (i) $V_n \subset V_{n+1}, n \in \mathbf{Z}$
- (ii) $f \in V_n \Leftrightarrow f(2 \cdot) \in V_{n+1}, n \in \mathbf{Z}$
- (iii) $\bigcap_{n \in \mathbf{Z}} V_n = \{0\}$
- (iv) $\bigcup_{n \in \mathbf{Z}} V_n$ is dense in $L^2(\mathbf{R})$
- (v) There is a function $\varphi \in V_0$ such that $\{\varphi(\cdot - \ell): \ell \in \mathbf{Z}\}$ is an orthonormal basis for V_0 .

The function φ is called the *scaling function* of the MRA. It is known (cf. [5, 14, 31, 40]) that given an MRA with scaling function φ there is a (nonunique) orthonormal dyadic wavelet ψ for which $\{\psi(\cdot - \ell): \ell \in \mathbf{Z}\}$ is an orthonormal basis for $W_0 := V_0 \ominus V_0$.

The proof of Proposition 3.3 is *very* elementary from an operator-theoretic point of view. In apparent contrast to Theorem 3.2, it is *not* new to function-theoretic wavelet theory, at least for MRA (multiresolution analysis) wavelets. The following result is at least implicit in, for instance, [14]. Its proof (at least to *me*) is not anywhere near as transparent as the

proof of Proposition 3.3. For the connection, note that $\widehat{T} = M_{e^{it}}$ implies that

$$\mathcal{F}w^*(T)\mathcal{F}^{-1} = \{M_f: f \in L^\infty(\mathbf{R}), f \text{ is } 2\pi\text{-translation-periodic}\}. \quad (24)$$

Proposition 3.3'. *Let ψ be an orthonormal dyadic MRA wavelet, and suppose $f \in L^\infty(\mathbf{R})$ is a unimodular 2π -periodic function. Then $\mathcal{F}^{-1}(f\widehat{\psi})$ is also an MRA wavelet.*

The proof of Proposition 3.3 easily abstracts to abstract unitary systems of the form

$$\mathcal{U} = \mathcal{U}_0\mathcal{U}_1 \quad (25)$$

where \mathcal{U}_0 and \mathcal{U}_1 are groups. In [9] we treated it in the abstract as well as the concrete fashion.

Let us define a *functional wavelet multiplier* to be any unimodular measurable function $f(s)$ with the property that $f(s)\widehat{\psi}(s)$ is the Fourier transform of a dyadic orthonormal wavelet whenever ψ is a dyadic orthonormal wavelet. That is, a functional wavelet multiplier is a unimodular function f for which $\mathcal{F}^{-1}M_h\mathcal{F}$ is a wavelet multiplier in the above sense. Motivated by the results Theorem 3.2 and Proposition 3.3, in Problem C' of [9] we asked: If h is a unimodular function in $L^\infty(\mathbf{R})$ with the property that $\mathcal{F}^{-1}(h\mathcal{F}(\psi))$ is a wavelet for every wavelet ψ , does h necessarily factor $f = f_1f_2$, where f_1 is 2-dilation-periodic and f_2 is 2π -translation periodic?

Very recently [8], in joint work with Xingde Dai, my student Qing Gu, and Dai's student Rufeng Liang, we showed that the answer to Problem C' in [9] is *no*. Our solution also showed that the answer to Problem D in [9] is negative. In his thesis work, Gu has answered Problem E in [9] positively for the "finite" case.

Problem C' was given as a subproblem of Problem C in [9] which asked the corresponding problem in the context of wandering vector multipliers for abstract unitary systems of the form (25) above. In this abstract context, negative answers have been obtained by my student Deguang Han [25] in which he studied wandering vector multipliers for unitary systems generated by a pair of generators of an irrational rotation C^* -algebra, and Li, McCarthy and Timotin [38] in which a different counterexample was constructed.

In [8] we obtained a negative answer to Problem C', and pushed it further [8, Theorem 4] to obtain a *complete* characterization of the functional wavelet multipliers.

Theorem 3.4. *Let $f \in L^\infty(\mathbf{R})$ be unimodular. Then f is a functional wavelet multiplier if and only if the function $k(s) := f(2s)/f(s)$ is 2π -translation-periodic.*

Note that if f is a unimodular function which is either 2-dilation-periodic or 2-translation-periodic, or a product of these two types, then f trivially satisfies the property of Theorem 3.4. However, there are many other functions having this property. Denote this class as (DT) . It is clearly a group under multiplication. The following (26) is a constructive characterization of all elements the class (DT) which take the value 1 on the Littlewood-Paley set. Every unimodular DT -function factors uniquely as the product of a unimodular dilation-periodic function and a unimodular function of the form (26).

Proposition 3.5. (A Formula.) Let $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$, and let $k(s)$ be any unimodular measurable 2π -translation-periodic function on \mathbf{R} . Define $f(s)$ on \mathbf{R} by:

$$f(s) := \begin{cases} 1, & s \in E \\ k(2^{-1}s) \dots k(2^{-n}s), & s \in 2^n E, n \geq 1 \\ \frac{k(s) \dots k(2^{n-1}s)}{k(s) \dots k(2^{n-1}s)}, & s \in 2^{-n} E, n \geq 1 \\ 1, & s = 0. \end{cases} \quad (26)$$

Then $f \in (DT)$ and $f(2s)/f(s) = k(s)$, a.e. $s \in \mathbf{R}$. Moreover, the map given by this formula from the family of unimodular 2π -translation-periodic functions to the family of unimodular DT -functions which are 1 on $E \cup \{0\}$ is one-to-one and onto.

Let $h \in L^2(\mathbf{R})$. We will define the *phase* of h to be the real-valued function $\alpha(s)$ defined on the support of \hat{h} determined uniquely a.e. modulo 2π by the equation

$$\hat{h}(s) = e^{i\alpha(s)} |\hat{h}(s)|. \quad (27)$$

It is known [5, Section 5.5] that *linear phase* can be a useful property for a wavelet to have. (This means $\alpha(s) = as + b$ for some $a, b \in \mathbf{R}$.) If ψ is a wavelet, we will say that a phase function $\alpha(s)$ is *attainable* for ψ if $e^{i\alpha(s)} |\hat{\psi}(s)|$ is the Fourier transform of a wavelet. A few years ago Charles Chui raised the question to us of whether linear phase is always attainable. One of the main purposes of the paper [8] was to answer this question. We showed that the answer is "yes" for MRA wavelets, but "no" in general. Our main theorem from [8] shows that the phase $s/2$ is always attainable for an MRA wavelet.

Theorem 3.6. Let ψ be an MRA wavelet. Then there is a unimodular function f in the class (DT) such that $\hat{\psi}(s) = e^{\frac{is}{2}} f(s) |\hat{\psi}(s)|$, $s \in \mathbf{R}$. Thus the function $\hat{\eta}(s) = e^{\frac{is}{2}} |\hat{\psi}(s)|$ is also the Fourier transform of an MRA-wavelet. We have $\{\eta: \eta \text{ is a wavelet with } |\hat{\eta}| = |\hat{\psi}|\} = \{\mathcal{F}^{-1}(g\hat{\psi}): g \text{ is a unimodular } DT\text{-function}\}$.

Define $g(s)$ on \mathbf{R} by

$$g(s) = \begin{cases} e^{-\frac{is}{2}}, & s \in [(2n)2\pi, (2n+1)2\pi), n \in \mathbf{Z} \\ -e^{-\frac{is}{2}}, & s \in [(2n+1)2\pi, (2n+2)2\pi), n \in \mathbf{Z}. \end{cases} \quad (28)$$

Then $g(s)$ is 2π -translation-periodic, so is a functional wavelet multiplier. So a consequence of Theorem 3.6 is the following result, which is apparently new to wavelet theory.

Corollary 3.7. *Let ψ be an MRA wavelet. Then there is an MRA wavelet η with $|\hat{\eta}| = |\widehat{\psi}|$ such that $\hat{\eta}$ is a real-valued function.*

Next in [8] we obtained some delimiting counterexamples for non-MRA wavelets. We used examples and techniques contained in Chapters 4 and 5 of the present article and [9, 21, 23, 29, 30] in obtaining these. In [23] a criterion which is contained in this article as Theorem 5.9 was used to verify Examples 3.8 and 3.9 below together with the phase information. We refer the reader to these verifications for some insight as to how certain of these computations can be done in practice.

First we give an example of a wavelet (in fact a path of wavelets parameterized by ϵ) for which the phases $\frac{1}{4}s$ and $\frac{3}{4}s$ are attainable, but not the phase $\frac{1}{2}s$.

Example 3.8. Let $0 < \epsilon < \frac{\pi}{6}$, and let

$$h(s) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & s \in \left[-\frac{8}{3}\pi, -2\pi - 2\epsilon\right) \cup \left[-\pi + \epsilon, -\frac{2}{3}\pi\right) \\ & \cup \left[\frac{2}{5}\pi, \frac{2}{3}\pi\right) \cup \left[\frac{8}{3}, 3\pi - \epsilon\right) \cup \left[6\pi + 2\epsilon, \frac{32}{5}\pi\right) \\ \frac{1}{2\sqrt{\pi}}, & s \in [-2\pi - 2\epsilon, -2\pi + 2\epsilon) \cup [3\pi - \epsilon, 3\pi + \epsilon) \\ & \cup [6\pi - 2\epsilon, 6\pi + 2\epsilon) \\ -\frac{1}{2\sqrt{\pi}}, & s \in [-\pi - \epsilon, -\pi + \epsilon) \\ 0, & \text{otherwise.} \end{cases}$$

Then

- (i) $h(s)$ is the Fourier transform of a wavelet, and
- (ii) if $a \in \mathbf{R}$ with $0 \leq a \leq 1$, then $e^{ias}|h(s)|$ is the Fourier transform of a wavelet if and only if $a = \frac{1}{4}$ or $\frac{3}{4}$

Next we give a wavelet (a 2-parameter family of them) for which no linear phase function is attainable.

