

INTERPOLATION MAPS AND CONGRUENCE DOMAINS FOR WAVELET SETS

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ABSTRACT. It is proven that if an interpolation map between two wavelet sets preserves the union of the sets, then the pair must be an interpolation pair. We also construct an example of a pair of wavelet sets for which the congruence domains of the associated interpolation map and its inverse are equal, and yet the pair is not an interpolation pair. The first result solves affirmatively a problem that the second author had posed several years ago, and the second result solves an intriguing problem of D. Han. The key to this counterexample is a special technical lemma on constructing wavelet sets. Several other applications of this result are also given. In addition, some problems are posed. We also take the opportunity to give some general exposition on wavelet sets and operator-theoretic interpolation of wavelets.

Dedicated to Larry Baggett for his great friendship, his love of mathematics, and his continued support of young mathematicians.

1. INTRODUCTION

An orthonormal wavelet is a single function ψ in $\mathcal{L}^2(\mathbb{R}^n)$ whose translates by all members of a full-rank lattice followed by dilates by all integral powers of a real expansive matrix on \mathbb{R}^n generates an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^n)$. By the term *wavelet set* we mean a measurable subset $E \subset \mathbb{R}^n$ with the property that the inverse Fourier Transform of its normalized characteristic function, $\mathcal{F}^{-1}(\frac{1}{(2\pi)^{\frac{n}{2}}} \chi_E)$, is an orthonormal wavelet. In [DL] an operator-theoretic technique for working with certain problems concerning wavelets was introduced that was called *operator-theoretic interpolation*. If ψ and η are orthonormal wavelets in the same space, and $\psi_{n,l}$ and $\eta_{n,l}$ are the corresponding wavelet bases, then the unitary operator determined by the mapping $\psi_{n,l}$ to $\eta_{n,l}$ was called the *interpolation unitary* between ψ and η . These interpolation operators associated with ordered pairs of wavelets play an essential role in the theory. They are associated with the von Neumann subalgebras of the so-called *local commutant* space, whose unitary groups provide natural parameterizations of certain families of wavelets. In the special case where the interpolation operator is involutive (i.e. has square I) the pair of wavelets is

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called an *interpolation pair* of wavelets. One surprising feature of the theory is that interpolation pairs occur not infrequently.

In general, interpolation unitaries can be hard to work with. However, in the special case where ψ and η are *s-elementary wavelets* (also called *MSF-wavelets* with phase 0, or *wavelet-set wavelets*), the interpolation unitary takes the form of a *composition operator* with measure-preserving symbol. Every pair of wavelet sets gives rise to a measure preserving transformation on the underlying measure space in a natural way, called the *interpolation map* determined by the pair. The interpolation theory for such wavelets and their associated wavelet sets is, in many concrete cases, computable by hands-on experimental paper-and-pencil computations. This permits experimentation in the form of testing of hypotheses in potential theorems for more general types of wavelets. The simplest case is where a pair of wavelet sets has the property that the measure preserving transformation σ is an involution (i.e. $\sigma \circ \sigma = id$). In this case the composition unitary has square I , so the pair of wavelets is indeed an interpolation pair. The pair of wavelet sets is, by analogy, called an *interpolation pair of wavelet sets*. More generally, an interpolation family of wavelets (and analogously, of wavelet sets) is a finite (or even infinite) family of wavelets for which the associated family of interpolation unitaries (interpolation maps) forms a group.

It is appropriate to give a bit of background and history that serves to indicate why interpolation pairs of wavelet sets are relevant to the theory of wavelets. More exposition on this, including specific details and statements of theorems involved, can be found in the semi-expository articles [La2], [La3], [La4], and [La5]. In [DL], for any interpolation pair of wavelet sets (E, F) , the authors constructed a 2×2 complex matrix valued function (called the Coefficient Criterion, see [DL], Proposition 5.4, and also [La2], section 5.22) which specifies precisely when a function f on \mathbb{R} with frequency support contained in $E \cup F$ is an orthonormal wavelet. If $supp\{\hat{f}\}$ is contained in $E \cup F$, then this criterion shows that f is an orthonormal wavelet iff this matrix valued function is a unitary matrix (a.e.), and it is a Riesz wavelet iff it is an invertible matrix (a.e). Moreover, it shows that a Parseval frame wavelet (resp. Riesz frame wavelet) with frequency support contained in $E \cup F$ is necessarily an orthonormal wavelet (resp. Riesz wavelet). It follows that the set of orthonormal wavelets with frequency support contained in the union of $E \cup F$, where (E, F) is an interpolation pair of wavelet sets, is pathwise connected in $\mathcal{L}^2(\mathbb{R})$. The set of Fourier Transforms of this set is also connected in the \mathcal{L}^∞ norm on the frequency space.

These results were the main motivating factor in posing the first open problem discussed in [DL], namely the question of whether the set of all orthonormal wavelets in $\mathcal{L}^2(\mathbb{R})$ is norm-pathwise connected. This was the same problem that was posed completely independently by G. Weiss and his research group in [HWW1],[HWW2]

for different reasons. Their reasons included the interesting discovery that certain wavelet sets (rather, the associated MSF wavelets) could be "smoothed" in a continuous fashion to obtain wavelets that were continuous in the frequency domain. It turned out that our operator-interpolation approach, in certain key cases, was equivalent to the "smoothing" approach of G. Weiss, and the cases involved included the derivation of Y. Meyer's classic family of wavelets that are compactly supported and continuous in the frequency domain. Exploring common interest in the relationships between smoothing of a wavelet set on the one hand, and operator-theoretic interpolation between a pair of wavelet sets on the other hand, and the general "connectedness" problem that was motivated by both approaches independently (as described above) led to the formation of the WUTAM Consortium (short for Washington University and Texas A&M University) and the joint work [Wut] of the consortium, in which the connectedness problem was shown to have a positive answer for the case of MRA wavelets.

The basic idea behind operator-interpolation is elementary. If x and y are elements of a vector space V , we say that a vector z is linearly interpolated from x and y if z is a convex combination of x, y . More generally, it is convenient to allow arbitrary linear combinations. So the set of vectors interpolated by x and y is the linear span of x and y . More generally, we can say that z is linearly interpolated from a collection \mathcal{F} of vectors if z is a linear combination of vectors from the family. And more generally yet, if the vector space V is a left module over some operator algebra \mathcal{D} we can consider linear combinations from \mathcal{F} with coefficients that are operators from \mathcal{D} , called *modular* linear combinations. If z is a modular linear combination of x, y , then we say that z is derived from x and y by operator-theoretic-interpolation (or operator-interpolation for short). In the case of wavelets, the operator algebra \mathcal{D} is the von Neumann algebra of all bounded linear operators acting on $\mathcal{L}^2(\mathbb{R}^n)$ that commute with the dilation and translation unitary operators for the wavelet system. When we conjugate this with the Fourier transform (which is unitary), so we are working in $\mathcal{L}^2(\mathbb{R}^n)$ as the frequency space, and if we denote this conjugated algebra by $\hat{\mathcal{D}}$, then $\hat{\mathcal{D}}$ is an algebra of *multiplication operators* on $\mathcal{L}^2(\mathbb{R}^n)$. In particular, it is a commutative algebra. If a wavelet η is a modular linear combination of wavelets ψ and ν with coefficients which are operators in \mathcal{D} , then we say η is derived by operator-interpolation between ψ and ν . Not all modular linear combinations of ψ and ν are orthonormal wavelets. They are all *Bessel* wavelets to be sure, but a certain unitarity condition needs to be satisfied to be an orthonormal wavelet. Let V_{ψ}^{ν} be the interpolation unitary from ψ to ν . If A and B are operators in \mathcal{D} and $\eta = A\psi + B\nu$, the necessary and sufficient condition for η to be an orthonormal wavelet is that the operator $U := A + BV_{\psi}^{\nu}$ needs to be unitary. (More generally, for a frame wavelet the criterion is that U must be surjective, and for a Riesz wavelet U must be invertible.)

The reason that interpolation pairs of orthonormal wavelets are special is that if (ψ, ν) is an interpolation pair, and if A and B are operators in \mathcal{D} such that $A * A + B * B = I$, then under certain circumstances $A\psi + B\nu$ can also be an orthonormal wavelet. In particular, if $\theta \in [0, 2\pi]$ is arbitrary, then $\eta := \cos\theta\psi + i\sin\theta\nu$ is an orthonormal wavelet. Indeed, in this case the operator U above is just $\cos\theta I + i\sin\theta V_\psi^\nu$, and since $(V_\psi^\nu)^2 = I$ it follows that $UU^* = I$, so U is unitary, as required by the criterion. Letting θ vary continuously it follows, in particular, that ψ and ν are pathwise connected via a path of orthogonal wavelets.

