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A New Technique for Error Analysis of Finite Element Approximations of Parabolic Problems with Non-smooth Initial Data

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We propose a new technique for analyzing the error of finite element approximations of parabolic problems with non-smooth initial data. For homogeneous equation we prove optimal L^2-error estimate of order $O\left(h^2/t\right)$ for $t > 0$ when the given initial data is in L^2. Further, for non-homogeneous parabolic equation with zero initial data we establish an optimal error estimate of order $O(h^2)$ in L^2. Thus, we get the results of Luskin and Rannacher from [6] by a new technique that does not require error estimates in negative Sobolev norms.

1. Introduction

In this paper, we study the semi-discrete finite element method for solving initial-boundary value problem of the form

$$\begin{aligned} u_t + Au &= f(x,t) \text{ in } \Omega \times (0,\infty), \\ u &= 0 \text{ on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) &= u_0 \text{ in } \Omega. \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and $u_t = \partial u/\partial t$. Further, A is a selfadjoint, uniformly positive definite second order elliptic partial differential operator of the form

$$A = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial}{\partial x_i}) + a_0(x)I.$$

The non-homogeneous term f and the coefficients of A are assumed to be smooth functions of x.

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Finite element approximations for this type of problems have been studied in the past three decades (see, e.g. ([1], [2], [4], [5], [7], [9], [10], and [11]). Optimal convergence and superconvergence estimates have been established by using Ritz projection, duality argument, semi-group theory, and various types of a priori estimates. Self-adjoint problems with non-smooth initial data were analyzed in [1], [2] and [11], while nonself-adjoint problems were considered in [6], [8], and [4]. The semigroup theory used in [4], [11] and [8] and energy types of estimates in conjunction with duality argument used in [6] led to optimal error estimate in L^2 for $u_0 \in L^2(\Omega)$. For the non-homogeneous differential equation ($f \neq 0$) with homogeneous initial data ($u_0 = 0$), Luskin and Rannacher [6] obtained a quasi-optimal error estimate in L^2 that is uniform in time. A crucial role in their analysis play the error estimates in negative Sobolev norms.

In the present paper we used an elementary energy technique and duality argument in time to derive optimal error estimates in L^2 for both smooth and non-smooth initial data. More precisely, for homogeneous equation, a new energy formulation is used to derive optimal error estimate in L^2 when $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. In the case of non-smooth initial data i.e., when $u_0 \in L^2(\Omega)$, energy method and parabolic duality argument is used to obtain error estimate of order $O(h^2/t)$ for $t > 0$. An attractive feature of the proposed technique is that unlike [6], it does not require error estimates in negative Sobolev norms while dealing with non-smooth initial data. Further, for the non-homogeneous equation with zero initial condition, we were able to established optimal order error estimate in L^2, uniformly in time, provided that $\|f(0)\|$ and $\|f_t(t)\|$ remain bounded.

The paper is organized as follows. In Section 2, we introduce some notations, define the relevant function spaces, and recall some basic estimates from the literature. Section 3 is devoted to the error estimates for smooth initial data. Finally, error estimates with non-smooth initial data are discussed in Section 4.

2. Preliminaries

Let $H_0^1(\Omega) = \{\phi \in H^1(\Omega) \mid \ \phi = 0 \text{ on } \partial\Omega\}$. The standard weak formulation may be stated as follows: Find $u : [0, \infty) \to H_0^1(\Omega)$ such that

$$(u_t, \phi) + A(u, \phi) = (f, \phi) \quad \forall \phi \in H_0^1(\Omega) \tag{2.2}$$

with $u(0) = u_0$, where the bilinear form $A(\cdot, \cdot)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$ is given by

$$A(u, \phi) = \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} + a_0(x) u \phi \right) dx.$$

Here and below, we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the L^2 inner product and the corresponding norm on $L^2(\Omega)$. Further, $H^m = H^m(\Omega)$, $m \in$ **Z**, is the standard Sobolev space with a norm denoted by $\|\cdot\|_m$.