Example 3.9. Let $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 + \epsilon_2 < \frac{4}{7}\pi$ and $4\epsilon_1 + \frac{\epsilon_2}{2} < \frac{2}{7}\pi$, and let

$$h(s) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & s \in \left[-\frac{8}{7}\pi + \epsilon_1, -\frac{4}{7}\pi - \epsilon_2\right) \cup \left[\frac{4}{7}\pi, \frac{6}{7}\pi\right) \\ & \cup \left[\frac{24}{7}\pi + 4\epsilon_1, \frac{26}{7}\pi - \frac{\epsilon_2}{2}\right) \cup \left[\frac{26}{7}\pi, \frac{32}{7}\pi\right) \\ \frac{1}{2\sqrt{\pi}}, & s \in \left[-\frac{32}{7}\pi, -\frac{32}{7}\pi + 4\epsilon_1\right) \cup \left[-\frac{4}{7}\pi - \epsilon_2, -\frac{4}{7}\pi\right) \\ & \cup \left[\frac{6}{7}\pi, \frac{6}{7}\pi + \epsilon_1\right) \cup \left[\frac{24}{7}\pi, \frac{24}{7}\pi + 4\epsilon_1\right) \\ & \cup \left[\frac{26}{7}\pi - \frac{\epsilon_2}{2}, \frac{26}{7}\pi\right) \cup \left[\frac{52}{7}\pi - \epsilon_2, \frac{52}{7}\pi\right) \\ -\frac{1}{2\sqrt{2}}, & s \in \left[-\frac{8}{7}\pi, -\frac{8}{7}\pi + \epsilon_1\right) \cup \left[-\frac{2}{7}\pi - \frac{\epsilon_2}{2}, -\frac{2}{7}\pi\right) \\ 0, & \text{otherwise.} \end{cases}$$

Then

- (i) $h(s)$ is the Fourier transform of a wavelet, and
- (ii) there is no real number $a \in \mathbb{R}$ such that $e^{ias}|h(s)|$ is the Fourier transform of a wavelet.

Both of these Examples 3.8 and 3.9 can be obtained (and this is essentially the way they are in fact analyzed in our theory) by “multiplying” a wavelet whose Fourier transform is the normalized characteristic function of a set (an *s-elementary* or *MSF-wavelet*) by a unitary in an “interpolation von Neumann algebra”, computed for a pair of such “wavelet sets”, which lies in the local commutant of $\mathcal{U}_{D,T}$ at the initial wavelet. It is a non-commutative von Neumann algebra, but of a simple form isomorphic to a subalgebra of 2×2 matrix valued functions which are “twisted” with respect to a certain measure-preserving transformation of the line. These examples represent the simplest and most “hands-on-computational” aspects of the general theory. Even so, they are not so easy to work out.

Now we can better describe the way in which von Neumann algebras enter into formulas and characterizations of classes of wavelets. If ψ is an orthonormal wavelet then the local commutant $\mathcal{C}_\psi(D, T)$ is a strongly closed linear space which contains many von Neumann algebras as subsets. So any method of constructing and parameterizing a von Neumann algebra M embedded in $\mathcal{C}_\psi(D, T)$ yields a family of wavelets (“new” wavelets from the “old” wavelet ψ) parameterized by the unitary group of M under the map $U \rightarrow U\psi$. It turns out that concrete *formulas*, which are new to wavelet theory, can be obtained in this way.

To facilitate exposition, let us define a **-wavelet bundle* to be a family of orthonormal wavelets parameterized by a von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ and a single orthonormal wavelet ψ of the form

$$\mathcal{B}_\psi(M) = \{U\psi : U \in \mathcal{U}(M)\}. \quad (29)$$

(The von Neumann algebra M need not be contained in $\mathcal{C}_\psi(D, T)$ for any ψ . For instance $M = w^*(T)$ satisfies this. Two examples are thus $\mathcal{B}_\psi(\{D, T\}')$ and $\mathcal{B}_\psi(w^*(T))$.

For any $\psi \in \mathcal{W}(D, T)$ the space $\mathcal{C}_\psi(D, T)$ is *not* an algebra and *not* self-adjoint (both of these assertions are nontrivial to prove) and many properties of its structure are mysterious. Complete knowledge about the set of unitaries in $\mathcal{C}_\psi(D, T)$ would yield complete knowledge about the set $\mathcal{W}(D, T)$ of all dyadic orthonormal wavelets. Also, complete knowledge about the set of invertibles in $\mathcal{C}_\psi(D, T)$ would yield complete knowledge about the set

$$\mathcal{RW}(D, T) \tag{30}$$

of all Riesz wavelets. So there is ample motivation to analyze $\mathcal{C}_\psi(D, T)$. A von Neumann algebra is the SOT-closed linear span of its unitaries. This motivates:

Problem 3.A. Is $\mathcal{C}_\psi(D, T)$ the SOT-closed linear span of its unitaries? (Lemma 3.1 (iii) shows that if this is ever true it is always true.)

Another problem of even greater potential significance is:

Problem 3.B. Is $\mathcal{C}_\psi(D, T)$ the SOT-closure of the set of invertible operators it contains? (Again, by Lemma 3.1 (iii) this is either always true or never true.)

Here by *invertible* we mean invertible in $\mathcal{B}(\mathcal{H})$, not necessarily that it has an inverse in $\mathcal{C}_\psi(D, T)$. A positive answer would imply, by Proposition 4, that the set of all Riesz wavelets is *dense* in $L^2(\mathbb{R})$. I have it on some authority that a “yes” answer would be rather astounding to certain wavelet theorists. The answer is probably no. A “yes” answer would be marvelous, because, as I understand, it would suggest that very “good” wavelets are always possible for certain application problems: one could then choose a wavelet “close” to the prototype signal (i.e. L^2 -function) for a problem.

If ψ, η are two orthonormal wavelets there is a simple relation between the local commutants $\mathcal{C}_\psi(D, T)$ and $\mathcal{C}_\eta(D, T)$. We have

$$\mathcal{C}_\psi(D, T) = \mathcal{C}_\eta(D, T) \cdot V \tag{31}$$

where V is the unique unitary in $\mathcal{C}_\psi(D, T)$ with $V\psi = \eta$. This shows that Problems A and B have positive answers for $\mathcal{C}_\psi(D, T)$ if and only if they have positive answers for $\mathcal{C}_\eta(D, T)$.

Let us call a $*$ -wavelet bundle $\mathcal{B}_\psi(M)$ *abelian* if the von Neumann algebra M is abelian. Then $\mathcal{B}_\psi(\{D, T\}')$ and $\mathcal{B}_\psi\{w^*(T)\}$ are abelian. The main application of the theory developed in [9] concerning operator-theoretic interpolation of wavelets involved examples where M is non-abelian. We showed that M could be taken to be isomorphic to the cross-product of

$\{D, T\}'$ by a cyclic group of order 2 or 3. The group is commutative but the von Neumann algebra is, of course, noncommutative. Recently, my student Gu [21] extended this to arbitrary (not necessarily commutative) finite groups. See the remarks after (50) and before (61). Of course, M can be taken to be any von Neumann algebra contained in $\mathcal{C}_\psi(D, T)$ for some ψ . So far, the examples we know of are all type I. But examples are hard to construct.

Problem 3.C. If $\psi \in \mathcal{W}(D, T)$, can $\mathcal{C}_\psi(D, T)$ contain a von Neumann algebra not of type I? In particular, can it contain a type II or type III factor?

A positive answer would certainly be interesting from an operator-algebraic point of view. Some evidence for a "yes" answer is that we have been able to show that, for certain ψ , $\mathcal{C}_\psi(D, T)$ contains an isometry V , which is not a unitary, for which $V^* \in \mathcal{C}_\psi(D, T)$. However, in our case $V^2 \notin \mathcal{C}_\psi(D, T)$, so $w^*(V) \not\subseteq \mathcal{C}_\psi(D, T)$. In the *abstract* theory for wandering vectors for unitary systems which, say, contain i.c.c. groups as right or left factors the theory can be very rich in the sense of Problem 3.3. But in the concrete wavelet theory for one (or higher) dimensions we do not know enough at present to even venture a good guess.

4. Wavelets of Computationally Elementary Form

We now give an account of s -elementary and MSF -wavelets. The two most elementary dyadic orthonormal wavelets are the *Haar wavelet* and *Shannon's wavelet* (also called the Littlewood-Paley wavelet).

The Haar wavelet is the function

$$\psi_H(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases} \quad (32)$$

In this case it is very easy to see that the dilates/translates

$$\{2^{\frac{n}{2}} \psi_H(2^n - \ell) : n, \ell \in \mathbf{Z}\}$$

are orthonormal, and an elementary argument shows that their span is dense in $L^2(\mathbf{R})$.

Shannon's wavelet is the $L^2(\mathbf{R})$ -function with Fourier transform $\widehat{\psi}_S = \frac{1}{\sqrt{2\pi}} \chi_{E_0}$ where

$$E_0 = [-2\pi, -\pi) \cup [\pi, 2\pi). \quad (33)$$

The argument that $\widehat{\psi}_S$ is a wavelet is in a way even more transparent than for the Haar wavelet. And it has the advantage of generalizing nicely. For

a simple argument, start from the fact that the exponents

$$\{e^{i\ell s}: \ell \in \mathbf{Z}\}$$

restricted to $[0, 2\pi]$ and normalized by $\frac{1}{\sqrt{2\pi}}$ is an orthonormal basis for $L^2[0, 2\pi]$. Write $E_0 = E_- \cup E_+$ where $E_- = [-2\pi, -\pi)$, $E_+ = [\pi, 2\pi)$. Since $\{E_- + 2\pi, E_+\}$ is a partition of $[0, 2\pi)$ and since the exponentials $e^{i\ell s}$ are invariant under translation by 2π , it follows that

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_{E_0} : \ell \in \mathbf{Z} \right\} \quad (34)$$

is an orthonormal basis for $L^2(E_0)$. Since $\widehat{T} = M_{e^{-is}}$, this set can be written

$$\{\widehat{T}^\ell \widehat{\psi}_s : \ell \in \mathbf{Z}\}. \quad (35)$$

Next, note that any "dyadic interval" of the form $J = [b, 2b)$, for some $b > 0$ has the property that $\{2^n J : n \in \mathbf{Z}\}$, is a partition of $(0, \infty)$. Similarly, any set of the form

$$\mathcal{K} = [-2a, -a) \cup [b, 2b) \quad (36)$$

for $a, b > 0$, has the property that

$$\{2^n \mathcal{K} : n \in \mathbf{Z}\}$$

is a partition of $\mathbf{R} \setminus \{0\}$. It follows that the space $L^2(\mathcal{K})$, considered as a subspace of $L^2(\mathbf{R})$, is a complete wandering subspace for the dilation unitary $(Df)(s) = \sqrt{2} f(2s)$. For each $n \in \mathbf{Z}$,

$$D^n(L^2(\mathcal{K})) = L^2(2^{-n}\mathcal{K}). \quad (37)$$

So $\bigoplus_n D^n(L^2(\mathcal{K}))$ is a direct sum decomposition of $L^2(\mathbf{R})$. In particular E_0 has this property. So

$$D^n \left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_{E_0} : \ell \in \mathbf{Z} \right\} = \left\{ \frac{e^{2^n i\ell s}}{\sqrt{2\pi}} \Big|_{2^{-n}E_0} : \ell \in \mathbf{Z} \right\} \quad (38)$$

is an orthonormal basis for $L^2(2^{-n}E_0)$ for each n . It follows that

$$\{D^n \widehat{T}^\ell \widehat{\psi}_s : n, \ell \in \mathbf{Z}\}$$

is an orthonormal basis for $L^2(\mathbf{R})$. Hence $\{D^n T^\ell \psi_s : n, \ell \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mathbf{R})$, as required.