For wavelet set wavelets, i.e. MSF wavelets with phase 0, more is true, and the operator-algebraic geometry involved is fairly rich. For *any* pair of wavelet sets E and F , with associated MSF wavelets ψ_E and ψ_F , the interpolation unitary $V_{\psi_E}^{\psi_F}$ normalizes the von Neumann algebra \mathcal{D} in the sense that $V_{\psi_E}^{\psi_F} \mathcal{D} (V_{\psi_E}^{\psi_F})^* = \mathcal{D}$. This was proven in Chapt 5 of [DL], and was a key result of that memoir. (We note that it is an open question (see [La2], Problem 4) as to whether arbitrary interpolation operators (i.e. for non-wavelet set wavelets) normalize \mathcal{D} .) The reason that interpolation pairs of wavelet sets, are even more special than general interpolations of wavelets is the following: Firstly, (E, F) is an interpolation pair of wavelet sets if and only if the pair of wavelets (ψ_E, ψ_F) is an interpolation pair of wavelets. And secondly, since in this case the interpolation unitary normalizes \mathcal{D} , and since $(V_{\psi_E}^{\psi_F})^2 = I$, it follows that the set of operators $\{A + BV_{\psi_E}^{\psi_F} \mid A, B \in \mathcal{D}\}$ is closed under multiplication and is in fact a von Neumann algebra. (For a more general interpolation pair of wavelets whose interpolation operator normalizes \mathcal{D} the same thing is true.) Since the unitary group of a von Neumann algebra is pathwise connected in the operator norm, the interpolated family of wavelets is also connected. Much more is true. Since the elements of \mathcal{D} are multiplication operators in a certain family (the dilation-periodic operators), if we write $A = M_f$ and $B = M_g$ we obtain $(A + BV_{\psi_E}^{\psi_F})\psi_E = f\psi_E + g\psi_F$. Thus the outcome is an actual formula as well as a criterion (the Coefficient Criterion mentioned above), for constructing all orthonormal wavelets whose frequency support is contained in the union $E \cup F$ of a given interpolation pair (E, F) of wavelet sets. By choosing f and g appropriately, so they are continuous and vanish on the boundary of the union $E \cup F$, and agree on $E \cap F$, and satisfy the unitarity condition referred to above, one can obtain wavelets in this fashion which are smooth in the frequency-domain. Not all interpolation pairs (E, F) can be so *smoothed*. But some can. As alluded to above, Y. Meyer's famous class of orthonormal wavelets which are continuous and compactly supported in the frequency domain can be derived in this way from a special interpolation pair of wavelet sets: namely the pair $E = [-\frac{8\pi}{3}, -\frac{4\pi}{3}) \cup [\frac{2\pi}{3}, \frac{4\pi}{3})$ and $F = [-\frac{4\pi}{3}, -\frac{2\pi}{3}) \cup [\frac{4\pi}{3}, \frac{8\pi}{3})$. Even for cases in which smoothing cannot work (for instance, $E \cup F$ may have too many boundary points), the operator algebra involved can be interesting.

The purpose of this paper is to provide solutions to two related problems concerning dyadic orthonormal wavelet sets in the line. One problem asked whether a certain containment relation for an interpolation map implies that the associated pair of wavelet sets is an interpolation pair. This was posed by the second author in a VIGRE seminar course at Texas A&M several years ago. We answer this question affirmatively, and we also observe that the analogous result does not hold for a general interpolation family of wavelet sets. The second problem was posed by D. Han, who asked whether the equality of the congruence domains of an interpolation map and its inverse implies that the associated pair of wavelet sets is an interpolation pair. We were able to give a counterexample to this problem. Our work on this interesting problem motivated a useful lemma on constructing wavelet sets, which is apparently different from those methods which have appeared in the literature to date, and which is used in this counterexample as well as in the construction of several wavelet sets concerning some related questions, and also some wavelet sets in the plane. Much of the work presented in this article is material from the doctoral dissertation of the first author [Zh], which has not appeared elsewhere.

Much work has been accomplished on the topic of wavelet sets since the mid 1990's. We have outlined some of the background and history in the opening paragraph of Section 5, for the interested reader. Our main results in this paper are for the special case $n = 1$ with the dilation scale factor 2 and integer translates (the dyadic case). In Section 5, we apply one of our techniques to the construction of certain dyadic wavelet sets in the plane. Based on our results, some further directions are suggested for higher dimensions.

2. TWO PROBLEMS

A *dyadic orthonormal wavelet* is a function $\psi \in \mathcal{L}^2(\mathbb{R})$ (Lebesgue measure) with the property that the set $\{2^{\frac{n}{2}}\psi(2^n \cdot -l) \mid n, l \in \mathbb{Z}\}$ forms an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$. More generally, if A is any real invertible $n \times n$ matrix, then a single function $\psi \in L^2(\mathbb{R}^n)$ is an orthonormal wavelet for A if

$$\{|\det A|^{\frac{n}{2}}\psi(A^n \cdot -l) \mid n \in \mathbb{Z}, l \in \mathbb{Z}^{(n)}\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. If A is *expansive* (equivalently, all eigenvalues of A are required to have absolute value strictly greater than 1) then it was shown in [DLS1] that orthonormal wavelets for A always exist.

By the support of a measurable function we mean the set of points in its domain at which it does not vanish. By the support of an element f of $\mathcal{L}^2(\mathbb{R})$, we mean the support of any measurable representative of f , which is well-defined in the measure algebra of equivalence classes of sets modulo null sets. By the *frequency support* of a function we mean the support of its Fourier Transform.

Let \mathcal{F} denote the n -dimensional Fourier transform on $\mathcal{L}^2(\mathbb{R}^n)$ defined by

$$(\mathcal{F}f)(s) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-s \circ t} f(t) dm$$

for all $f \in L^2(\mathbb{R}^n)$. Here, $s \circ t$ denotes the real inner product. A measurable set $E \subseteq \mathbb{R}^n$ is a wavelet set for A if

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\mu(E)}} \chi_E\right)$$

is an orthonormal wavelet for A .

A sequence of measurable sets $\{E_n\}$ is called a measurable partition of E if $E = E \Delta (\bigcup_n E_n)$ is a null set and $E_n \cap E_m$ has measure zero if $n \neq m$, where Δ denotes the symmetric difference of sets. Measurable subsets E and F of \mathbb{R} are called 2π -translation congruent to each other, denoted by $E \sim_{2\pi} F$, if there exists a measurable partition $\{E_n\}$ of E , such that $\{E_n + 2n\pi\}$ is a measurable partition of F . Similarly, E and F are called 2-dilation congruent to each other, denoted by $E_2 \sim F$, if there is a measurable partition $\{E_n\}$ of E , such that $\{2^n E_n\}$ is a measurable partition of F . A measurable set E is called a 2π -translation generator of a measurable partition of \mathbb{R} if $\{E + 2n\pi\}_{n \in \mathbb{Z}}$ forms a measurable partition of \mathbb{R} . Similarly, a measurable set F is called a 2-dilation generator of a measurable partition of \mathbb{R} if $\{2^n F\}_{n \in \mathbb{Z}}$ forms a measurable partition of \mathbb{R} .

Lemma 4.3 in [DL] gives the following characterization of wavelet sets, which was also obtained independently in [FW] using different techniques.

Let $E \subseteq \mathbb{R}$ be a measurable set. Then E is a wavelet set if and only if E is both a 2π -translation generator of a measurable partition of \mathbb{R} and a 2-dilation generator of a measurable partition of \mathbb{R} .

Again from [DL], suppose that E, F are wavelet sets, and let $\sigma : E \rightarrow F$ be the bijective map (modulo null sets) implementing the 2π -translation congruence. Then σ can be extended to a bijective (modulo null sets) measurable map on \mathbb{R} by defining $\sigma(0) = 0$ and $\sigma(s) = 2^n \sigma(2^{-n}s)$ for each $s \in 2^n E, n \in \mathbb{Z}$. This map is denoted by σ_E^F and called the *interpolation map* for the ordered pair of wavelet sets (E, F) .

An ordered pair of wavelet sets (E, F) is called an *interpolation pair* if $\sigma_E^F \circ \sigma_E^F := (\sigma_E^F)^2 = id_{\mathbb{R}}$. In general, an *interpolation family of wavelet sets* is a family \mathcal{F} of wavelet sets such that $\{\sigma_E^F \mid F \in \mathcal{F}\}$ is a group under composition of maps for some $E \in \mathcal{F}$.

Question A. *Let E, F be wavelet sets. Does $\sigma_E^F(E \cup F) \subseteq E \cup F$ imply that $(\sigma_E^F)^2 = id_{\mathbb{R}}$?*

We answer Question A affirmatively. We then give an elementary example that shows that this result need not hold for a *triple* of wavelet sets; However, there is a

natural modification which does make sense for n -tuples, and we pose it as an open question (Question C).

Given a pair of wavelet sets, it is not obvious at all upon initial inspection whether they actually form an interpolation pair. And constructing interpolation pairs can be hard. A basic problem from [DL] that still remains open is the question: Given an arbitrary dyadic wavelet set E in \mathbb{R}^1 , is there necessarily a second distinct wavelet set F such that (E, F) is an interpolation pair? Partly to address this problem, and partly for intrinsic interest, Han introduced and studied properties of congruence domains in [Han1] and [Han2]. Given a pair of wavelet sets (E, F) , the *domain of 2π -congruence* of σ_E^F , denoted by \mathcal{D}_E^F , is the set of all points $s \in \mathbb{R}$ such that $\sigma_E^F(s) - s$ is an integral multiple of 2π . There is a close relation between this and the interpolation map of a pair of wavelet sets. Han asked the following question.

Question B. *Let E, F be wavelet sets. Does $\mathcal{D}_E^F = \mathcal{D}_F^E$ imply that $(\sigma_E^F)^2 = id_{\mathbb{R}}$?*

If (E, F) is an interpolation pair, then it is easily verified that the domains of 2π -congruence of (E, F) and (F, E) are the same. So the above question just asks if the converse is true. In many cases it is true. But it is not universally true. We answer Question B negatively by constructing a counterexample. The key is a special lemma on constructing wavelet sets, which is also used to build examples for several other questions.