We shall assume that A satisfies the following conditions:

(A1) full elliptic regularity: i.e. there exists a positive constant α such that

$$\|A\phi\| \geq \alpha \|\phi\|_2, \quad \phi \in H^2(\Omega) \cap H_0^1(\Omega); \tag{2.3}$$

(A2) the bilinear from $A(\phi,\phi)$ is coercive in H^1: i.e. there exists a positive constant c such that

$$A(\phi,\phi) \geq c\|\phi\|_1^2, \quad \phi \in H_0^1(\Omega). \tag{2.4}$$

For the finite element Galerkin approximation, we assume that we are given a family $\{S_h\}$, $0 < h < 1$, of finite dimensional subspaces of $H_0^1(\Omega)$ such that for $r = 1, 2$

$$\inf_{\chi \in S_h} \{\|\phi - \chi\| + h\|\phi - \chi\|_1\} \leq Ch^r \|\phi\|_r, \quad \phi \in H^r(\Omega) \cap H_0^1(\Omega). \tag{2.5}$$

Throughout this paper C denotes a generic positive constant which does not depend on the mesh parameter h.

The standard semi-discrete finite element approximation of (1.1) is then defined as a function $u_h : [0,\infty) \to S_h$ such that

$$\begin{aligned} (u_{ht}, \chi) + A(u_h, \chi) &= (f, \chi), \quad \forall \chi \in S_h, \\ u_h(0) &= P_h u_0, \end{aligned} \tag{2.6}$$

where $P_h u_0$ is the standard L^2-projection of u_0 onto S_h.

Below, we state some a priori bounds for the solution u satisfying (1.1) under appropriate regularity assumption on the initial function u_0. For a proof, we refer to [6].

Lemma 1. *Let u satisfy (1.1). If $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega)$ then*

$$\|u(t)\|^2 + \int_0^t \|u(s)\|_1^2 ds \leq C\left(\|u_0\|^2 + \int_0^t \|f(s)\|^2 ds\right).$$

Moreover, when $u_0 \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$, we have

$$\|u(t)\|_1^2 + \int_0^t \{\|u_s(s)\|^2 + \|u(s)\|_2^2\} ds \leq C\left(\|u_0\|_1^2 + \int_0^t \|f(s)\|^2 ds\right).$$

Lemma 2. *Let u satisfy (1.1) with $f = 0$, and let $0 \leq i, j, k \leq 2$. If $0 \leq k + 2j - i \leq 2$, then*

$$t^i \left\| \frac{\partial^j u}{\partial t^j}(t) \right\|_k^2 \leq C\|u_0\|_{k+2j-i}^2.$$

Further, if $0 \le k + 2j - i - 1 \le 2$, *then*

$$\int_0^t s^i \left\| \frac{\partial^j u}{\partial s^j}(s) \right\|_k^2 ds \le C\|u_0\|_{k+2j-i-1}^2.$$

Following the arguments of Lemmas 1-2 on the discrete level, it is easy to derive the following stability estimates for the solution u_h satisfying (2.6).

Lemma 3. *Let* u_h *satisfy (2.6). If* $u_0 \in L^2(\Omega)$ *and* $f \in L^2(\Omega)$ *then*

$$\|u_h(t)\|^2 + \int_0^t \|u_h(s)\|_1^2 ds \le C \left(\|u_h(0)\|^2 + \int_0^t \|f(s)\|^2 ds \right).$$

Moreover, when $u_0 \in H_0^1(\Omega)$ *and* $f \in L^2(\Omega)$, *we have*

$$\|u_h(t)\|_1^2 + \int_0^t \|u_{hs}(s)\|^2 ds \le C \left(\|u_h(0)\|_1^2 + \int_0^t \|f(s)\|^2 ds \right).$$