The Haar wavelet can be generalized, and in fact Daubechies well-known continuous compactly-supported wavelet is a generalization of the Haar

wavelet. However, known generalization of the Haar wavelet are all complicated and difficult to work with in hand-computations.

For our work, in order to proceed with developing an operator algebraic theory that had a chance of directly impacting concrete function-theoretic wavelet theory we needed a large supply of examples of wavelets which were elementary enough to work with. First, we found another “Shannon-type” wavelet in the literature. This was the Journe wavelet, which we found described on p. 136 in Daubechies book [14]. Its Fourier transform is $\widehat{\psi}_J = \frac{1}{\sqrt{2\pi}}\chi_{E_J}$, where

$$E_J = \left[-\frac{32\pi}{7}, -4\pi\right) \cup \left[-\pi, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi\right) \cup \left[4\pi, \frac{32\pi}{7}\right).$$

Then, thinking the old adage “where there’s smoke there’s fire!”, we painstakingly worked out many more examples. So far, these are the basic building blocks in the *concrete* part of our theory. By this we mean the part of our theory that has had some type of direct impact on function-theoretic wavelet theory.

We define a *wavelet set* to be a measurable subset E of \mathbf{R} for which $\frac{1}{\sqrt{2\pi}}\chi_E$ is the Fourier transform of a wavelet. The wavelet $\widehat{\psi}_E := \frac{1}{\sqrt{2\pi}}\chi_E$ is called *s-elementary* in [9].

It turns out that this class of wavelets was also discovered and systematically explored completely independently, and in about the same time period, by Guido Weiss (Washington University), his colleague and former student E. Hernandez (U. Madrid), and his students X. Fang and X. Wang. In [17, 29, 30] they are called MSF (minimally supported frequency) wavelets. In signal processing, the parameter s , which is the independent variable for $\widehat{\psi}$, is the *frequency* variable, and the variable t , which is the independent variable for ψ , is the *time* variable. No function with support a subset of a wavelet set E of strictly smaller measure can be the Fourier transform of a wavelet.

Problem 4.A. Must the support of the Fourier transform of a wavelet contain a wavelet set? This question is open for dimension 1. It makes sense for any finite dimension.

From the argument above describing why Shannon’s wavelet is, indeed, a wavelet, it is clear that *sufficient* conditions for E to be a wavelet set are

- (i) the normalized exponentials $\frac{1}{\sqrt{2\pi}}e^{i\ell s}$, $\ell \in \mathbf{Z}$, when restricted to E should constitute an orthonormal basis for $L^2(E)$ (in other words E is a *spectral set* for the integer lattice \mathbf{Z}),

and

(ii) the family $\{2^n E: n \in \mathbf{Z}\}$ of dilates of E by integral powers of 2 should constitute a measurable partition (i.e. a partition modulo null sets) of \mathbf{R} .

These conditions are also necessary. In fact if a set E satisfies (i), then for it to be a wavelet set it is obvious that (ii) must be satisfied. To show that (i) must be satisfied by a wavelet set E , consider the vectors

$$\widehat{D}^n \widehat{\psi}_E = \frac{1}{\sqrt{2\pi}} \chi_{2^{-n}E}, \quad n \in \mathbf{Z}.$$

Since $\widehat{\psi}_E$ is a wavelet these must be orthogonal, and so the sets $\{2^n E: n \in \mathbf{Z}\}$ must be disjoint modulo null sets. It follows that $\{\frac{1}{\sqrt{2\pi}} e^{i\ell s}|_E: \ell \in \mathbf{Z}\}$ is not only an orthonormal set of vectors in $L^2(E)$, it must also *span* $L^2(E)$.

It is known from the theory of spectral sets (as an elementary special case) that a measurable set E satisfies (i) if and only if it is a generator of a measurable partition of \mathbf{R} under translation by 2π (i.e. iff $\{E + 2\pi n: n \in \mathbf{Z}\}$ is a measurable partition of \mathbf{R}). This result generalizes to spectral sets for the integral lattice in \mathbf{R}^n . For this elementary special case a direct proof is not hard.

We say that measurable sets E, F are *translation congruent modulo 2π* if there is a measurable bijection $\phi: E \rightarrow F$ such that $\phi(s) - s$ is an integral multiple of 2π for each $s \in E$; or equivalently, if there is a measurable partition $\{E_n: n \in \mathbf{Z}\}$ of E such that

$$\{E_n + 2n\pi: n \in \mathbf{Z}\} \quad (39)$$

is a measurable partition of F . Analogously, define measurable sets G and H to be *dilation congruent modulo 2* if there is a measurable bijection $\tau: G \rightarrow H$ such that for each $s \in G$ there is an integer n , depending on s , such that $\tau(s) = 2^n s$; or equivalently, if there is a measurable partition $\{G_n\}_{-\infty}^{\infty}$ of G such that

$$\{2^n G\}_{-\infty}^{\infty} \quad (40)$$

is a measurable partition of H . (Translation and dilation congruency modulo other positive numbers of course make sense as well.)

The following lemma is useful.

Lemma 4.1. *Let $f \in L^2(\mathbf{R})$, and let $E = \text{supp}(f)$. Then f has the property that*

$$\{e^{ins} f: n \in \mathbf{Z}\}$$

is an orthonormal basis for $L^2(E)$ if and only if

- (i) E is congruent to $[0, 2\pi)$ modulo 2π , and
- (ii) $|f(s)| = \frac{1}{\sqrt{2\pi}}$ a.e. on E .

If E is a measurable set which is 2π -translation congruent to $[0, 2\pi)$, then since

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_{[0, 2\pi)} : \ell \in \mathbf{Z} \right\}$$

is an orthonormal basis for $L^2[0, 2\pi]$ and the exponentials $e^{i\ell s}$ are 2π -invariant, as in the case of Shannon's wavelet it follows that

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_E : \ell \in \mathbf{Z} \right\}$$

is an orthonormal basis for $L^2(E)$. Also, if E is 2π -translation congruent to $[0, 2\pi)$, then since

$$\{[0, 2\pi) + 2\pi n : n \in \mathbf{Z}\}$$

is a measurable partition of \mathbf{R} , so is

$$\{E + 2\pi n : n \in \mathbf{Z}\}.$$

These arguments can be reversed.

We say that a measurable subset $G \subset \mathbf{R}$ is a *2-dilation generator* of a *partition* of \mathbf{R} if the sets

$$2^n G := \{2^n s : s \in G\}, \quad n \in \mathbf{Z} \quad (41)$$

are disjoint and $\mathbf{R} \setminus \cup_n 2^n G$ is a null set. Also, we say that $E \subset \mathbf{R}$ is a *2π -translation generator* of a *partition* of \mathbf{R} if the sets

$$E + 2n\pi := \{s + 2n\pi : s \in E\}, \quad n \in \mathbf{Z}, \quad (42)$$

are disjoint and $\mathbf{R} \setminus \cup_n (E + 2n\pi)$ is a null set.

Lemma 4.2. *A measurable set $E \subseteq \mathbf{R}$ is a 2π -translation generator of a partition of \mathbf{R} if and only if, modulo a null set, E is translation congruent to $[0, 2\pi)$ modulo 2π . Also, a measurable set $G \subseteq \mathbf{R}$ is a 2-dilation generator of a partition of \mathbf{R} if and only if, modulo a null set, G is dilation congruent modulo 2 to the set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.*

The following is a useful criterion for wavelet sets. It was published independently by Dai-Larson in [9] and by Fang-Wang in [17] at about the same time in December, 1994. In fact, it is amusing that the two papers had been submitted within two days of each other; only much later did we even learn of each others work and of this incredible timing.

Proposition 4.3. *Let $E \subseteq \mathbf{R}$ be a measurable set. Then E is a wavelet set if and only if E is both a 2-dilation generator of a partition (modulo null sets) of \mathbf{R} and a 2π -translation generator of a partition (modulo null sets) of \mathbf{R} . Equivalently, E is a wavelet set if and only if E is both translation congruent to $[0, 2\pi)$ modulo 2π and dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ modulo 2.*

Note that a set is 2π -translation congruent to $[0, 2\pi)$ iff it is 2π -translation congruent to $[-2\pi, \pi) \cup [\pi, 2\pi)$. So the last sentence of Proposition 4.3 can be stated: A measurable set E is a wavelet set if and only if it is both 2π -translation and 2-dilation congruent to the Littlewood-Paley set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.