3. SOLUTION TO QUESTION A

The following theorem provides an answer to Question A.

Theorem 1. *Let (E, F) be wavelet sets. The following two statements are equivalent:*

- (i) (E, F) is an interpolation pair,
- (ii) $\sigma_E^F(E \cup F) \subseteq E \cup F$.

Proof. (i) \Rightarrow (ii). Suppose that (E, F) is an interpolation pair. Then $(\sigma_E^F)^2 = id_{\mathbb{R}}$. Since $(\sigma_E^F)^{-1} = \sigma_F^E$, it follows that $\sigma_E^F = \sigma_F^E$. Observe that $\sigma_E^F(E) = F$, $\sigma_F^E(F \setminus E) = E \setminus F$, and so

$$\sigma_E^F(E \cup F) = \sigma_E^F(E) \cup \sigma_E^F(F \setminus E) = F \cup \sigma_F^E(F \setminus E) = F \cup E \setminus F = E \cup F.$$

(ii) \Rightarrow (i). Suppose that $\sigma_E^F(E \cup F) \subseteq E \cup F$. Since $\sigma_E^F(E) = F$ and σ_E^F is bijective (modulo null sets), we must have $\sigma_E^F(F \setminus E) \subseteq E \setminus F$. We will prove this by way of contradiction, and a diagram is included at the end of the proof to help in navigating between E and F , with arrows representing σ_E^F .

Assume that (E, F) is not an interpolation pair. Let $G_0 = \{s \in E \mid (\sigma_E^F)^2(s) \neq s\}$. Then $G_0 \subseteq E \setminus F$ is Lebesgue measurable and has positive measure. Since

$\{G_0 \cap (F - 2n\pi)\}_{n \in \mathbb{Z}}$ forms a measurable partition of G_0 , it follows that $G_0 \cap (F - 2n_1\pi)$ has positive measure for some $n_1 \in \mathbb{Z}$. Denote this set by G_1 . Then $\sigma_E^F(G_1) = G_1 + 2n_1\pi \subseteq F \setminus E$. Since $E \setminus F$ is 2-dilation congruent to $F \setminus E$, following the similar discussion, there exists a measurable subset G_2 of G_1 with positive measure, such that $2^{-k_1}(G_2 + 2n_1\pi) \subseteq E \setminus F$ for some integer k_1 . Then, there exists a measurable subset G_3 of G_2 with positive measure such that $\sigma_E^F(2^{-k_1}(G_3 + 2n_1\pi)) = 2^{-k_1}(G_3 + 2n_1\pi) + 2n_2\pi$ for some integer n_2 . Thus,

$$\begin{aligned} (\sigma_E^F)^2(G_3) &= \sigma_E^F(G_3 + 2n_1\pi) = 2^{k_1} \cdot \sigma_E^F(2^{-k_1}(G_3 + 2n_1\pi)) \\ &= 2^{k_1} \cdot (2^{-k_1}(G_3 + 2n_1\pi) + 2n_2\pi) = G_3 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi \subseteq E \setminus F. \end{aligned}$$

Since both $G_3, G_3 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi \subseteq E \setminus F$ and they are distinct by assumption, we must have $n_1 + 2^{k_1} \cdot n_2 \notin \mathbb{Z}$, which implies that $k_1 \leq -1$. Similarly, there exists a measurable subset G_4 of G_3 with positive measure, such that $\sigma_E^F(G_4 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi) = G_4 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi + 2n_3\pi$ for some integer n_3 . Then,

$$\begin{aligned} (\sigma_E^F)^2(2^{-k_1}(G_4 + 2n_1\pi)) &= \sigma_E^F(2^{-k_1}(G_4 + 2n_1\pi) + 2n_2\pi) \\ &= 2^{-k_1} \cdot \sigma_E^F(G_4 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi) = 2^{-k_1} \cdot (G_4 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi + 2n_3\pi) \\ &= 2^{-k_1}(G_4 + 2n_1\pi) + 2n_2\pi + 2^{-k_1} \cdot 2n_3\pi \subseteq E \setminus F. \end{aligned}$$

Since both $2^{-k_1}(G_4 + 2n_1\pi)$ and $2^{-k_1}(G_4 + 2n_1\pi) + 2n_2\pi + 2^{-k_1} \cdot 2n_3\pi$ are contained in $E \setminus F$ and $n_2 + 2^{-k_1} \cdot n_3$ must be an integer, we must have $n_2 + 2^{-k_1} \cdot n_3 = 0$. Then, since σ_E^F maps both G_4 and $G_4 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi$ to $G_4 + 2n_1\pi$, G_4 and $G_4 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi$ must be the same, then $(\sigma_E^F)^2(s) = s, \forall s \in G_4 \subseteq G_0$, contradicting the assumption. Thus, $(\sigma_E^F)^2 = id_{\mathbb{R}}$ almost everywhere.

$$\begin{array}{ccc} E \setminus F : & G_0 & 2^{-k_1}(G_2 + 2n_1\pi) \\ & \downarrow & \downarrow \\ F \setminus E : & G_1 + 2n_1\pi & 2^{-k_1}(G_3 + 2n_1\pi) + 2n_2\pi \\ & \downarrow & \downarrow \\ E \setminus F : & G_3 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi & 2^{-k_1}(G_4 + 2n_1\pi) + 2n_2\pi + 2^{-k_1} \cdot 2n_3\pi \\ & \downarrow & \downarrow \\ F \setminus E : & G_4 + 2n_1\pi + 2^{k_1} \cdot 2n_2\pi + 2n_3\pi & \dots \end{array}$$

□

Let \mathcal{F} be a general interpolation family of wavelet sets, and fix $E_1 \in \mathcal{F}$. Then for arbitrary $E_2, E_3 \in \mathcal{F}$, we have $\sigma_{E_1}^{E_2} \circ \sigma_{E_1}^{E_3} = \sigma_{E_1}^{E_4}$ for some $E_4 \in \mathcal{F}$. Thus $\sigma_{E_1}^{E_2}(E_3) = \sigma_{E_1}^{E_2} \circ \sigma_{E_1}^{E_3}(E_1) = \sigma_{E_1}^{E_4}(E_1) = E_4$. It follows that $\sigma_{E_1}^{E_\lambda}(\bigcup_{E \in \mathcal{F}} E) \subseteq \bigcup_{E \in \mathcal{F}} E$ for each $E_\lambda \in \mathcal{F}$. However, the converse may be false as the following example shows.

Example 2. Let (E, F) be an interpolation pair of wavelet sets and suppose $G \subset E \cup F$ is a wavelet set contained in the union which is distinct from E and F . Observe that $E \setminus F$ is both 2π -translation congruent and 2-dilation congruent to

$F \setminus E$, thus $G \subset E \cup F$ must contain $E \cap F$ to be a wavelet set. Since $\sigma_E^G = id_{\mathbb{R}}$ on $E \cap G$ and $\sigma_E^G = \sigma_E^F$ on $E \setminus G$, $(\sigma_E^G)^2 = id_{\mathbb{R}}$. Similarly, $(\sigma_F^G)^2 = id_{\mathbb{R}}$. Then, $\sigma_E^F \circ \sigma_E^G = \sigma_F^G \neq id_{\mathbb{R}}$, which implies that $\{id_{\mathbb{R}}, \sigma_E^F, \sigma_E^G\}$ is not a group under composition of maps. Hence $\{E, F, G\}$ is not an interpolation family. However, notice that $\{E \setminus G, G \setminus F, E \cap F, G \setminus E, F \setminus G\}$ forms a measurable partition of $E \cup F$. $E \setminus G$ is both 2π -translation and 2-dilation congruent to $G \setminus E$ and $F \setminus G$ is both 2π -translation and 2-dilation congruent to $G \setminus F$ which is contained in $E \cap G$. We have $\sigma_E^F(E \cup F \cup G) = \sigma_E^F(E \cup F) = E \cup F$, and

$$\begin{aligned}
 \sigma_E^G(E \cup F \cup G) &= \sigma_E^G(E \cup F) = \sigma_E^G(E) \cup \sigma_E^G(F \setminus E) \\
 &= G \cup \sigma_E^G(G \setminus E) \cup \sigma_E^G(F \setminus G) = G \cup E \setminus G \cup F \setminus G = E \cup F.
 \end{aligned}$$

□

Observe that in the above example, if let $H = E \setminus G \cup (E \cap F) \cup F \setminus G$, then $H \subseteq E \cup F$ is also a wavelet set and $\sigma_E^H = \sigma_E^F \circ \sigma_E^G = \sigma_E^G \circ \sigma_E^F$. Then $\{E, F, G, H\}$ is an interpolation family since $\{\sigma_E^E = id_{\mathbb{R}}, \sigma_E^F, \sigma_E^G, \sigma_E^H\}$ is actually isomorphic to the Klein four group. This motivates the following question which seems difficult and which we pose as an open problem:

Question C. Let \mathcal{F} be a finite collection of wavelet sets. Fix $E_1 \in \mathcal{F}$. Suppose $\sigma_{E_1}^{E_\lambda}(\bigcup_{E \in \mathcal{F}} E)$ is contained in $\bigcup_{E \in \mathcal{F}} E$ for each $E_\lambda \in \mathcal{F}$. Is \mathcal{F} a subset of a finite interpolation family of wavelet sets? That is, can \mathcal{F} always be extended to a finite interpolation family?