Lemma 4. *Let* u_h *satisfy (2.6) with* $f = 0$. *Then we have*

$$\begin{aligned} t^k \|u_h(t)\|^2 + \int_0^t s^k \|u_h(s)\|_1^2 ds, &\le C\|u_h(0)\|^2, \quad k = 0, 2, \\ \int_0^t s\|u_{hs}(s)\|^2 + t\|u_h(t)\|_1^2 ds &\le C\|u_h(0)\|^2. \end{aligned}$$

3. Error estimates for smooth initial data

In this section, we obtain an optimal error estimate in L^2 when the initial data $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. First, we recall the associated Ritz projection defined as $R_h : H_0^1(\Omega) \to S_h$ by

$$A(R_h u - u, \chi) = 0, \ \ \forall\, \chi \in S_h. \tag{3.7}$$

Set $\rho = R_h u - u$. It is quite standard to prove that (cf. [3]) ρ and its temporal derivatives satisfy the following error estimates.

Lemma 5. *Let* ρ *satisfy (3.7). Then*

$$\begin{aligned} &\|\rho(t)\| + h\|\rho(t)\|_1 \le Ch^2\|u(t)\|_2, \\ &\|\rho_t(t)\| + h\|\rho_t(t)\|_1 \le Ch^2\|u_t(t)\|_2. \end{aligned}$$

For the analysis we split the error $e(t) = u(t) - u_h(t)$ as

$$e(t) = (R_h u - u_h) - (R_h u - u) = \theta - \rho.$$

Since the estimate of $\|\rho\|$ is already known it is enough to estimate θ. Using (2.2), (2.6) and (3.7), it is easy to verify that θ satisfies an error equation of the form

$$(\theta_t, \chi) + A(\theta, \chi) = -(\rho_t, \chi), \;\; \forall \chi \in S_h. \tag{3.8}$$

Now we define $\hat{\rho}(t) = \int_0^t \rho(\tau)d\tau$ and note that $\hat{\rho}(0) = 0$ and $\hat{\rho}_t = \rho$. Then, integrating (3.7) from 0 to t we obtain an equation for $\hat{\rho}$

$$A(\hat{\rho}(t), \chi) = 0, \quad \chi \in S_h. \tag{3.9}$$

Further, integrating (2.2) and (2.6) from 0 to t and then using (3.9) and $u_h(0) = P_h u_0$ we obtain an error equation in $\hat{\theta}$ of the form

$$(\hat{\theta}_t, \chi) + A(\hat{\theta}, \chi) = -(\rho, \chi), \;\; \chi \in S_h, \tag{3.10}$$

where $\hat{\theta}(t) = \int_0^t \theta(s)ds$.

Below, we prove a sequence of lemmas that will lead to the desired result.

Lemma 6. *Let $\hat{\theta}$ satisfy (3.10) and $u_h(0) = P_h u_0$. Then there is a positive constant C such that*

$$\int_0^t \|\theta(s)\|^2 ds + \|\hat{\theta}(t)\|_1^2 \leq C \int_0^t \|\rho(s)\|^2 ds.$$

Proof. Take $\chi = \theta$ in (3.10) and integrate from 0 to t to have

$$\int_0^t (\theta, \theta)ds + \frac{1}{2}A(\hat{\theta}, \hat{\theta}) \leq \int_0^t \|\rho\|\|\theta\|ds.$$

Now use standard kickback argument to complete the rest of the proof. □

Lemma 7. *Let θ satisfy (3.8) and $u_h(0) = P_h u_0$. Then there is a positive constant C independent of h such that*

$$t\|\theta(t)\|^2 + \int_0^t s\|\theta(s)\|_1^2 ds \leq C \int_0^t \{\|\rho(s)\|^2 + s^2\|\rho_s(s)\|^2\}ds.$$

Proof. Take $\chi = t\theta$ in (3.8) to have

$$\frac{1}{2}\frac{d}{dt}\{t(\theta, \theta)\} + tA(\theta, \theta) = \frac{1}{2}\|\theta\|^2 - t(\rho_t, \theta)$$

Integrating from 0 to t, we obtain

$$t\|\theta(t)\|^2 + \int_0^t s\|\theta(s)\|_1^2 ds \leq C \int_0^t \{\|\theta\|^2 + s^2\|\rho_s\|^2\}ds.$$

Then, use Lemma 6 to complete the rest of the proof. □

The main result of this section is given in the following theorem.