Remark 4.4. If E is a wavelet set, and if $f(s)$ is any function with support E which has constant modulus $\frac{1}{\sqrt{2\pi}}$ on E , then $\mathcal{F}^{-1}(f)$ is a wavelet. Indeed, by Lemma 4.1 $\{\widehat{T}^\ell f: \ell \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(E)$, and since the sets $2^n E$ partition \mathbf{R} , so $L^2(E)$ is a complete wandering subspace for \widehat{D} , it follows that $\{\widehat{D}^n \widehat{T}^\ell f: n, \ell \in \mathbf{Z}\}$ must be an orthonormal basis for $L^2(\mathbf{R})$, as required. In [17, 29, 30] the term MSF-wavelet includes this type of wavelet. So MSF-wavelets can have arbitrary phase and s -elementary wavelets have phase 0. *Every* phase is attainable in the sense of chapter 3 for an MSF or s -elementary wavelet.

Example 4.5. It is usually easy to determine, using the dilation-translation criteria, in Proposition 4.3, whether a given finite union of intervals is a wavelet set. In fact, to verify that a given “candidate” set E is a wavelet set, it is clear from the above discussion and criteria that it suffices to do two things:

- (1) Show, by appropriate partitioning, that E is 2-dilation-congruent to a set of the form $[-2a, -a) \cup [b, 2b)$ for some $a, b > 0$.

and

- (2) Show, by appropriate partitioning, that E is 2π -translation-congruent to a set of the form $[c, c + 2\pi)$ for some real number c .

On the other hand, wavelet sets suitable for testing hypotheses, can be quite difficult to construct. There are very few “recipes” for wavelet sets, as it were. Many families of such sets have been constructed for reasons including perspective, experimentation, testing hypotheses, etc., including perhaps the pure enjoyment of doing the computations – which are somewhat “puzzle-like” in nature. The interested reader can consult the papers [9] and [17] and some of the subsequent articles [21, 29, 30] for a wide array of examples of such sets. In working with the theory it is nice (and in fact

necessary) to have a large supply of wavelets on hand that permit relatively simple analysis.

For this reason we take the opportunity here to present for the reader a collection of such sets, mainly taken from [9], leaving most of the "fun" in verifying that they are indeed wavelet sets to the reader.

We refer the reader to [10] for a proof of the existence of wavelet sets in $\mathbf{R}^{(n)}$, and a proof that there are sufficiently many to generate the Borel structure of $\mathbf{R}^{(n)}$. These results are true for arbitrary expansive dilation factors. Some concrete examples in the plane were subsequently obtained by Soardi and Weiland in [45], and others were obtained by Gu and Speegle (not yet written up). Two had also been obtained by Dai for inclusion in the revised concluding remarks section of our Memoir [9].

In these examples we will usually write intervals as half-open intervals $[\cdot, \cdot)$ because it is easier to verify the translation and dilation congruency relations (1) and (2) above when wavelet sets are written thus, even though in actuality the relations need only hold modulo null sets.

(i) As mentioned above, an example due to Journé of a wavelet which admits no multiresolution analysis is the s -elementary wavelet with wavelet set

$$\left[-\frac{32\pi}{7}, -4\pi\right) \cup \left[-\pi, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi\right) \cup \left[4\pi, \frac{32\pi}{7}\right).$$

To see that this satisfies the criteria, label these intervals, in order, as J_1, J_2, J_3, J_4 and write $J = \cup J_i$. Then

$$J_1 \cup 4J_2 \cup 4J_3 \cup J_4 = \left[-\frac{32\pi}{7}, -\frac{16\pi}{7}\right) \cup \left[\frac{16\pi}{7}, \frac{32\pi}{7}\right).$$

This has the form $[-2a, a) \cup [b, 2b)$ so is a 2-dilation generator of a partition of $\mathbf{R} \setminus \{0\}$. Then also observe that

$$\{J_1 + 6\pi, J_2 + 2\pi, J_3, J_4 - 4\pi\}$$

is a partition of $[0, 2\pi)$.

(ii) The Littlewood-Paley set can be generalized. For any $-\pi < \alpha < \pi$, the set

$$E_\alpha = [-2\pi + 2\alpha, -\pi + \alpha) \cup [\pi + \alpha, 2\pi + 2\alpha)$$

is a wavelet set. Indeed, it is clearly a 2-dilation generator of a partition of $\mathbf{R} \setminus \{0\}$, and to see that it satisfies the translation congruency criterion for $-\pi < \alpha \leq 0$ (the case $0 < \alpha < \pi$ is analogous) just observe that

$$\{[-2\pi + 2\alpha, 2\pi) + 4\pi, [-2\pi, -\pi + \alpha) + 2\pi, [\pi + \alpha, 2\pi + 2\alpha)\}$$

is a partition of $[0, 2\pi)$. It is clear that ψ_{E_α} is then a continuous (in $L^2(\mathbf{R})$ -norm) path of s -elementary wavelets. Note that

$$\lim_{\alpha \rightarrow \pi} \widehat{\psi}_{E_\alpha} = \frac{1}{\sqrt{2\pi}} \chi_{[2\pi, 4\pi)}.$$

This is *not* the Fourier transform of a wavelet because the set $[2\pi, 4\pi)$ is not a 2-dilation generator of a partition of $\mathbf{R} \setminus \{0\}$. So

$$\lim_{\alpha \rightarrow \pi} \psi_{E_\alpha}$$

is not an orthogonal wavelet. (It is what is known as a Hardy wavelet because it generates an orthonormal basis for $H^2(\mathbf{R})$ under dilation and translation.) This example demonstrates that $\mathcal{W}(D, T)$ is *not* closed in $L^2(\mathbf{R})$.

(iii) Journe's example above can be extended to a path. For $-\frac{\pi}{7} \leq \beta \leq \frac{\pi}{7}$ the set

$$J_\beta = \left[-\frac{32\pi}{7}, -4\pi + 4\beta\right) \cup \left[-\pi + \beta, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi + \beta\right) \cup \left[4\pi + 4\beta, 4\pi + \frac{4\pi}{7}\right)$$

is a wavelet set. The same argument in (i) establishes dilation congruency. For translation, the argument in (i) shows congruency to $[4\beta, 2\pi + 4\beta)$ which is in turn congruent to $[0, 2\pi)$ as required. Observe that here, as opposed to in (ii) above, the limit of ψ_{J_β} as β approaches the boundary point $\frac{\pi}{7}$ is a wavelet. Its wavelet set is a union of 3 disjoint intervals.

(iv) While the Littlewood-Paley and the Journe wavelets sets are *symmetric* by reflection through the origin (modulo the boundary, which is a null set), the paths in (ii) and (iii) consist of non-symmetric sets (except at 0). It is noteworthy that paths of *symmetric* wavelet sets also exist: For example, consider for $0 \leq \alpha \leq \frac{\pi}{3}$,

$$F_\alpha = \left[-\frac{8\pi}{3} + 2\alpha, -2\pi\right) \cup \left[-\frac{4\pi}{3} - 2\alpha, -\frac{4\pi}{3} + \alpha\right) \cup \left[-\pi, -\frac{2\pi}{3} - \alpha\right) \\ \cup \left[\frac{2\pi}{3} + \alpha, \pi\right) \cup \left[\frac{4\pi}{3} - \alpha, \frac{4\pi}{3} + 2\alpha\right) \cup \left[2\pi, \frac{8\pi}{3} - 2\alpha\right).$$

We leave to the reader the (easy) verification that F_α satisfies the dilation and translation congruency criteria so is a wavelet set. Note that $F_{\frac{\pi}{3}}$ is the Littlewood-Paley set. We have

$$F_0 = \left[-\frac{8\pi}{3}, -2\pi\right) \cup \left[-\pi, -\frac{2\pi}{3}\right) \cup \left[\frac{2\pi}{3}, \pi\right) \cup \left[2\pi, \frac{8\pi}{3}\right).$$

(v) The wavelet set

$$\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left[\pi, \frac{5\pi}{4}\right) \cup \left[\frac{7\pi}{4}, 2\pi\right) \cup \left[\frac{5\pi}{2}, 3\pi\right) \cup \left[\frac{13\pi}{4}, \frac{7\pi}{2}\right) \cup \left[6\pi, \frac{13\pi}{2}\right)$$

is the union of 6 disjoint intervals, all but one of which are positive. This illustrates that wavelet sets can be very asymmetric in structure.

(vi) Let $0 < \alpha < \beta < \gamma < \delta < \dots < \frac{\pi}{2}$. The sets of item (ii) admit further "splitting" into multiparameter families of wavelet sets:

$$E_{\alpha\beta} = [-2\pi, -2\pi + 2\alpha) \cup [-2\pi + 2\beta, \pi) \cup [-\pi + \alpha, -\pi + \beta) \\ \cup [\pi, \pi + \alpha) \cup [\pi + \beta, 2\pi) \cup [2\pi + 2\alpha, 2\pi + 2\beta)$$

$$E_{\alpha\beta\gamma} = [-2\pi + 2\alpha, -2\pi + 2\beta) \cup [-2\pi + 2\gamma, -\pi + \alpha) \cup [-\pi + \beta, -\pi + \gamma) \\ \cup [\pi + \alpha, \pi + \beta) \cup [\pi + \gamma, 2\pi + 2\alpha) \cup [2\pi + 2\beta, 2\pi + 2\gamma)$$

$$E_{\alpha\beta\gamma\delta} = [-2\pi, -2\pi + 2\alpha) \cup [-2\pi + 2\beta, -2\pi + 2\gamma) \cup [-2\pi + 2\delta, -\pi) \\ \cup [-\pi + \alpha, -\pi + \beta) \cup [-\pi + \gamma, -\pi + \delta) \cup [\pi, \pi + \alpha) \\ \cup [\pi + \beta, \pi + \gamma) \cup [\pi + \delta, 2\pi) \cup [2\pi + 2\alpha, 2\pi + 2\beta) \\ \cup [2\pi + 2\gamma, 2\pi + 2\delta).$$

This process can be continued. It is perhaps curious that $E_{\alpha\beta}$ and $E_{\alpha\beta\gamma}$ have 6 disjoint intervals, yet $E_{\alpha\beta\gamma\delta}$ has 10.

(vii) Another easily-checked family of wavelet sets is

$$G_{\alpha} = \left[-\frac{8\pi}{3}, -\frac{8\pi}{3} + 2\alpha\right) \cup \left[-\frac{4\pi}{3} + \alpha, -\frac{2\pi}{3}\right) \\ \cup \left[\frac{2\pi}{3}, \frac{2\pi}{3} + \alpha\right) \cup \left[\frac{4\pi}{3} + 2\alpha, \frac{8\pi}{3}\right)$$

for $0 \leq \alpha < \frac{\pi}{3}$.