4. A COUNTEREXAMPLE TO QUESTION B

Before giving a counterexample to Question B, we present a characterization of interpolation pairs of wavelet sets and equality of 2π -congruence domains of associated interpolation maps. This is actually the motivation that led us to find a counterexample.

Given a pair of wavelet sets (E, F) , there exists a measurable partition $\{E_n\}_{n \in \mathbb{Z}}$ of E such that $\{E_n + 2n\pi\}_{n \in \mathbb{Z}}$ is a measurable partition of F . Similarly, there exists another measurable partition $\{E^k\}_{k \in \mathbb{Z}}$ of E such that $\{2^k E^k\}_{k \in \mathbb{Z}}$ is another measurable partition of F . Denote $E_n \cap E^k$ by $E_{n,k}$, then $\{E_{n,k}\}_{n,k \in \mathbb{Z}}$ is a measurable partition of E , and both $\{E_{n,k} + 2n\pi\}_{n,k \in \mathbb{Z}}$ and $\{2^k E_{n,k}\}_{n,k \in \mathbb{Z}}$ are measurable partitions of F . Observe that if $E_{n,k}$ has positive measure, then $n = 0$ whenever $k = 0$, and $E_{n,k} \cap (2^l E_{m,l} - 2n\pi)$, $n, k, m, l \in \mathbb{Z} \setminus 0$ forms a partition of $E \setminus F$.

Theorem 3. Let E, F be wavelet sets, and let $\{E_{n,k}\}_{n,k \in \mathbb{Z}}$ be as defined above. Then:

(i) (E, F) is an interpolation pair if and only if $E_{n,k} \cap (2^l E_{m,l} - 2n\pi)$ with positive measure for some nonzero integers n, k, m, l implies that $n + 2^l \cdot m = 0$. Furthermore, for such a set, we also have $k = -l$.

(ii) $\mathcal{D}_E^F = \mathcal{D}_F^E$ if and only if $E_{n,k} \cap (2^l E_{m,l} - 2n\pi)$ with positive measure for some nonzero integers n, k, m, l implies that $[n] + l = [m]$. Here, $[n]$ denotes the smallest integer k such that $2^k \cdot n \in \mathbb{Z}$.

Proof. (i) Suppose that (E, F) is an interpolation pair. Then $(\sigma_E^F)^2 = id$. Since

$$\begin{aligned} (\sigma_E^F)^2(E_{n,k} \cap (2^l E_{m,l} - 2n\pi)) &= \sigma_E^F((E_{n,k} + 2n\pi) \cap 2^l E_{m,l}) \\ &= 2^l \cdot \sigma_E^F(2^{-l}(E_{n,k} + 2n\pi) \cap E_{m,l}) \\ &= 2^l \cdot ((2^{-l}(E_{n,k} + 2n\pi) \cap E_{m,l}) + 2m\pi) \\ &= (E_{n,k} + 2n\pi + 2^l \cdot 2m\pi) \cap 2^l(E_{m,l} + 2m\pi), \end{aligned}$$

it follows that if $E_{n,k} \cap (2^l E_{m,l} - 2n\pi)$ has positive measure, then $2n\pi + 2^l \cdot 2m\pi = 0$, i.e., $n + 2^l \cdot m = 0$.

Conversely, suppose that $E_{n,k} \cap (2^l E_{m,l} - 2n\pi), n, k, m, l \neq 0$ having positive measure implies that $n + 2^l \cdot m = 0$. Observe that $\{E_{n,k} \cap (2^l E_{m,l} - 2n\pi)\}_{n,k,m,l \neq 0}$ forms a measurable partition of $E \setminus F$. Then, a similar argument shows that $(\sigma_E^F)^2 = id_{\mathbb{R}}$.

Furthermore, if $E_{n,k} \cap (2^l E_{m,l} - 2n\pi)$ having positive measure implies that $n + 2^l \cdot m = 0$, then $(E_{m,l} + 2m\pi) \cap 2^{-l} E_{n,k} = 2^{-l} \cdot (E_{n,k} \cap (2^l E_{m,l} - 2n\pi))$ also has positive measure. Since $E_{m,l} + 2m\pi \subseteq F$ and $2^{-l} E_{n,k} \subseteq F$ only if $k = -l$, we must have $k = -l$.

(ii) Observe that

$$\begin{aligned} \mathcal{D}_E^F &= \dot{\bigcup}_{n,k \neq 0} (\dot{\bigcup}_{j \geq [n]} 2^j E_{n,k}) \dot{\cup} \dot{\bigcup}_j 2^j (E \cap F) \\ &= \dot{\bigcup}_{n,k,m \neq 0} (\dot{\bigcup}_{j \geq [n]} 2^j (E_{n,k} \cap 2^{-k}(E_m + 2m\pi))) \dot{\cup} \dot{\bigcup}_j 2^j (E \cap F), \end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_F^E &= \dot{\bigcup}_{n,k \neq 0} \left(\dot{\bigcup}_{j \geq [n]} 2^j(E_{n,k} + 2n\pi) \right) \dot{\cup} \dot{\bigcup}_j 2^j(E \cap F) \\
&= \dot{\bigcup}_{n,k,m,l \neq 0} \left(\dot{\bigcup}_{j \geq [n]} 2^j((E_{n,k} + 2n\pi) \cap 2^l E_{m,l}) \right) \dot{\cup} \dot{\bigcup}_j 2^j(E \cap F) \\
&= \dot{\bigcup}_{n,k,m,l \neq 0} \left(\dot{\bigcup}_{j \geq [n]} 2^{j+l}(E_{m,l} \cap 2^{-l}(E_{n,k} + 2n\pi)) \right) \dot{\cup} \dot{\bigcup}_j 2^j(E \cap F) \\
&= \dot{\bigcup}_{n,k,m,l \neq 0} \left(\dot{\bigcup}_{j \geq [m]} 2^{j+k}(E_{n,k} \cap 2^{-k}(E_{m,l} + 2m\pi)) \right) \dot{\cup} \dot{\bigcup}_j 2^j(E \cap F) \\
&= \dot{\bigcup}_{n,k,m \neq 0} \left(\dot{\bigcup}_{j \geq [m]} 2^{j+k}(E_{n,k} \cap 2^{-k}(E_m + 2m\pi)) \right) \dot{\cup} \dot{\bigcup}_j 2^j(E \cap F),
\end{aligned}$$

where $\dot{\cup}$ denotes disjoint union. The next to last equality comes from changing indices. Thus, $\mathcal{D}_E^F = \mathcal{D}_F^E$ implies that $[m] + k = [n]$, if $E_{n,k} \cap 2^{-k}(E_m + 2m\pi)$ has positive measure, or equivalently, if $E_{m,l} \cap (2^k E_{n,k} - 2m\pi)$ has positive measure for some $l \neq 0$, since $2^k(E_{n,k} \cap 2^{-k}(E_m + 2m\pi)) - 2m\pi = (2^k E_{n,k} - 2m\pi) \cap E_m = (2^k E_{n,k} - 2m\pi) \cap (\dot{\bigcup}_l E_{m,l})$. The converse direction can be shown by reversing the above discussion. \square

The basic idea we use is to construct two certain measurable subsets of \mathbb{R} , which are both 2π -translation and 2-dilation congruent to each other, yet the interpolation map restricted to the union of 2-dilates of which does not have the property that its square equals the identity map, and then to construct the remaining pieces of wavelet sets for these two sets so that the congruency domains match up. This type of approach was used by D. Speegle in [S1] and Q. Gu in [Gu1], and a necessary and sufficient condition has also been given for a measurable set being contained in some wavelet set in [IP]. However, the methods and the constructions we use in the present paper are completely different from those in [S1], [Gu1] and [IP].

Theorem 4. *The answer to Question B is no.*

Before proving Theorem 4, we require a technical lemma. This will also be used in constructing several other examples in the next section.

Lemma 5. *Let $E, F \subseteq \mathbb{R}^n$ be Lebesgue measurable sets with finite positive measure. If there exist $n_1, n_2, k_1, k_2 \in \mathbb{Z}^{(n)}$, such that modulo null sets,*

$$2^{k_1} F \subseteq E + 2n_1\pi \quad \text{and} \quad E + 2n_2\pi \subseteq 2^{k_2} F,$$

and $(E + 2n_2\pi) \cap 2^{k_1} F$ is a null set, then there exists a Lebesgue measurable set $G \subseteq (E + 2n_2\pi) \cup 2^{k_1} F$ such that G is both 2π -translation congruent to E and

2-dilation congruent to F . In fact G can be taken as:

$$G = \bigcup_{i=0}^{\infty} S^i(2^{k_1}F \setminus 2^{k_1-k_2}(E + 2n_2\pi)) \\ \cup (E + 2n_2\pi) \setminus \left(\bigcup_{i=1}^{\infty} 2^{k_2-k_1} \cdot S^i(2^{k_1}F \setminus 2^{k_1-k_2}(E + 2n_2\pi)) \right),$$

where $S(x) = 2^{k_1-k_2}(x + 2(n_2 - n_1)\pi)$, $\forall x \in \mathbb{R}^n$.