Theorem 1. *Let u and u_h, respectively, satisfy (1.1) and (2.6) with $f \equiv 0$. Then for $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_h(0) = P_h u_0$, we have*

$$\|e(t)\| \leq Ch^2 \|u_0\|_2.$$

Proof. By triangle inequality, we write

$$t^{1/2}\|e(t)\| \leq t^{1/2}\|\rho(t)\| + t^{1/2}\|\theta(t)\|.$$

For the first term on the right, use of Lemma 5 and a priori estimates in Lemma 2 yields

$$t^{1/2}\|\rho(t)\| \leq Ch^2 t^{1/2}\|u_0\|_2.$$

By Lemma 7, Lemma 5 and a priori estimates in Lemma 2, the second term is bounded by

$$\begin{aligned} t^{1/2}\|\theta(t)\| &\leq C\left(\int_0^t \{\|\rho(s)\|^2 + s^2\|\rho_s(s)\|^2\}ds\right)^{1/2} \\ &\leq Ch^2\left(\int_0^t \{\|u(s)\|_2^2 + s^2\|u_s(s)\|_2^2\}ds\right)^{1/2} \\ &\leq Ch^2 t^{1/2}\|u_0\|_2. \end{aligned}$$

Altogether these estimates yield the desired result and this completes the proof. □

Remark. We note that the similar result (Theorem 1) has been established before in [6] in which parabolic duality argument is used in a crucial way and the initial approximation is is assumed to be $u_h(0) = R_h u_0$. From the proof of Theorem 1, it is clear that we can choose $u_h(0)$ as the L^2 projection of u_0 into S_h instead of the elliptic projection $R_h u_0$ and use of duality argument is avoided in the proof.

4. Error estimates for non-smooth initial data

This section we establish an optimal error estimate in L^2 for non-smooth initial data, i.e. $u_0 \in L^2(\Omega)$. Below, we shall prove a sequence of lemmas which will be used to derive the error estimates for non-smooth initial data.

Lemma 8. *Let $\hat{\rho}$ satisfy (3.9). Then*

$$\|\hat{\rho}\| + h\|\hat{\rho}\|_1 \leq Ch^2\|u_0\|.$$

Proof. By (3.9), we have

$$c\|\hat{\rho}\|_1^2 \leq A(\hat{\rho}, \hat{\rho}) = A(\hat{\rho}, \hat{\rho} - \chi)$$

In view of (2.5)

$$\|\hat{\rho}\|_1 \leq C \inf_{\underline{\chi} \in S_h} \|\hat{u} - \underline{\chi}\|_1 \leq Ch\|\hat{u}\|_2, \tag{4.11}$$

where $\underline{\chi} = \chi - R_h\hat{u}$. Now it remains to estimate $\|\hat{u}\|_2$. Integrating (1.1) from 0 to t, we obtain for $f = 0$

$$A\hat{u} = u_0 - u(t), \quad t \geq 0.$$

An use of elliptic regularity and Lemma 2 yields

$$\|\hat{u}\|_2 \leq C\left(\|u_0\| + \|u(t)\|\right) \leq C\|u_0\|, \quad t \geq 0. \tag{4.12}$$

Combine (4.11) and (4.12) to obtain the estimate for $\|\hat{\rho}\|_1$.