(viii) Let $A \subseteq [\pi, \frac{3\pi}{2})$ be an arbitrary measurable subset. Then there is a wavelet set W , such that $W \cap [\pi, \frac{3\pi}{2}) = A$. For the construction, let

$$B = [2\pi, 3\pi) \setminus 2A, \\ C = \left[-\pi, -\frac{\pi}{2}\right) \setminus (A - 2\pi) \\ \text{and } D = 2A - 4\pi.$$

Let

$$W = \left[\frac{3\pi}{2}, 2\pi \right) \cup A \cup B \cup C \cup D.$$

We have $W \cap \left[\pi, \frac{3\pi}{2} \right) = A$. Observe that the sets $\left[\frac{3\pi}{2}, 2\pi \right)$, A, B, C, D , are disjoint. Also observe that the sets

$$\left[\frac{3\pi}{2}, 2\pi \right), A, \frac{1}{2}B, 2C, D,$$

are disjoint and have union $[-2\pi, -\pi) \cup [\pi, 2\pi)$. In addition, observe that the sets

$$\left[\frac{3\pi}{2}, 2\pi \right), A, B - 2\pi, C + 2\pi, D + 2\pi,$$

are disjoint and have union $[0, 2\pi)$. Hence W is a wavelet set.

(ix) Let $A \subseteq \left(\frac{8\pi}{3}, 3\pi \right)$ be an arbitrary measurable subset. Then there is a *symmetric* (by reflection through the origin) wavelet set W such that $W \cap \left(\frac{8\pi}{3}, 3\pi \right) = A$. For the construction, let

$$B = -\frac{1}{2}A + 2\pi \quad \text{and} \quad C = [\pi, 2\pi) \setminus \left(2B \cup \frac{1}{2}A \right).$$

We claim that the symmetric set

$$W = -(A \cup B \cup C) \cup (A \cup B \cup C)$$

satisfies our requirements. Observe that the sets A, B, C are disjoint and contained in $(0, \infty)$. Then observe that the sets $\frac{1}{2}A, 2B, C$ are disjoint and have union $[\pi, 2\pi)$, so W is 2-dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ modulo a null set. Then note that

$$\frac{1}{2}A = -B + 2\pi \quad \text{and} \quad 2B = -A + 4\pi.$$

So the sets $-A + 4\pi, -B + 2\pi, C$ are disjoint and have union $[\pi, 2\pi)$, and the sets $A - 4\pi, B - 2\pi, -C$ are disjoint and have union $[-2\pi, -\pi)$. So W is 2π -translation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$, and hence to $[0, 2\pi)$. This shows that W is a wavelet set. By the construction we have $W \cap \left(\frac{8\pi}{3}, 3\pi \right) = A$.

(x) Wavelet sets for arbitrary (not necessarily integral) dilation factors other than 2 exist. For instance, if $d \geq 2$ is arbitrary, let

$$\begin{aligned} A &= \left[-\frac{2d\pi}{d+1}, -\frac{2\pi}{d+1} \right), \\ B &= \left[\frac{2\pi}{d^2-1}, \frac{2\pi}{d+1} \right), \\ C &= \left[\frac{2d\pi}{d+1}, \frac{2d^2\pi}{d^2-1} \right) \end{aligned}$$

and let $G = A \cup B \cup C$. Then G is d -wavelet set. To see this, note that $\{A + 2\pi, B, C\}$ is a partition of an interval of length 2π . So G is 2π -translation-congruent to $[0, 2\pi)$. Also, $\{A, B, d^{-1}C\}$ is a partition of the set $[-d\alpha, -\alpha) \cup [\beta, d\beta)$ for $\alpha = \frac{2\pi}{d^2-1}$, and $\beta = \frac{2\pi}{d^2-1}$, so from this form it follows that $\{d^n G: n \in \mathbf{Z}\}$ is a partition of $\mathbf{R} \setminus \{0\}$. Hence if $\psi := \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_G)$, it follows that $\{d^{\frac{n}{2}}\psi(d^{n/2}t - \ell): n, \ell \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mathbf{R})$, as required.

(xi) There exist unbounded wavelet sets. Let $\{A_n: n = 0, 1, 2, \dots\}$ be a measurable partition of $[\pi, 2\pi)$. Then the sets $\{A_n\}$ are disjoint, and for $n \geq 1$ we have $2^{-n}A_n \subseteq [0, 2\pi)$. Let $B_1 = \bigcup_{n=1}^{\infty} 2^{-n}A_n$. Then let

$$\begin{aligned} B &= \bigcup_{n=0}^{\infty} 2^{-n}A_n \\ C &= [-2\pi, -\pi) \setminus (B_1 - 2\pi) \\ D &= \bigcup_{n=1}^{\infty} 2^n(2^{-n}A_n - 2\pi). \end{aligned}$$

Note that the set D is unbounded. We leave to the reader the verification that

$$W = B \cup C \cup D$$

satisfies the dilation-translation congruency criteria so is a wavelet set.

We note that an example of an unbounded wavelet set (i.e. an MSF-wavelet with unbounded support) was also obtained independently by Fang and Wang in [17]. It is a different example. In [4, 17, 31] the term *band-limited* is used to denote wavelets and other L^2 -functions whose Fourier transforms have bounded support.

The following gives an operator algebraic characterization of MSF (s -elementary) wavelets. The proof is contained in [9, Chapter 4], and was more difficult than might be expected. It was very difficult for us to obtain, partly because that we did not, at first, believe that it was true. It is perhaps good for perspective in this present context. It is an open question whether the analogue of this is true in n -dimensions.

Theorem 4.6. *Let $\psi \in \mathcal{W}(\mathcal{U}_{D,T})$. Let V_ψ be the unique unitary operator in $\mathcal{C}_\psi(\mathcal{U}_{D,T})$ with $V_\psi\psi = T\psi$. Then $TV_\psi = V_\psi T$ if and only if*

$$|\widehat{\psi}| = \frac{1}{\sqrt{2\pi}}\chi_E$$

for some wavelet set E .

5. Operator-Theoretic Interpolation of Wavelets

If ψ, η are wavelets let $V := V_\psi^\eta$ be the (unique) unitary operator in $\mathcal{C}_\psi(D, T)$ with $V\psi = \eta$. We call this the *interpolation unitary* for the or-

dered pair (ψ, η) . Suppose that V normalizes $\{D, T\}'$ in the sense that

$$V^*(\{D, T\}')V = \{D, T\}'. \quad (43)$$

In this case the algebra, before closure, generated by $\{D, T\}'$ and V is the set of all finite sums (polynomials) of the form $\sum A_n V^n$, with coefficients $A_n \in \{D, T\}'$. The closure in the strong operator topology is a von Neumann algebra. Now suppose further that *every power* of V is contained in $\mathcal{C}_\psi(D, T)$. This occurs only in special cases, yet it occurs frequently enough to yield some general methods. Then since $\mathcal{C}_\psi(D, T)$ is a SOT-closed linear subspace which is closed under left multiplication by $\{D, T\}'$ this von Neumann algebra is contained in $\mathcal{C}_\psi(D, T)$, so its unitary group parameterizes a path-connected subset of $\mathcal{W}(D, T)$ that contains ψ and η via the correspondence $U \rightarrow U\psi$. We say that wavelets in this set are *interpolated* from (ψ, η) , and that (ψ, η) admits operator-theoretic interpolation.

It turns out that if ψ and η are s -elementary wavelets, then indeed V_ψ^η normalizes $\{D, T\}'$. Moreover, V_ψ^η has a very special form: after conjugating with the Fourier transform, it is a composition operator with symbol a a natural and very computable measure-preserving transformation of \mathbf{R} . In fact, it is precisely this special form for V_ψ^η that allows us to make the computation that it normalizes $\{D, T\}'$. On the other hand, we know of no pair (ψ, η) of wavelets for which V_ψ^η fails to normalize $\{D, T\}'$. The difficulty is simply that in general it is very hard to do the computations.

Problem 5.A. If $\{\psi, \eta\}$ is a pair of dyadic orthonormal wavelets, does the interpolation unitary V_ψ^η normalize $\{D, T\}'$? As mentioned above, the answer is yes if ψ and η are s -elementary wavelets.

Let E and F be arbitrary wavelet sets. Let $\sigma: E \rightarrow F$ be the 1-1, onto map implementing the 2π -translation congruence. Since E and F both generate partitions of $\mathbf{R} \setminus \{0\}$ under dilation by powers of 2, we may extend σ to a 1-1 map of \mathbf{R} onto \mathbf{R} by defining $\sigma(0) = 0$, and

$$\sigma(s) = 2^n \sigma(2^{-n}s) \quad \text{for } s \in 2^n E, \quad n \in \mathbf{Z}. \quad (44)$$

We adopt the notation σ_E^F for this, and call it the *interpolation map* for the ordered pair (E, F) .

Lemma 5.1. In the above notation, σ_E^F is a measure-preserving transformation from \mathbf{R} onto \mathbf{R} .

Proof. Let $\sigma := \sigma_E^F$. Let $\Omega \subseteq \mathbf{R}$ be a measurable set. Let $\Omega_n = \Omega \cap 2^n E$, $n \in \mathbf{Z}$, and let $E_n = 2^{-n}\Omega_n \subseteq E$. Then $\{\Omega_n\}$ is a partition of Ω , and we have $m(\sigma(E_n)) = m(E_n)$ because the restriction of σ to E is measure-preserving.