Proof. By hypothesis,

$$2^{k_1-k_2}(E + 2n_2\pi) \subseteq 2^{k_1}F \subseteq E + 2n_1\pi.$$

Since $\mu(2^{k_1-k_2}(E + 2n_2\pi)) = 2^{k_1-k_2}\mu(E) \leq \mu(E + 2n_1\pi) = \mu(E)$, $k_1 - k_2 \leq 0$. If $k_1 = k_2$, then $E + 2n_2\pi \subseteq 2^{k_1}F$, contradicts the hypothesis that $(E + 2n_2\pi) \cap 2^{k_1}F$ is a null set. Thus, $k_1 - k_2 < 0$. Construct a sequence of measurable sets $\{G_i\}_{i \in \mathbb{N}}$ as follows. Let

$$G_0 = 2^{k_1}F \setminus 2^{k_1-k_2}(E + 2n_2\pi) \\ = 2^{k_1}F \setminus 2^{k_1-k_2}(E + 2n_1\pi + 2(n_2 - n_1)\pi) \subseteq (E + 2n_1\pi) \setminus S(E + 2n_1\pi).$$

Let $G_i = S^i(G_0) \subseteq S^i(E + 2n_1\pi) \setminus S^{i+1}(E + 2n_1\pi)$, for each $i \in \mathbb{N}$. Notice that the measure of G_i is bounded by $2^{i \cdot (k_1 - k_2)} \cdot \mu(E)$, which will approach 0 as i approaches infinity. Furthermore, $S(E + 2n_1\pi) \subseteq E + 2n_1\pi$, and it follows that the G_i 's are disjoint and $\bigcup_{i=0}^{\infty} G_i \subseteq 2^{k_1}F$. Also by definition, $\bigcup_{i=1}^{\infty} G_i \subseteq S(E + 2n_1\pi)$ and $2^{k_2-k_1}G_{i+1} = G_i + 2(n_2 - n_1)\pi$, $\forall i \geq 0$. Let

$$G = \left(\bigcup_{i=0}^{\infty} G_i \right) \dot{\cup} (E + 2n_2\pi) \setminus \left(\bigcup_{i=1}^{\infty} 2^{k_2-k_1}G_i \right).$$

Then, since $2^{k_1-k_2}((E + 2n_2\pi) \setminus (\bigcup_{i=1}^{\infty} 2^{k_2-k_1}G_i)) = S(E + 2n_1\pi) \setminus (\bigcup_{i=1}^{\infty} G_i)$, and $\{G_i\}_{i=0}^{\infty} \cup \{S(E + 2n_1\pi) \setminus (\bigcup_{i=1}^{\infty} G_i)\}$ constitutes a measurable partition of $2^{k_1}F$, it's clear that G is 2-dilation congruent to F . On the other hand, since

$$\bigcup_{i=1}^{\infty} 2^{k_2-k_1}G_i = \bigcup_{i=0}^{\infty} (G_i + 2(n_2 - n_1)\pi),$$

G is 2π -translation congruent to E . □

Proof of Theorem 4. Let $E_1 = [\frac{33}{16}\pi, \frac{34}{16}\pi)$, $E_2 = [\frac{33}{8}\pi, \frac{34}{8}\pi) + 12\pi$, $E_3 = [\frac{33}{4}\pi, \frac{34}{4}\pi) + 8\pi$ and $E_4 = [\frac{33}{2}\pi, \frac{34}{2}\pi) + 96\pi$. Let $F_1 = [\frac{33}{4}\pi, \frac{34}{4}\pi)$, $F_2 = [\frac{33}{16}\pi, \frac{34}{16}\pi) + 6\pi$, $F_3 = [\frac{33}{2}\pi, \frac{34}{2}\pi) + 16\pi$ and $F_4 = [\frac{33}{8}\pi, \frac{34}{8}\pi) + 24\pi$. Notice that E_1, E_2, E_3, E_4 are both 2π -translation congruent and 2-dilation congruent to disjoint pieces of $[-2\pi, -\pi) \cup$

$[\pi, 2\pi)$, and so are F_1, F_2, F_3, F_4 . Furthermore,

$$\begin{aligned} E_1 \cup E_2 \cup E_3 \cup E_4 &\sim_{2\pi} F_1 \cup F_2 \cup F_3 \cup F_4 \sim_{2\pi} \left[\frac{\pi}{16}, \pi\right), \\ E_1 \cup E_2 \cup E_3 \cup E_4 &\mathcal{I} \sim F_1 \cup F_2 \cup F_3 \cup F_4 \mathcal{I} \sim \\ &\left[\pi + \frac{\pi}{128}, \pi + \frac{\pi}{16}\right) \cup \left[\frac{7}{4}\pi + \frac{\pi}{128}, \frac{7}{4}\pi + \frac{\pi}{64}\right). \end{aligned}$$

Let G be the measurable set determined by Lemma 5 and the following two containment relations:

$$\begin{aligned} \frac{1}{32}([\pi, \pi + \frac{\pi}{128}) \cup [\pi + \frac{\pi}{16}, \frac{7}{4}\pi + \frac{\pi}{128}) \cup [\frac{7}{4}\pi + \frac{\pi}{64}, 2\pi)) &\subseteq [0, \frac{\pi}{16}), \\ [0, \frac{\pi}{16}) + 8\pi &\subseteq 8 \cdot ([\pi, \pi + \frac{\pi}{128}) \cup [\pi + \frac{\pi}{16}, \frac{7}{4}\pi + \frac{\pi}{128}) \cup [\frac{7}{4}\pi + \frac{\pi}{64}, 2\pi)). \end{aligned}$$

Then, G is both 2π -translation congruent to $[0, \frac{\pi}{16})$ and 2-dilation congruent to $[\pi, \pi + \frac{\pi}{128}) \cup [\pi + \frac{\pi}{16}, \frac{7}{4}\pi + \frac{\pi}{128}) \cup [\frac{7}{4}\pi + \frac{\pi}{64}, 2\pi)$. Hence,

$$E := [-2\pi, -\pi) \cup G \cup E_1 \cup E_2 \cup E_3 \cup E_4,$$

$$F := [-2\pi, -\pi) \cup G \cup F_1 \cup F_2 \cup F_3 \cup F_4$$

are both wavelet sets. The associated interpolation map σ_E^F is defined by:

$$\sigma_E^F(s) = \begin{cases} s + 6\pi & \text{if } s \in E_1 \\ s + 12\pi & \text{if } s \in E_2 \\ s - 8\pi & \text{if } s \in E_3 \\ s - 80\pi & \text{if } s \in E_4 \\ s & \text{if } s \in [-2\pi, -\pi) \cup G. \end{cases}$$

Straightforward computation shows that

$$\mathcal{D}_E^F = (-\infty, 0) \cup \bigcup_{k \in \mathbb{Z}} 2^k G \cup \bigcup_{k \geq 0} (E_1 \cup \frac{1}{2}E_2 \cup \frac{1}{4}E_3 \cup \frac{1}{8}E_4)$$

and

$$\mathcal{D}_F^E = (-\infty, 0) \cup \bigcup_{k \in \mathbb{Z}} 2^k G \cup \bigcup_{k \geq 0} (F_2 \cup \frac{1}{2}F_4 \cup \frac{1}{4}F_1 \cup \frac{1}{8}F_3).$$

Observe that $F_2 = \frac{1}{2}E_2$, $\frac{1}{2}F_4 = \frac{1}{8}E_4$, $\frac{1}{4}F_1 = E_1$, and $\frac{1}{8}F_3 = \frac{1}{4}E_3$, which implies that $\mathcal{D}_E^F = \mathcal{D}_F^E$. However, since

$$(\sigma_E^F)^2(E_1) = \sigma_E^F(E_1 + 6\pi) = \sigma_E^F(\frac{1}{2}E_2) = \frac{1}{2}\sigma_E^F(E_2) = \frac{1}{2}(E_2 + 12\pi) = E_1 + 12\pi,$$

(E, F) is not an interpolation pair. \square

It turns out that if one of E, F is the Shannon set, then the equality of integral domains does imply that (E, F) is an interpolation pair.

Proposition 6. *Let E, F be wavelet sets. If $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$, then $\mathcal{D}_E^F = \mathcal{D}_F^E$ implies that (E, F) is an interpolation pair.*

Proof. Let $E_n = E \cap (F - 2n\pi)$, $n \in \mathbb{Z}$. Then $\{E_n\}_{n \in \mathbb{Z}}$ is a measurable partition of E and $\{E_n + 2n\pi\}$ is a measurable partition of F . Let $E_n^- = E_n \cap [-2\pi, -\pi)$, $E_n^+ = E_n \cap [\pi, 2\pi)$. Then, we have the following diagram.

$$\begin{array}{c}
 E : \quad \underbrace{\begin{array}{ccccccc} & E_n^- & & & E_n^+ & & \\ & -2\pi & -\pi & 0 & \pi & 2\pi & \end{array}} \\
 \\
 F : \quad \underbrace{\begin{array}{cccccccccccc} \cdots & E_{-1}^- - 2\pi & E_{-2}^+ - 4\pi & E_0^- & E_{-1}^+ - 2\pi & E_1^- + 2\pi & E_0^+ & E_2^- + 4\pi & E_1^+ + 2\pi & \cdots \\ & -4\pi & -3\pi & -2\pi & -\pi & 0 & \pi & 2\pi & 3\pi & 4\pi \end{array}}
 \end{array}$$

Observe that

$$\mathcal{D}_E^F = \dot{\bigcup}_{n \neq 0, j \geq [n]} 2^j (E_n^- \cup E_n^+) \dot{\cup} \dot{\bigcup}_{j \in \mathbb{Z}} 2^j E_0.$$