Next, we shall use duality argument to estimate $\|\hat{\rho}\|$. Let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$\begin{aligned} A\psi &= \hat{\rho} \ \text{ in } \Omega, \\ \psi &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{4.13}$$

Note that by (2.3) the solution ψ satisfies the following a priori estimate

$$\|\psi\|_2 \leq C\|\hat{\rho}\|. \tag{4.14}$$

Multiply (4.13) by $\hat{\rho}$. Then, taking L^2 inner-product over Ω and using (3.9), we obtain

$$\|\hat{\rho}\|^2 = A(\hat{\rho}, \psi - \chi) \leq C\|\hat{\rho}\|_1 \inf_{\chi \in S_h} \|\psi - \chi\|_1.$$

The desired estimate now follows from (2.5), (4.14) and the estimate of $\|\hat{\rho}\|_1$. This completes the rest of the proof. □

Lemma 9. *Let u and u_h be the solution of (1.1) and (2.6), respectively. Then for $u_0 = 0$, we have*

$$\int_0^t \|u(s) - u_h(s)\|_1^2 ds \leq Ch^2 \int_0^t \|f\|^2 ds.$$

Proof. The proof is contained in (page 100, [6]).

Define $\hat{e}(t) = \int_0^t e(s)ds$. In order to obtain optimal L^2-error estimate in the case non-smooth data, it is convenient to prove first an estimate of $\|\hat{e}\|$. This is accomplished via parabolic duality argument. For this purpose, we now consider the following backward problem: for a fixed $t > 0$ and given any $\bar{f} \in L^2(\Omega)$, let $v(s) \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$v_s - Av = \bar{f}, \ \ s \leq t, \quad v(t) = g. \tag{4.15}$$

The associated weak formulation is then find $v : [0, t) \to H_0^1(\Omega)$ such that

$$(\phi, v_s) - A(\phi, v) = (\phi, \bar{f}), \quad \forall \phi \in H_0^1(\Omega), \ s \leq t \tag{4.16}$$

with $v(t) = g$. Similarly, the finite element approximation of the above problem is: find $v_h : [0, t) \to S_h$ such that

$$(\chi, v_{hs}) - A(\chi, v_h) = (\chi, \bar{f}), \tag{4.17}$$

$\forall \chi \in S_h, \ s \le t$ with $v_h(t) = g_h$, where g_h is a suitable approximation of g in S_h to be defined later.

Remark: With a simple change of variables in the proofs of Lemmas 1-2, it is easy to obtain a priori bounds for the backward solutions v and v_h (cf. [6]).

Lemma 10. *Assume that $u_0 \in L^2(\Omega)$ and $f = 0$. Then there is a constant C independent of h such that*

$$\|\hat{e}(t)\| \le Ch^2 \|u_0\| \tag{4.18}$$

holds.

Proof. Let $w(s) \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the backward problem

$$w_s - Aw = \hat{e}, \ \ s \le t, \quad w(t) = 0. \tag{4.19}$$

With a simple change of variables in the proofs of Lemmas 1- 2 and their discrete analogies and Lemma 9, it is an easy to check that the solution $w(s)$ and its semi-discrete approximation $w_h(s)$, which may be stated in a similar to (4.16)-(4.17) manner, satisfy the following estimate

$$\int_0^t \{\|w_s - w_{hs}\|^2 + h^{-2}\|w - w_h\|_1^2\} ds \le C \int_0^t \|\hat{e}\|^2 ds. \tag{4.20}$$

We form L^2-inner product of (4.19) with e to obtain

$$\frac{1}{2}\frac{d}{ds}\|\hat{e}(s)\|^2 = \frac{d}{ds}(e, w_h) + (e, w_s - w_{hs}) - A(e, w - w_h), \tag{4.21}$$

where we have used the fact that (recall that $f = 0$)

$$(e_t, \chi) + A(e, \chi) = 0, \quad \chi \in S_h. \tag{4.22}$$

Multiply both side of (4.21) by s and integrate from 0 to t to have

$$\begin{aligned} \frac{1}{2}t\|\hat{e}(t)\|^2 &= \frac{1}{2}\int_0^t \|\hat{e}(s)\|^2 ds + \int_0^t (e, w_h) ds + \int_0^t s(\rho, w_s - w_{hs}) ds \\ &\quad - \int_0^t sA(\rho, w - w_h) ds = \frac{1}{2}\int_0^t \|\hat{e}(s)\|^2 ds + I_1 + I_2 + I_3. \end{aligned}$$