So

$$\begin{aligned}
m(\sigma(\Omega)) &= \sum_n m(\sigma(\Omega_n)) = \sum_n m(2^n \sigma(E_n)) \\
&= \sum_n 2^n m(\sigma(E_n)) = \sum_n 2^n m(E_n) \\
&= \sum_n m(2^n E_n) = \sum_n m(\Omega_n) = m(\Omega). \quad \blacksquare
\end{aligned}$$

A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is called *2-homogeneous* if $f(2s) = 2f(s)$ for all $s \in \mathbf{R}$. Equivalently, f is 2-homogeneous iff $f(2^n s) = 2^n f(s)$, $s \in \mathbf{R}$, $n \in \mathbf{Z}$. Such a function is completely determined by its values on any subset of \mathbf{R} which generates a partition of $\mathbf{R} \setminus \{0\}$ by 2-dilation. So σ_E^F is the (unique) 2-homogeneous extension of the 2π -transition congruence $E \rightarrow F$. The set of all 2-homogeneous measure-preserving transformations of \mathbf{R} clearly forms a group under composition. Also, the composition of a 2-dilation-periodic function f with a 2-homogeneous function g is (in either order) 2-dilation periodic. We have $f(g(2s)) = f(2g(s)) = f(g(s))$ and $g(f(2s)) = g(f(s))$. These facts will be useful.

Now let

$$U_E^F := U_{\sigma_E^F}, \quad (45)$$

where if σ is any measure-preserving transformation of \mathbf{R} then U_σ denotes the composition operator defined by $U_\sigma f = f \circ \sigma^{-1}$, $f \in L^2(\mathbf{R})$. Clearly $(\sigma_E^F)^{-1} = \sigma_F^E$ and $(U_E^F)^* = U_F^E$. We have $U_E^F \hat{\psi}_E = \hat{\psi}_F$ since $\sigma_E^F(E) = F$. That is,

$$U_E^F \hat{\psi}_E = \hat{\psi}_E \circ \sigma_F^E = \frac{1}{\sqrt{2\pi}} \chi_E \circ \sigma_F^E = \frac{1}{\sqrt{2\pi}} \chi_F = \hat{\psi}_F.$$

Proposition 5.2. Let E and F be arbitrary wavelet sets. Then $U_E^F \in \mathcal{C}_{\hat{\psi}_E}(\hat{D}, \hat{T})$. Hence $\mathcal{F}^{-1} U_E^F \mathcal{F}$ is the interpolation unitary for the ordered pair (ψ_E, ψ_F) .

Proof. Write $\sigma = \sigma_E^F$ and $U_\sigma = U_E^F$. We have $U_\sigma \hat{\psi}_E = \hat{\psi}_F$ since $\sigma(E) = F$. We must show

$$U_\sigma \hat{D}^n \hat{T}^\ell \hat{\psi}_E = \hat{D}^n \hat{T}^\ell U_\sigma \hat{\psi}_E, \quad , n, \ell \in \mathbf{Z}.$$

We have

$$\begin{aligned}
(U_\sigma \hat{D}^n \hat{T}^\ell \hat{\psi}_E)(s) &= (U_\sigma \hat{D}^n e^{-i\ell s} \hat{\psi}_E)(s) \\
&= U_\sigma 2^{-\frac{n}{2}} e^{-i\ell 2^{-n} s} \hat{\psi}_E(2^{-n} s) \\
&= 2^{-\frac{n}{2}} e^{-i\ell 2^{-n} \sigma^{-1}(s)} \hat{\psi}_E(2^{-n} \sigma^{-1}(s)) \\
&= 2^{-\frac{n}{2}} e^{-i\ell \sigma^{-1}(2^{-n} s)} \hat{\psi}_E(\sigma^{-1}(2^{-n} s))
\end{aligned}$$

$$= 2^{-\frac{n}{2}} e^{-i\ell\sigma^{-1}(2^{-n}s)} \widehat{\psi}_F(2^{-n}s).$$

This last term is nonzero iff $2^{-n}s \in F$, in which case $\sigma^{-1}(2^{-n}s) = \sigma_F^E(2^{-n}s) = 2^{-n}s + 2\pi k$ for some $k \in \mathbf{Z}$ since σ_F^E is a 2π -translation-congruence on F . It follows that $e^{-i\ell\sigma^{-1}(2^{-n}s)} = e^{-i\ell 2^{-n}s}$. Hence we have

$$\begin{aligned} (U_\sigma \widehat{D}^n \widehat{T}^\ell \widehat{\psi}_E)(s) &= 2^{-\frac{n}{2}} e^{-i\ell 2^{-n}s} \widehat{\psi}_F(2^{-n}s) \\ &= (\widehat{D}^n \widehat{T}^\ell \widehat{\psi}_F)(s) \\ &= (\widehat{D}^n \widehat{T}^\ell U_\sigma \widehat{\psi}_E)(s). \end{aligned}$$

We have shown $U_E^F \in \mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$. Since $U_E^F \widehat{\psi}_E = \widehat{\psi}_F$, the uniqueness part of Proposition 2.1 shows that $\mathcal{F}^{-1} U_E^F \mathcal{F}$ must be the interpolation unitary for (ψ_E, ψ_F) . ■

Proposition 5.3. *Let E and F be arbitrary wavelet sets. Then the interpolation unitary for the ordered pair (ψ_E, ψ_F) normalizes $\{D, T\}'$.*

Proof. By Proposition 5.2 we may work with U_E^F in the Fourier transform domain. By Theorem 3.2, the generic element of $\{\widehat{D}, \widehat{T}\}'$ has the form M_h for some 2-dilation-periodic function $h \in L^\infty(\mathbf{R})$. Write $\sigma = \sigma_E^F$ and $U_\sigma = U_E^F$. Then

$$U_\sigma^{-1} M_h U_\sigma = M_{h \circ \sigma^{-1}}. \quad (46)$$

So since the composition of a 2-dilation-periodic function with a 2-homogeneous function is 2-dilation-periodic, the proof is complete. ■

It can also be shown ([9, Theorem 5.2 (iii)]) that if E, F are wavelet sets with $E \neq F$ then U_E^F is not contained in the double commutant $\{\widehat{D}, \widehat{T}\}''$. So since U_E^F and $\{\widehat{D}, \widehat{T}\}'$ are both contained in the local commutant of $\mathcal{U}_{\widehat{D}, \widehat{T}}$ at $\widehat{\psi}_E$, this proves that $\mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$ is nonabelian. In fact (see [9, Proposition 1.8]) this can be used to show that $\mathcal{C}_\psi(D, T)$ is nonabelian for every wavelet ψ . We suspected this, but we could not prove it until we discovered the “right” way of doing the needed computations using s -elementary wavelets.

The above shows that a pair (E, F) of wavelet sets (or, rather, their corresponding s -elementary wavelets) admits operator-theoretic interpolation if and only if $\text{Group}\{U_E^F\}$ is contained in the local commutant $\mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$, since the requirement that U_E^F normalizes $\{\widehat{D}, \widehat{T}\}'$ is automatically satisfied. It is easy to see that this is equivalent to the condition that for each $n \in \mathbf{Z}$, σ^n is a 2π -congruence of E in the sense that $(\sigma^n(s) - s)/2\pi \in \mathbf{Z}$ for all $s \in E$, which in turn implies that $\sigma^n(E)$ is a wavelet set for all n . Here $\sigma = \sigma_E^F$. This property holds trivially if σ is *involutive* (i.e. $\sigma^2 = \text{identity}$).

In cases where "torsion" is present, so $(\sigma_E^F)^k$ is the identity map for some finite integer k , the von Neumann algebra generated by $\{\widehat{D}, \widehat{T}\}'$ and $U := U_E^F$ has the simple form

$$\left\{ \sum_{n=0}^k M_{h_n} U^n : h_n \in L^\infty(\mathbf{R}) \text{ with } h_n(2s) = h_n(s), \quad s \in \mathbf{R} \right\},$$

and so each member of this "interpolated" family of wavelets has the form

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^k h_n(s) \chi_{\sigma^n(E)} \quad (47)$$

for 2-dilation periodic "coefficient" functions $\{h_n(s)\}$ which satisfy the necessary and sufficient condition that the operator

$$\sum_{n=0}^k M_{h_n} U^n \quad (48)$$

is unitary.

A standard computation shows that the map θ sending $\sum_0^k M_{h_n} U^n$ to the $k \times k$ function matrix (h_{ij}) given by

$$h_{ij} = h_{\alpha(i,j)} \circ \sigma^{-i+1} \quad (49)$$

where $\alpha(i, j) = (i+1)$ modulo k , is a $*$ -isomorphism. This matricial algebra is the cross-product of $\{D, T\}'$ by the $*$ -automorphism $ad(U_E^F)$ corresponding to conjugation with U_E^F . For instance, if $k = 3$ then θ maps

$$M_{h_1} + M_{h_2} U_E^F + M_{h_3} (U_E^F)^2$$

to

$$\begin{pmatrix} h_1 & h_2 & h_3 \\ h_3 \circ \sigma^{-1} & h_1 \circ \sigma^{-1} & h_2 \circ \sigma^{-1} \\ h_2 \circ \sigma^{-2} & h_3 \circ \sigma^{-2} & h_1 \circ \sigma^{-2} \end{pmatrix}. \quad (50)$$

This shows that $\sum_0^k M_{h_n} U^n$ is a unitary operator iff the scalar matrix $(h_{ij})(s)$ is unitary for almost all $s \in \mathbf{R}$. Unitarity of this matrix-valued function is called the *Coefficient Criterion* in [9], and the functions h_i are called the interpolation coefficients. This leads to formulas for families of wavelets which are new to wavelet theory. They are $*$ -wavelet bundles in the sense of section 3. (See (29).)

If E is a wavelet set, define an *interpolation family* of wavelet sets based at E to be a family \mathcal{F} of wavelet sets, with $E \in \mathcal{F}$, with the property that

$\{\sigma_E^F: F \in \mathcal{J}\}$ is a group under composition. Then $\mathcal{G}_{\mathcal{F}} := \{U_E^F: F \in \mathcal{F}\}$ is a group of unitaries in $\mathcal{C}_{\psi_E}(\widehat{D}, \widehat{T})$ which is isomorphic to the group $\{\sigma_E^F: F \in \mathcal{F}\}$ of measure-preserving transformations of \mathbf{R} , and every unitary in $\mathcal{G}_{\mathcal{F}}$ normalizes $\{\widehat{D}, \widehat{T}\}'$. So the von Neumann algebra generated by $\{\widehat{D}, \widehat{T}\}'$ and $\mathcal{G}_{\mathcal{F}}$ lies in $\mathcal{C}_{\psi_E}(\widehat{D}, \widehat{T})$, hence its unitary group yields a $*$ -wavelet bundle (29). In [9], Problem E asked which groups were attainable in this way. Gu partially answered this question by showing that every finite group is attainable. He showed that for each n , the permutation group S_n on n generators is attainable. So the cross-product of $\{D, T\}'$ by any finite group is attainable as a subalgebra of $\mathcal{C}_{\psi}(D, T)$ for some wavelet ψ . This was also discussed previously just before Problem 3.C.