Then

$$\begin{aligned}
 \mathcal{D}_E^F \cap [0, \pi) &= \dot{\bigcup}_{n \neq 0, [n] \leq j < 0} 2^j E_n^+ \dot{\cup} \dot{\bigcup}_{j < 0} 2^j E_0^+ \\
 &= \dot{\bigcup}_{j < 0} 2^j E_0^+ \dot{\cup} \frac{1}{2} E_{\pm 2}^+ \dot{\cup} \frac{1}{2} E_{\pm 4}^+ \dot{\cup} \frac{1}{4} E_{\pm 4}^+ \dot{\cup} \frac{1}{2} E_{\pm 6}^+ \dot{\cup} \frac{1}{2} E_{\pm 8}^+ \dot{\cup} \frac{1}{4} E_{\pm 8}^+ \dot{\cup} \cdots,
 \end{aligned}$$

here $E_{\pm n} := E_n \cup E_{-n}$. $\mathcal{D}_E^F \cap [\pi, 2\pi) = [\pi, 2\pi)$. Similarly,

$$\mathcal{D}_F^E = \dot{\bigcup}_{n \neq 0, j \geq [n]} 2^j ((E_n^- \cup E_n^+) + 2n\pi) \dot{\cup} \dot{\bigcup}_{j \in \mathbb{Z}} 2^j E_0.$$

Then, following from the diagram,

$$\mathcal{D}_F^E \cap [0, \pi) = (E_1^- + 2\pi) \dot{\cup} \dot{\bigcup}_{j < 0} 2^j E_0^+,$$

$$\mathcal{D}_F^E \cap [\pi, 2\pi) = E_0^+ \dot{\cup} \left(\dot{\bigcup}_{j \geq 1} 2^j (E_1^- + 2\pi) \cap [\pi, 2\pi) \right) \dot{\cup} \frac{1}{2} (E_2^- + 4\pi) \dot{\cup} \frac{1}{4} (E_4^- + 8\pi) \dot{\cup} \cdots.$$

Comparing $\mathcal{D}_E^F \cap [0, \pi)$ and $\mathcal{D}_F^E \cap [0, \pi)$, we have

$$(1) \quad E_1^- + 2\pi = \frac{1}{2} E_{\pm 2}^+ \dot{\cup} \frac{1}{2} E_{\pm 4}^+ \dot{\cup} \frac{1}{4} E_{\pm 4}^+ \dot{\cup} \frac{1}{2} E_{\pm 6}^+ \dot{\cup} \frac{1}{2} E_{\pm 8}^+ \dot{\cup} \frac{1}{4} E_{\pm 8}^+ \dot{\cup} \cdots.$$

Then, it is easy to see that $(\dot{\bigcup}_{j \geq 1} 2^j (E_1^- + 2\pi)) \cap [\pi, 2\pi) = \dot{\bigcup}_{n \in 2\mathbb{Z} \setminus 0} E_n^+$. Comparing $\mathcal{D}_E^F \cap [\pi, 2\pi)$ and $\mathcal{D}_F^E \cap [\pi, 2\pi)$, we have

$$(2) \quad \dot{\bigcup}_{n \in 2\mathbb{Z}+1} E_n^+ \subseteq \frac{1}{2} (E_2^- + 4\pi) \dot{\cup} \frac{1}{4} (E_4^- + 8\pi) \dot{\cup} \cdots.$$

By symmetry, we also have

$$(3) \quad E_{-1}^+ - 2\pi = \frac{1}{2} E_{\pm 2}^- \dot{\cup} \frac{1}{2} E_{\pm 4}^- \dot{\cup} \frac{1}{4} E_{\pm 4}^- \dot{\cup} \frac{1}{2} E_{\pm 6}^- \dot{\cup} \frac{1}{2} E_{\pm 8}^- \dot{\cup} \frac{1}{4} E_{\pm 8}^- \dot{\cup} \cdots,$$

$$(4) \quad \dot{\bigcup}_{n \in 2\mathbb{Z}+1} E_n^- \subseteq \frac{1}{2} (E_{-2}^+ - 4\pi) \dot{\cup} \frac{1}{4} (E_{-4}^+ - 8\pi) \dot{\cup} \cdots.$$

(1) and (4) imply that $E_1^- + 2\pi = \frac{1}{2}E_{-2}^+$, and E_n^- has measure zero for each odd integer n except 1, and E_n^+ has measure zero for each even integer n except $-2, 0$. (2) and (3) imply that $E_{-1}^+ - 2\pi = \frac{1}{2}E_2^-$, and E_n^+ has measure zero for each odd integer n except -1 , and E_n^- has measure zero for each even integer n except $0, 2$. Thus, F must have the following form

$$F = (E_{-2}^+ - 4\pi) \cup E_0^- \cup (E_{-1}^+ - 2\pi) \cup (E_1^- + 2\pi) \cup E_0^+ \cup (E_2^- + 4\pi).$$

Using the fact that $E_1^- + 2\pi = \frac{1}{2}E_{-2}^+$ and $E_{-1}^+ - 2\pi = \frac{1}{2}E_2^-$, it is easy to verify that $(\sigma_E^F)^2 = id_{\mathbb{R}}$. \square

5. SOME EXAMPLES OF WAVELET SETS

Wavelet sets are useful as examples and counterexamples. Many examples of them in the real line were given in [DL] exactly for that purpose, for experimentation and testing hypotheses, and many open questions still remain in that setting. The existence of wavelet sets in the plane, and more generally in \mathbb{R}^n , was first proven by Dai, Larson and Speegle [DLS1] in the summer of 1994 during a course at Texas A&M taught by the second author using the manuscript of [DL] as a text. The proof in [DLS1] was abstract and covered dual congruence results for more general types of dual-dynamical systems. But given the definitions, the proof was constructive, implicitly giving an algorithm (a *casting-outward* technique) for constructing examples of wavelet sets for arbitrary expansive matrices and full rank translation lattices on \mathbb{R}^n . However, this method produced only wavelet sets that were unbounded and had 0 as a limit point, and were not very well described by diagrams or pictures. In other words, they were wavelet sets but not "nice" wavelet sets. In 1995 concrete examples of dyadic wavelet sets in the plane, which were bounded and bounded away from 0, and could be nicely diagrammed, were constructed by Soardi and Weiland [SW]. At about the same time, Dai and Larson constructed two \mathbb{R}^2 dyadic wavelet sets (denoted the "four-corners set" and the "wedding cake set"), for inclusion in a final version of [DL] in response to a referee's suggestion. In 1996-97, a number of authors constructed many other concrete examples of wavelet sets in the plane, and in higher dimensions. Included are the wavelet sets computed and diagrammed in the articles [BMM], [BL1], [BL2], [DLS2], [Za], [C], [GH], [Gu2], [Han2], [S2]. The last three are the Ph.D. theses of Gu, Han and Speegle, respectively. Importantly, Baggett-Medina-Merrill [BMM], and Benedetto-Leon [BL2], independently found two interesting completely different constructive characterizations of all wavelet sets for expansive matrices in \mathbb{R}^n . Open questions remain, especially questions concerning the existence of wavelet sets with special properties, and algorithms for constructing special classes of such sets.

In this section, we will apply the techniques introduced in Lemma 5 to construct an unbounded symmetric wavelet set and counterexamples to two related questions

on wavelet sets, in addition to new constructions of some known wavelet sets.

Example 7. Let $G_n = [\frac{2^n-1}{2^n}\pi, \frac{2^{n+1}-1}{2^{n+1}}\pi) + (2^{n+2} - 2)\pi, n > 0$, then

$$G_n \subseteq [(2^{n+2} - 2)\pi, (2^{n+2} - 1)\pi) = 2^{n+1}[\pi + \frac{2^n - 1}{2^n}\pi, \pi + \frac{2^{n+1} - 1}{2^{n+1}}\pi).$$

Observe that $\dot{\bigcup}_{n>0}(G_n - (2^{n+2} - 2)\pi) = [\frac{\pi}{2}, \pi)$ and $\dot{\bigcup}_{n>0}\frac{1}{2^{n+1}}G_n \subseteq \dot{\bigcup}_{n>0}[\pi + \frac{2^n-1}{2^n}\pi, \pi + \frac{2^{n+1}-1}{2^{n+1}}\pi) = [\frac{3}{2}\pi, 2\pi)$. Let $E_0 = [0, \frac{\pi}{2})$, let $F_0 = [\pi, 2\pi) \setminus (\dot{\bigcup}_{n=1}^{\infty}\frac{1}{2^{n+1}}G_n)$. Notice that since $F_0 \supseteq [\pi, \frac{3}{2}\pi)$,

$$E_0 + 2\pi \subseteq 2F_0 \quad \text{and} \quad \frac{1}{4}F_0 \subseteq E_0,$$

and $(E_0 + 2\pi) \cap \frac{1}{4}F_0 = \emptyset$. Let G_0 be the set obtained by applying Lemma 5. Then $\dot{\bigcup}_{n=0}^{\infty}G_n$ is both 2π -translation congruent to $[0, \pi)$ and 2-dilation congruent to $[\pi, 2\pi)$. Hence, by symmetry, $-(\dot{\bigcup}_{n=0}^{\infty}G_n) \dot{\cup} (\dot{\bigcup}_{n=0}^{\infty}G_n)$ is an unbounded symmetric wavelet set. \square

We note that the first example of an unbounded symmetric wavelet set was given in Proposition 2.14 of [FW]. The construction involved some significant computations and explanation. Example 4.5 (xi) of [DL] gave a different unbounded wavelet set whose construction required little explanation, but it was not symmetric.