Since $\hat{e}(0) = 0 = w_h(t)$, we obtain using (4.20)

$$\begin{aligned} |I_1| &= |-\int_0^t (\hat{e}, w_{hs}) ds| \le C \int_0^t \|\hat{e}\| \|w_{hs}\| ds \\ &\le \left(\int_0^t \|\hat{e}\|^2 ds\right)^{1/2} \left(\int_0^t \|w_{hs}\|^2 ds\right)^{1/2} \\ &\le C \int_0^t \|\hat{e}(s)\|^2 ds. \end{aligned}$$

For I_2 and I_3, an application of Lemma 5, (4.20) and a priori estimates yields

$$\begin{aligned}|I_2|+|I_3| &\leq \int_0^t s\|\rho\|\|w_s-w_{hs}\|ds+\int_0^t s\|\rho\|_1\|w-w_h\|_1 ds\\ &\leq Ch^4\int_0^t s^2\|u\|_2^2 ds+C\int_0^t\{\|w_s-w_{hs}\|^2+h^{-2}\|w-w_h\|_1^2\}ds\\ &\leq Ch^4 t\|u_0\|^2+C\int_0^t\|\hat{e}\|^2 ds.\end{aligned}$$

Altogether now leads to

$$t\|\hat{e}(t)\|^2\leq Ch^4t\|u_0\|^2+C\int_0^t\|\hat{e}(s)\|^2ds. \tag{4.23}$$

It now remains to estimate $\int_0^t\|\hat{e}\|^2ds$. Multiply (4.19) by $\hat{e}$ and integrate by parts with respect to x to get

$$\|\hat{e}(s)\|^2=\frac{d}{ds}(\hat{e},w_h)+(\hat{e},w_s-w_{hs})-A(\hat{e},w-w_h). \tag{4.24}$$

Here, we have used the relation $(e,\chi)+A(\hat{e},\chi)=0$,which is obtained by integrating (4.22) from 0 to t and using $u_h(0)=P_hu_0$. Now integrate (4.24) from 0 to t and note that $\hat{e}(0)=0=w_h(t)$ to get

$$\int_0^t\|\hat{e}(s)\|^2ds=\int_0^t(\hat{\rho},w_s-w_{hs})ds-\int_0^t A(\hat{\rho},w-w_h)ds.$$

Using Lemma 8 and (4.20), it now follows that

$$\begin{aligned}\int_0^t\|\hat{e}(s)\|^2ds &\leq \int_0^t\|\hat{\rho}\|\|w_s-w_{hs}\|ds+\int_0^t\|\hat{\rho}\|_1\|w-w_h\|_1ds\\ &\leq C(\epsilon)\int_0^t\{\|\hat{\rho}\|^2+h^2\|\hat{\rho}\|_1^2\}ds\\ &\quad+\epsilon\int_0^t\{\|w_s-w_{hs}\|^2+h^{-2}\|w-w_h\|_1^2\}ds\\ &\leq Cth^4\|u_0\|^2+\epsilon C\int_0^t\|\hat{e}\|^2ds.\end{aligned}$$

Choose ϵ appropriately so that

$$\int_0^t\|\hat{e}\|^2ds\leq Cth^4\|u_0\|^2. \tag{4.25}$$

Now combine (4.25) and (4.23) to complete the rest of the proof. □

Remark. Defining the error $\bar{e}=v-v_h$ associated with the backward problem (4.16) and its finite element approximation (4.17), set $\tilde{\bar{e}}(s)=-\int_s^t\bar{e}(\tau)d\tau$, $s\leq t$.