The involutive ($k = 2$) case seems to be common. Let us say that a pair of wavelet sets (E, F) is an *interpolation pair* if $(\sigma_E^F)^2 = \text{identity}$. In this case $\sigma_E^F = \sigma_F^E$.

Example 5.4. Let $E = [-\frac{8\pi}{3}, -\frac{4\pi}{3}] \cup [\frac{2\pi}{3}, \frac{4\pi}{3}]$ and $F = [-\frac{4\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{4\pi}{3}, \frac{8\pi}{3}]$. Then (E, F) is an interpolation pair, and $E \cup F = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}]$, which is the support set of Meyer's well-known class of (Fourier transforms of) wavelets. To see that (E, F) is an interpolation pair, note that

$$\begin{aligned} \left[-\frac{4\pi}{3}, -\frac{2\pi}{3}\right] &= \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] - 2\pi = \frac{1}{2} \left[-\frac{8\pi}{3}, -\frac{4\pi}{3}\right] \quad \text{and} \\ \left[\frac{4\pi}{3}, \frac{8\pi}{3}\right] &= \left[-\frac{8\pi}{3}, -\frac{4\pi}{3}\right] + 4\pi = 2 \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right], \end{aligned}$$

so

$$\sigma_E^F(s) = \begin{cases} s - 2\pi & \text{on } \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \\ s + 4\pi & \text{on } \left[-\frac{8\pi}{3}, -\frac{4\pi}{3}\right]. \end{cases} \quad (51)$$

Hence for $s \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$, $2\sigma(s) \in \left[-\frac{8\pi}{3}, -\frac{4\pi}{3}\right]$, so

$$\sigma^2(s) = \frac{1}{2}\sigma(2\sigma(s)) = \frac{1}{2}[2\sigma(s) + 4\pi] = \frac{1}{2}[2(s - 2\pi) + 4\pi] = s.$$

Similarly, $\sigma^2(s) = s$ on $\left[-\frac{8\pi}{3}, -\frac{4\pi}{3}\right]$. So since σ^2 is 2-homogeneous, this implies $\sigma^2 = \text{id}$ on \mathbf{R} .

The above is a special case ($\alpha = \pi$) of the following:

Example 5.5. Let $E = \left[-\frac{8}{3}\alpha, -\frac{4}{3}\alpha\right] \cup \left[2\pi - \frac{4}{3}\alpha, 4\pi - \frac{8}{3}\alpha\right]$ and $F = \left[-\frac{4}{3}\alpha, -\frac{2}{3}\alpha\right] \cup \left[2\pi - \frac{2}{3}\alpha, 4\pi - \frac{4}{3}\alpha\right]$ for some $0 \leq \alpha \leq \pi$. Partition these by

$$E_1 = \left[-\frac{8}{3}\alpha, -\frac{4}{3}\alpha\right], E_2 = \left[2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha\right],$$

$$E_3 = \left[2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha \right) \text{ and } F_1 = \left[-\frac{4}{3}\alpha, -\frac{2}{3}\alpha \right),$$

$$F_2 = \left[2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha \right), F_3 = \left[4\pi - \frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha \right).$$

It is easy to see that these are wavelet sets: E clearly generates a partition of $\mathbf{R} \setminus \{0\}$ under dilation by 2, and since $\{E_1 + 2\pi, E_2, E_3\}$ is a partition of $\left[2\pi - \frac{8}{3}\alpha, 4\pi - \frac{8}{3}\alpha \right)$, and this latter set generates a partition of \mathbf{R} under 2π -translation, so does E . Thus E is a wavelet set by the dilation-translation congruency criteria. Similarly, F is a wavelet set. Now consider the relations

$$F_1 = E_2 - 2\pi = \frac{1}{2}E_1$$

$$F_2 = E_3$$

$$F_3 = E_1 + 4\pi = 2E_2.$$

Thus on E ,

$$\sigma_E^F(s) = \begin{cases} s + 4\pi, & s \in E_1 \\ s - 2\pi, & s \in E_2 \\ s, & s \in E_3. \end{cases} \quad (52)$$

If $s \in E_1$, then $\sigma(s) = s + 4\pi \in 2E_2$, so $\sigma^2(s) = 2\sigma\left(\frac{1}{2}\sigma(s)\right) = 2\left[\frac{1}{2}\sigma(s) - 2\pi\right] = \sigma(s) - 4\pi = s$. If $s \in E_2$, then $\sigma(s) = s - 2\pi \in \frac{1}{2}E_1$, so $\sigma^2(s) = \frac{1}{2}\sigma(2\sigma(s)) = \frac{1}{2}[2\sigma(s) + 4\pi] = \sigma(s) + 2\pi = s$. And if $s \in E_3$ then $\sigma^2(s) = \sigma(s) = s$. So $\sigma^2 = id$ on E , hence on \mathbf{R} . So (E, F) is an interpolation pair. Observe that $E \cup F = \left[-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha\right) \setminus H_\alpha$, where H_α is the "hole" of Theorem 5.8. ■

The Coefficient Criterion for the case $k = 2$ yields:

Proposition 5.6. *If (E, F) is an interpolation pair then*

$$\widehat{\psi}(s) = h_1(s)\widehat{\psi}_E(s) + h_2(s)\widehat{\psi}_F(s) \quad (53)$$

is the Fourier Transform of an orthogonal wavelet whenever h_1 and h_2 are 2-dilation-periodic functions on \mathbf{R} with the property that the matrix

$$H(s) := \begin{pmatrix} h_1 & h_2 \\ h_2 \circ \sigma_E^F & h_1 \circ \sigma_E^F \end{pmatrix} \quad (54)$$

is unitary (a.e.). Moreover, unitarity a.e. of $H(s)$ on \mathbf{R} is equivalent to unitarity of $H(s)$ on any measurable set $G \subseteq \mathbf{R}$ which has the property that

$$\bigcup_{n \in \mathbf{Z}} (G \cup \sigma_E^F(G)) = \mathbf{R} \quad (55)$$

modulo a null set. In particular, it is sufficient to check (54) on E or F .

Proof. We need only prove the reduction to G . Since h_1, h_2 are 2-dilation-periodic we have, for a.e. $s \in \mathbf{R}$,

$$H(2^n s) = H(s), n \in \mathbf{Z}, \quad \text{and} \quad H(2^n \sigma_E^F(s)) = H(\sigma_E^F(s)) =$$

$$\begin{pmatrix} h_1(\sigma_E^F(s)) & h_2(\sigma_E^F(s)) \\ h_2(\sigma_E^F(\sigma_E^F(s))) & h_1(\sigma_E^F(\sigma_E^F(s))) \end{pmatrix}$$

which upon using the identity $\sigma_E^F \circ \sigma_E^F = \text{identity}$ and interchanging rows and columns yields $H(s)$. So unitarity of $H(s)$ on G implies unitarity on $\bigcup_{n \in \mathbf{Z}} (G \cup \sigma_E^F(G)) = \mathbf{R}$. ■

We shall show that Meyer's (family of) wavelets have the above form. Meyer's class is (cf. [14], p. 117):

$$\hat{\psi}_{Me}(s) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{\frac{is}{2}} \cos \left[\frac{\pi}{2} \nu \left(-\frac{3s}{4\pi} - 1 \right) \right], & s \in \left[-\frac{8\pi}{3}, -\frac{4\pi}{3} \right), \\ \frac{1}{\sqrt{2\pi}} e^{\frac{is}{2}} \sin \left[\frac{\pi}{2} \nu \left(-\frac{3s}{2\pi} - 1 \right) \right], & s \in \left[-\frac{4\pi}{3}, -\frac{2\pi}{3} \right) \\ \frac{1}{2\pi} e^{\frac{is}{2}} \sin \left[\frac{\pi}{2} \nu \left(\frac{3s}{2\pi} - 1 \right) \right], & s \in \left[\frac{2\pi}{3}, \frac{4\pi}{3} \right) \\ \frac{1}{\sqrt{2\pi}} e^{\frac{is}{2}} \cos \left[\frac{\pi}{2} \nu \left(\frac{3s}{4\pi} - 1 \right) \right], & s \in \left[\frac{4\pi}{3}, \frac{8\pi}{3} \right) \\ 0 & \text{otherwise} \end{cases} \quad (56)$$

for $s \in \mathbf{R}$, where ν is a real-valued function which satisfies the relation

$$\nu(s) + \nu(1-s) = 1, \quad s \in \mathbf{R}.$$

Normally, one chooses ν so that $\hat{\psi}_{Me}$ has desired regularity properties. If ν is taken with $\nu(s) = 0$ for $s \leq 0$ and $\nu(s) = 1$ for $s \geq 1$, then if ν is continuous, or in class C^k , or C^∞ , then the function $\hat{\psi}_{Me}$ is in the same class. Any choice of a measurable real valued function ν satisfying $\nu(s) + \nu(1-s) = 1$ yields a (perhaps "badly behaved") wavelet, however.