The following is a counterexample to a question posed by the second author in [La1]:

Let E be a wavelet set, and suppose that $G \subseteq 2E \cup E \cup \frac{1}{2}E$ is also a wavelet set. Is (E, G) an interpolation pair?

Example 8. Let $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$, and $G \subseteq 2E \cup E \cup \frac{1}{2}E = [-4\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, 4\pi)$ be a wavelet set. Suppose that $G_1 = G \cap [3\pi, 4\pi)$ has positive measure, then $G_1 - 2\pi \subseteq [\pi, 2\pi) \subseteq E$. $\forall s \in G_1 - 2\pi$, $(\sigma_E^G)^2(s) = \sigma_E^G(s + 2\pi) = 2 \cdot \sigma_E^G(\frac{1}{2}s + \pi) = 2 \cdot (\frac{1}{2}s + \pi + 2k\pi) = s + 2\pi + 4k\pi$, for some $k \in \mathbb{Z}$. Thus, $(\sigma_E^G)^2(s) \neq s$ for each $s \in G_1 - 2\pi$. Therefore, (E, G) is not an interpolation pair if $G \cap [3\pi, 4\pi)$ has positive measure. Since

$$\frac{1}{2}[\pi, \frac{7}{4}\pi) \subseteq [0, \pi) \quad \text{and} \quad [0, \pi) + 2\pi \subseteq 2[\pi, \frac{7}{4}\pi),$$

and $\frac{1}{2}[\pi, \frac{7}{4}\pi) \cap ([0, \pi) + 2\pi) = \emptyset$, by Lemma 5, there exists $G_0 \subseteq [\frac{\pi}{2}, \frac{7}{8}\pi) \cup [2\pi, 3\pi)$, such that G_0 is 2π -translation congruent to $[0, \pi)$ and 2-dilation congruent to $[\pi, \frac{7}{4}\pi)$. Let $G = [-\pi, -\frac{\pi}{2}) \cup G_0 \cup [\frac{7}{2}\pi, 4\pi)$. Then G is a wavelet set and $G \subseteq 2E \cup E \cup \frac{1}{2}E$, but (E, G) is not an interpolation pair. \square

The next example answers another question from [La1]:

Let E be a wavelet set, and suppose that $G \subseteq (E - 2\pi) \cup E \cup (E + 2\pi)$ is also a wavelet set. Is (E, G) an interpolation pair?

Example 9. Let $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$ and $G \subseteq (E - 2\pi) \cup E \cup (E + 2\pi) = [-4\pi, -3\pi) \cup [-2\pi, 2\pi) \cup [3\pi, 4\pi)$ be a wavelet set. Suppose that $G \cap [-\pi, \pi)$ is not a null set. Without loss of generality, assume that $G_1 = G \cap [0, \pi)$ has positive measure. Then, for each $s \in G_1 - 2\pi \subset E$, $(\sigma_E^G)^2(s) = \sigma_E^G(s + 2\pi) = 2^{-k} \cdot \sigma_E^G(2^k(s + 2\pi))$, for some $k > 0$ such that $2^k(s + 2\pi) \in [\pi, 2\pi)$. Then $(\sigma_E^G)^2(s) = 2^{-k} \cdot (2^k(s + 2\pi) + 2n\pi) = s + 2\pi + 2^{-k} \cdot 2n\pi$ for some $n \in \{1, 0, -1\}$. Thus, $(\sigma_E^G)^2(s) \neq s$ for each $s \in G_1 - 2\pi$. Therefore, if $G \cap [-\pi, \pi)$ is not a null set, (E, G) is not an interpolation pair. Based on the above observation, let G_0 be the measurable set given by Lemma 5 and the following two containment relations:

$$[-2\pi, -\frac{3}{2}\pi) \subseteq [-2\pi, -\frac{11}{8}\pi) \cup [-\frac{9}{8}\pi, -\pi),$$

$$([-2\pi, -\frac{11}{8}\pi) \cup [-\frac{9}{8}\pi, -\pi)) - 2\pi \subseteq 2[-2\pi, -\frac{3}{2}\pi).$$

Then, $G := [\frac{5}{8}\pi, \frac{7}{8}\pi) \cup [\pi, \frac{5}{4}\pi) \cup [\frac{7}{2}\pi, 4\pi) \cup [-\frac{3}{4}\pi, -\frac{\pi}{2}) \cup G_0$ is a wavelet set, since

$$\begin{aligned} [0, 2\pi) &= ([-2\pi, -\frac{11}{8}\pi) + 2\pi) \dot{\cup} [\frac{5}{8}\pi, \frac{7}{8}\pi) \dot{\cup} ([-\frac{9}{8}\pi, -\pi) + 2\pi) \\ &\quad \dot{\cup} [\pi, \frac{5}{4}\pi) \dot{\cup} ([-\frac{3}{4}\pi, -\frac{\pi}{2}) + 2\pi) \dot{\cup} ([\frac{7}{2}\pi, 4\pi) - 2\pi), \end{aligned}$$

and

$$\begin{aligned} [-2\pi, -\pi) \cup [\frac{5}{8}\pi, \frac{5}{4}\pi) &= [-2\pi, -\frac{3}{2}\pi) \dot{\cup} 2 \cdot [-\frac{3}{4}\pi, -\frac{\pi}{2}) \\ &\quad \dot{\cup} [\frac{5}{8}\pi, \frac{7}{8}\pi) \dot{\cup} \frac{1}{4} \cdot [\frac{7}{2}\pi, 4\pi) \dot{\cup} [\pi, \frac{5}{4}\pi) \end{aligned}$$

is a 2-dilation generator of a measurable partition of \mathbb{R} . However, $G \cap [-\pi, \pi)$ is not a null set, hence $(\sigma_E^G)^2 \neq id_{\mathbb{R}}$. \square

Lemma 5 can be useful in constructing special wavelet sets, and in particular, parametric families of wavelet sets.

Example 10. Let $l, m, n \in \mathbb{N}$ be given. Let $E_+(m) = [0, 2^{-m+1}\pi)$, $F_+ = [\pi, 2\pi)$. Observe that

$$\frac{1}{2^m}F_+ \subseteq E_+(m) \quad \text{and} \quad E_+(m) + 2^l\pi \subseteq 2^lF_+,$$

and $\frac{1}{2^m}F_+ \cap (E_+(m) + 2^l\pi) = \emptyset$. Let $G_+(l, m)$ be the measurable set obtained by applying Lemma 5. Straightforward computation shows that

$$G_+(l, m) = (\frac{2^l}{2^{l+m} - 1}\pi, 2^{-m+1}\pi) \cup [2^l\pi, 2^l\pi + \frac{2^l}{2^{l+m} - 1}\pi).$$

Similarly, let $E_-(m) = [-2\pi + 2^{-m+1}\pi, 0)$, $F_-(m) = [-2\pi + 2^{-m+1}\pi, -\pi + 2^{-m}\pi)$. Observe that

$$F_-(m) \subseteq E_-(m) \quad \text{and} \quad E_-(m) - 2^{m+n}\pi + 2^n\pi \subseteq 2^{m+n}F_-(m).$$

By applying Lemma 5 again, we get

$$\begin{aligned} G_-(m, n) &= [-2^{m+n}\pi + 2^n\pi - \frac{2^{m+n} - 2^n}{2^{m+n} - 1}\pi, -2^{m+n}\pi + 2^n\pi) \\ &\cup [-2\pi + 2^{-m+1}\pi, -\frac{2^{m+n} - 2^n}{2^{m+n} - 1}\pi). \end{aligned}$$

Thus, for fixed l, m, n , $G_-(m, n) \cup G_+(l, m)$ is both 2π -translation congruent to $E_-(m) \cup E_+(m) = [-2\pi + 2^{-m+1}\pi, 2^{-m+1}\pi)$ and 2-dilation congruent to $F_-(m) \cup F_+$, hence is a wavelet set. \square

Notice that $G_-(l, m, n) \cup G_+(l, m, n)$, $l, m, n \geq 1$ is exactly the family of wavelet sets $K_{l, m, n}$, $l, m, n \geq 1$ introduced by X. Fang and X. Wang in [FW] (see [FW] Example (5)).

In the following examples, we will construct some known and new wavelet sets in \mathbb{R}^2 with dilation matrix A given by $2 \cdot id_{\mathbb{R}^2}$. It is known (see [DLS1] and [SW]) that $E \subseteq \mathbb{R}^2$ is a wavelet set for $2 \cdot id_{\mathbb{R}^2}$ if and only if E is both 2π -translation congruent to $[-\pi, \pi) \times [-\pi, \pi)$ and 2-dilation congruent to $[-2\pi, 2\pi) \times [-2\pi, 2\pi) \setminus [-\pi, \pi) \times [-\pi, \pi)$. Wavelet sets in higher dimensional spaces with respect to arbitrary real expansive matrices can also be obtained in the similar way.

Example 11. Consider the first quadrant. Since we have the following two relations:

$$[0, \pi) \times [0, \pi) \setminus [0, \frac{\pi}{2}) \times [0, \frac{\pi}{2}) \subseteq [0, \pi) \times [0, \pi),$$

$$[0, \pi) \times [0, \pi) + (2\pi, 2\pi) \subseteq 4 \cdot [0, \pi) \times [0, \pi) \setminus [0, \frac{\pi}{2}) \times [0, \frac{\pi}{2}),$$

and $[0, \pi) \times [0, \pi) \setminus [0, \frac{\pi}{2}) \times [0, \frac{\pi}{2}) \cap ([0, \pi) \times [0, \pi) + (2\pi, 2\pi)) = \emptyset$. Applying Lemma 5, we can construct a set W_1 which is both 2π -translation congruent to $[0, \pi) \times [0, \pi)$ and 2-dilation congruent to $[0, \pi) \times [0, \pi) \setminus [0, \frac{\pi}{2}) \times [0, \frac{\pi}{2})$. Symmetrically, construct sets W_2, W_3, W_4 in second, third and fourth quadrants, respectively. Then $W_1 \cup W_2 \cup W_3 \cup W_4$ is a wavelet set. Straightforward computation shows that it is exactly the “four corners set” in [DLS2]. \square

Example 12. Consider the right half plane. Since we have the following two relations:

$$[0, \pi) \times [-\pi, \pi) \setminus [0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq [0, \pi) \times [-\pi, \pi),$$

$$[0, \pi) \times [-\pi, \pi) + (2\pi, 0) \subseteq 4 \cdot [0, \pi) \times [-\pi, \pi) \setminus [0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2}),$$

and $[0, \pi) \times [-\pi, \pi) \setminus [0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2}) \cap ([0, \pi) \times [-\pi, \pi) + (2\pi, 0)) = \emptyset$. By Lemma 5, we can construct a set W_1 which is both 2π -translation congruent to $[0, \pi) \times [-\pi, \pi)$ and 2-dilation congruent to $[0, \pi) \times [-\pi, \pi) \setminus [0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$. Symmetrically, construct the set W_2 in the left half plane. Then $W_1 \cup W_2$ is a wavelet set. Straightforward computation shows that it is exactly the “wedding cake set” (Example 6.6.2 in [DL]

and also Figure 2 in[DLS2].) □

Example 13. Consider the left-top half plane (above the line $y = x$ in the left half plane). Let $E_1 = \{(x, y) | x \geq -\pi, y \leq \pi, y \geq x\}$, and $F_1 = \{(x, y) | x \geq -\pi, y \leq \pi, y \geq x\} \setminus \{(x, y) | x \geq -\frac{\pi}{2}, y \leq \frac{\pi}{2}, y \geq x\}$. Then we have the following two relations:

$$F_1 \subseteq E_1, \quad E_1 + (-2\pi, 2\pi) \subseteq 4 \cdot F_1,$$

and $F_1 \cap (E_1 + (-2\pi, 2\pi)) = \emptyset$. Now use Lemma 5 to construct a set W_1 which is both 2π -translation congruent to E_1 and 2-dilation congruent to F_1 . Symmetrically, construct a corresponding set W_2 in the right-bottom half plane. Then $W_1 \cup W_2$ is a wavelet set. The diagram for this is given in Figure 1. □

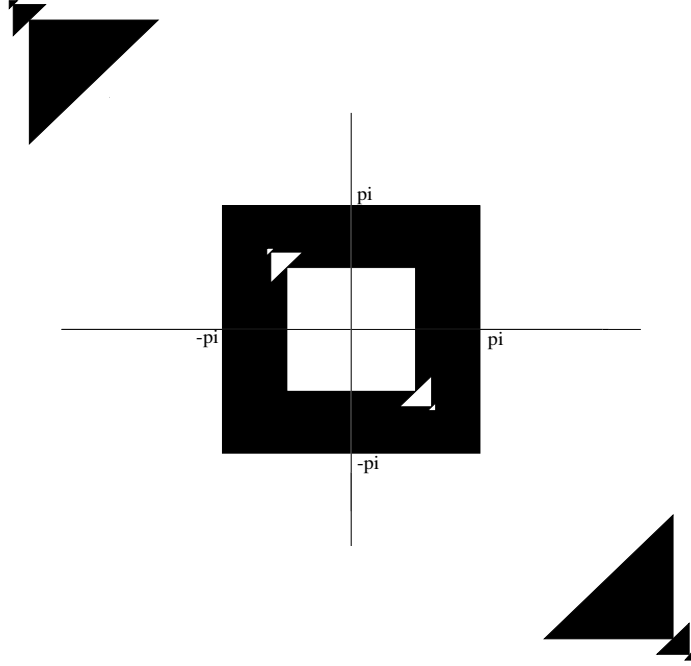


FIGURE 1 Pine Tree

Example 14. Consider the first quadrant. Let $E_1 = [-\frac{\pi}{2}, 0) \times [-\frac{\pi}{2}, 0) \cup [-\pi, -\frac{3}{4}\pi) \times [-\pi, -\frac{3}{4}\pi)$ and $F_1 = [\frac{3}{2}\pi, 2\pi) \times [\frac{3}{2}\pi, 2\pi)$. Then we have the following two relations:

$$F_1 \subseteq E_1 + (2\pi, 2\pi), \quad E_1 + (4\pi, 4\pi) \subseteq 2 \cdot F_1,$$

and $F_1 \cap (E_1 + (4\pi, 4\pi)) = \emptyset$. By Lemma 5, we can construct a set W_1 which is both 2π -translation congruent to E_1 and 2-dilation congruent to F_1 . Symmetrically, we can define E_2, E_3, E_4 and F_2, F_3, F_4 , and construct sets W_2, W_3, W_4 in the second, third and fourth quadrants, respectively. Let $B = [-\pi, \pi) \times [-\pi, \pi) \setminus ([-\frac{1}{2}\pi, \frac{1}{2}\pi) \times [-\frac{1}{2}\pi, \frac{1}{2}\pi) \cup [\frac{3}{4}\pi, \pi) \times [\frac{3}{4}\pi, \pi) \cup [-\pi, -\frac{3}{4}\pi) \times [\frac{3}{4}\pi, \pi) \cup [\frac{3}{4}\pi, \pi) \times [-\pi, -\frac{3}{4}\pi) \cup [[-\pi, -\frac{3}{4}\pi) \times [-\pi, -\frac{3}{4}\pi))$. Then since $B \cup E_1 \cup E_2 \cup E_3 \cup E_4 = [-\pi, \pi) \times [-\pi, \pi)$ and $2B \cup F_1 \cup$

$F_2 \cup F_3 \cup F_4 = [-2\pi, 2\pi) \times [-2\pi, 2\pi) \setminus [-\pi, \pi) \times [-\pi, \pi)$, $W_1 \cup W_2 \cup W_3 \cup W_4 \cup B$ is a wavelet set. Computation shows that it is one of the wavelet sets introduced in [SW]. The diagram for this is given in Figure 2. \square

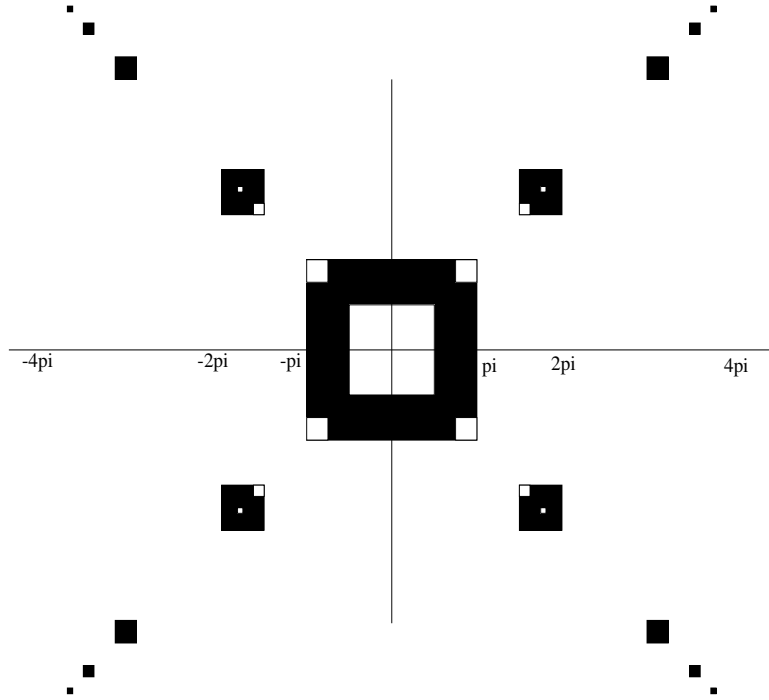


FIGURE 2 A Wavelet Set in [SW]

Remarks. (1) The idea of Lemma 5 was used in constructing a covering of \mathbb{R} by symmetric wavelet sets, and this was a key to subsequent work of the first author with Rzesotnik in [RZ].

(2) Further Directions: Interpolation maps, and interpolation pairs and more general interpolation families of wavelet sets, make sense and have been studied for matrix dilations in \mathbb{R}^n . Congruence domains also make sense for matrix dilations. Lemma 5 in this article was stated and proved for \mathbb{R}^n , and was applied to solve Question B in \mathbb{R}^1 , and also used to study examples of dyadic (i.e. for matrix dilation $2I$) wavelet sets in the plane. The results in this paper suggest some directions for further research. In particular, does Theorem 1 extend to matrix dilations in \mathbb{R}^n , especially for the cases where interpolation pairs are known to exist? It might be useful to try to extend Lemma 5 and also Theorem 3 to matrix dilations.

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