Proposition 5.7. *The wavelets $\hat{\psi}_{Me}$ have the interpolation form (53).*

Proof. The support of $\hat{\psi}_{Me}$ is $E \cup F$, where E and F are as in Example 5.4. Define h_1 on E by

$$h_1(s) = \begin{cases} e^{\frac{is}{2}} \cos \left[\frac{\pi}{2} \nu \left(-\frac{3s}{4\pi} - 1 \right) \right], & s \in \left[-\frac{8\pi}{3}, -\frac{4\pi}{3} \right) \\ e^{\frac{is}{2}} \sin \left[\frac{\pi}{2} \nu \left(\frac{3s}{2\pi} - 1 \right) \right], & s \in \left[\frac{2\pi}{3}, \frac{4\pi}{3} \right) \end{cases} \quad (57)$$

and extend h_1 to \mathbf{R} 2-dilation-periodically. (That is, set $h_1(s) = h_1(2^{-n}s)$ if $s \in 2^n E$, $n \in \mathbf{Z}$, and set $h(0) = 0$.) Similarly, define h_2 on F by

$$h_2(s) = \begin{cases} e^{\frac{is}{2}} \sin \left[\frac{\pi}{2} \nu \left(-\frac{3s}{2\pi} - 1 \right) \right], & s \in \left[-\frac{4\pi}{3}, -\frac{2\pi}{3} \right) \\ e^{\frac{is}{2}} \cos \left[\frac{\pi}{2} \nu \left(\frac{3s}{4\pi} - 1 \right) \right], & s \in \left[\frac{4\pi}{3}, \frac{8\pi}{3} \right) \end{cases} \quad (58)$$

and extend to \mathbb{R} 2-dilation periodically. Then

$$\widehat{\psi}(s) = h_1(s)\widehat{\psi}_E(s) + h_2(s)\widehat{\psi}_F(s). \quad (59)$$

Let $G = [-\frac{8\pi}{3}, -\frac{4\pi}{3}]$. Then $\sigma_E^F(G) = [-\frac{8\pi}{3}, -\frac{4\pi}{3}] + 4\pi = [\frac{4\pi}{3}, \frac{8\pi}{3}]$, so G satisfies the hypotheses of Proposition 5.6. For $s \in [-\frac{8\pi}{3}, -\frac{4\pi}{3}]$ we have

$$H(s) = \begin{pmatrix} h_1(s) & h_2(s) \\ h_2(\sigma_E^F(s)) & h_1(\sigma_E^F(s)) \end{pmatrix} = \begin{pmatrix} h_1(s) & h_2(s) \\ h_2(s+4\pi) & h_1(s+4\pi) \end{pmatrix},$$

and $s+4\pi \in [\frac{4\pi}{3}, \frac{8\pi}{3}]$. Using 2-dilation-periodicity we have

$$h_1(s) = e^{\frac{is}{4}} \sin \left[\frac{\pi}{2} \nu \left(\frac{3s}{4\pi} - 1 \right) \right], \quad s \in \left[\frac{4\pi}{3}, \frac{8\pi}{3} \right]$$

and

$$h_2(s) = e^{\frac{is}{4}} \sin \left[\frac{\pi}{2} \nu \left(-\frac{3s}{4\pi} - 1 \right) \right], \quad s \in \left[-\frac{8\pi}{3}, -\frac{4\pi}{3} \right].$$

So

$$\begin{aligned} H(s) &= \begin{pmatrix} e^{\frac{is}{2}} \cos \left[\frac{\pi}{2} \nu \left(-\frac{3s}{4\pi} - 1 \right) \right] & e^{\frac{is}{4}} \sin \left[\frac{\pi}{2} \nu \left(-\frac{3s}{4\pi} - 1 \right) \right] \\ e^{\frac{i(s+4\pi)}{2}} \cos \left[\frac{\pi}{2} \nu \left(\frac{3(s+4\pi)}{4\pi} - 1 \right) \right] & e^{\frac{i(s+4\pi)}{4}} \sin \left[\frac{\pi}{2} \nu \left(\frac{3(s+4\pi)}{4\pi} - 1 \right) \right] \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{is}{2}} \cos \left[\frac{\pi}{2} \nu \left(-\frac{3s}{4\pi} - 1 \right) \right] & e^{\frac{is}{4}} \sin \left[\frac{\pi}{2} \nu \left(-\frac{3s}{4\pi} - 1 \right) \right] \\ e^{\frac{is}{2}} \cos \left[\frac{\pi}{2} \nu \left(\frac{3s}{4\pi} + 2 \right) \right] & -e^{\frac{is}{4}} \sin \left[\frac{\pi}{2} \nu \left(\frac{3s}{4\pi} + 2 \right) \right] \end{pmatrix}. \end{aligned}$$

Now consider the equation $\nu(s) + \nu(-s) = 1$. This implies $\frac{\pi}{2} \nu \left(\frac{3s}{4\pi} + 2 \right) = \frac{\pi}{2} - \nu \left(-\frac{3s}{4\pi} - 1 \right)$. Let $\theta(s) = \frac{\pi}{2} \nu \left(-\frac{3s}{4\pi} - 1 \right)$. Then

$$H(s) = \begin{pmatrix} e^{\frac{is}{2}} \cos \theta & e^{\frac{is}{4}} \sin \theta \\ e^{\frac{is}{2}} \cos \left(\frac{\pi}{2} - \theta \right) & -e^{\frac{is}{4}} \sin \left(\frac{\pi}{2} - \theta \right) \end{pmatrix} = \begin{pmatrix} e^{\frac{is}{2}} \cos \theta & e^{\frac{is}{4}} \sin \theta \\ e^{\frac{is}{2}} \sin \theta & -e^{\frac{is}{4}} \cos \theta \end{pmatrix}$$

which is unitary. Hence $\widehat{\psi}_{Me}$ satisfies (53). \blacksquare

Beginning with the operator-theoretic interpolation result [9, Prop. 5.4], which is also Proposition 5.6 in this present article, together with a generalization to our setting we obtained of a result of Hernandez, Wang and Weiss [29], we have recently [23] settled affirmatively Problem F in [9] which asked for a converse of Proposition 5.6.

Theorem 5.8. [29, Thm. 2.1.] *Let $\psi \in L^2(\mathbb{R})$ with support $(\widehat{\psi}) \subseteq [-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$ for some $0 < \alpha < \pi$. Then $\widehat{\psi}$ is an orthonormal wavelet if and only if (a.e.):*

- (i) $|\widehat{\psi}(s)|^2 + |\widehat{\psi}(\frac{s}{2})|^2 = \frac{1}{2\pi}$ on $[4\pi - \frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha)$,
- (ii) $|\widehat{\psi}(s)| = \frac{1}{\sqrt{2\pi}}$ on $[2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha)$,
- (iii) $|\widehat{\psi}(s)|^2 + |\widehat{\psi}(s + 2\pi)|^2 = \frac{1}{2\pi}$ on $[-\frac{4}{3}\alpha, -\frac{2}{3}\alpha)$,
- (iv) $|\widehat{\psi}(s)| = |\widehat{\psi}(\frac{1}{2}s + 2\pi)|$ on $[-\frac{8}{3}\alpha, -\frac{4}{3}\alpha)$,
- (v) $\widehat{\psi}(s) = e^{i \cdot p(s)} |\widehat{\psi}(s)|$, where the phase function $p(s)$ satisfies $p(s) + p(2(s - 2\pi)) - p(2s) - p(s - 2\pi) = (2n(s) + 1)\pi$ on $[2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha) \cap (\text{supp } \widehat{\psi}) \cap (\frac{1}{2} \text{supp } \widehat{\psi})$ for some integer-valued measurable function $n(s)$, and
- (vi) $\widehat{\psi}(s) = 0$ on $H_\alpha = [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha)$.

We remark that in [29] the form of the Fourier transform used is $\widehat{f}(s) = \int_{\mathbf{R}} e^{-ist} f(t) dt$, and in the present article the normalization factor $\frac{1}{\sqrt{2\pi}}$ is used to make \mathcal{F} unitary. The appropriate numbers in the statement of Theorem 5.8 have been changed to reflect this.

We observed that Theorem 5.8 appeared to be close in content, if not in form, to Proposition 5.6. Indeed, an inspection of Theorem 5.8 suggests that its criteria might be interpreted as a unitarity requirement for a certain 2×2 matrix-valued function, even though its manner of proof is not suggestive of such an interpretation. This turns out to be correct. The family of wavelets in Theorem 5.8 is indeed a $*$ -wavelet bundle (defined in (29)).

If E is a wavelet set, let τ_E be the projection from \mathbf{R} onto E determined by 2π -translation. (That is, for a.e. $s \in \mathbf{R}$ there exists a unique $\tilde{s} \in E$ with $(s - \tilde{s})/2\pi \in \mathbf{Z}$; define $\tau_E(s) = \tilde{s}$.) Also, let δ_E be the projection from $\mathbf{R} \setminus \{0\}$ onto E determined by 2-dilation. (That is, for a.e. $s \in \mathbf{R} \setminus \{0\}$ there exists a unique $s' \in E$ with $s = 2^n s'$ for some $n \in \mathbf{Z}$; define $\delta_E(s) = s'$.) Then for E and F wavelet sets, the interpolation map σ_E^F takes the form

$$\sigma_E^F(s) = \begin{cases} \tau_F(s) & \text{if } s \in E \\ 2^n \tau_F(2^{-n}s) & \text{if } s \in 2^n E \\ 0 & \text{if } s = 0. \end{cases} \quad (60)$$

The main result of [23] is the following criterion, which improves Proposition 5.6, and generalizes Theorem 5.8 to the union of an arbitrary interpolation pair of wavelets sets. It is a complete characterization of the wavelets in the $*$ -wavelet bundle determined by E and F . Not all $*$ -wavelet bundles look like this. Most are going to be much more complicated. This one "arises" from the cyclic group of order 2. As indicated earlier after (50), other $*$ -wavelet bundles can "arise" likewise from arbitrary finite groups, and probably many infinite groups. As was noted in Chapter 3, (61) is the criterion that is most easily used in verifying examples such as Examples 3.8 and 3.9.

Theorem 5.9. ([23, Theorem 1].) Let (E, F) be an interpolation pair of wavelet sets, and let $\psi \in L^2(\mathbb{R})$ satisfy $\text{support}(\hat{\psi}) \subseteq E \cup F$. Then ψ is an orthogonal wavelet if and only if $|\hat{\psi}(s)| = \frac{1}{\sqrt{2\pi}}$ for a.e. $s \in E \cap F$ and the 2×2 matrix-valued function

$$\sqrt{2\pi} \begin{pmatrix} \hat{\psi} & \hat{\psi} \circ \delta_F \\ \hat{\psi} \circ \tau_F & \hat{\psi} \circ \delta_E \circ \tau_E \end{pmatrix} \quad (61)$$

is unitary a.e. on $E \setminus F$.

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