Then, for $g \in L^2(\Omega)$ and $\bar{f} = 0$, analogous to Lemma 10, it is easy to show that

$$\|\tilde{\bar{e}}\| \leq Ch^2 \|g\|. \tag{4.26}$$

We conclude this section by showing our main results in the following two theorems.

Theorem 2. *Let u and u_h be solutions of (1.1) and (2.6), respectively with $f = 0$. Assume that $u_0 \in L^2(\Omega)$ and $u_h(0) = P_h u_0$. Then there is a positive constant C independent of h such that*

$$\|e(t)\| \leq Ct^{-1}h^2\|u_0\|, \quad t > 0.$$

Proof. From (2.2), (2.6), (4.16) and (4.17) with $\bar{f} = 0$, we first note that

$$\frac{d}{ds}\left\{s^2[(u,v) - (u_h, v_h)]\right\} = 2s\left\{(u,v) - (u_h, v_h)\right\}$$

Integrate the above equation from 0 to t. Then, with $g_h = P_h g$, where P_h is the L^2-projection onto S_h defined by $(P_h g, \chi) = (g, \chi),\ \chi \in S_h$, we have

$$\begin{aligned}
\frac{1}{2}t^2(e(t), g) &= \int_0^t s\left\{(u(s), v(s)) - (u_h(s), v_h(s))\right\} ds \\
&= \int_0^t s(e(s), v)ds - \int_0^t s(e(s), \bar{e}(s))ds + \int_0^t s(u, \bar{e}(s))ds \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, we integrate by parts to have

$$I_1 = \int_0^t s(\hat{e}_s, v)ds = t(\hat{e}, g) - \int_0^t (\hat{e}, v)ds - \int_0^t s(\hat{e}, v_s)ds,$$

and hence, by Cauchy-Schwarz's inequality, Lemma 10 and Lemma 2 (with time reverse) we obtain

$$|I_1| \leq t\|\hat{e}\|\|g\| + \int_0^t \|\hat{e}\|\|v\|ds + \int_0^t s\|\hat{e}\|\|v_s\|ds \leq Cth^2\|u_0\|\|g\|.$$

Since $\tilde{\bar{e}}(t) = 0$, using (4.26) and the a priori estimates in Lemma 2, we can estimate I_3 as

$$|I_3| \leq \int_0^t \|\tilde{\bar{e}}\|\|u\|ds + \int_0^t s\|\tilde{\bar{e}}\|\|u_s\|ds \leq Cth^2\|u_0\|\|g\|.$$

Finally, for I_2, integration by parts leads to

$$I_2 = -\int_0^t s(\hat{e}_s, \bar{e}) = -t(\hat{e}, g - P_h g) + \int_0^t (\hat{e}, \bar{e})ds + \int_0^t s(\hat{e}, \bar{e}_s)ds.$$

Apply Cauchy-Schwarz's inequality, Lemma 10 and a priori estimates in Lemma 2 (with time reverse) to obtain

$$|I_2| \le t\|\hat{e}\|\|g\| + \int_0^t \|\hat{e}\|\|\bar{e}\|ds + \int_0^t s\|\hat{e}\|\|\bar{e}_s\|ds \le Cth^2\|u_0\|\|g\|.$$

Altogether these estimates yield the desired result and complete the proof. □

We now have the following result for the nonhomogeneous equation ($f \neq 0$).

Theorem 3. *Let u and u_h be solutions of (1.1) and (2.6), respectively. Assume that $u_0 = 0$ and $f(0), f_t(t) \in L^2(\Omega)$. Then there is a positive constant C independent of h such that*

$$\|e(t)\| \le Ch^2 \left(\|f(0)\| + \max_{[0,t]} \|f_s(s)\| \right).$$

Proof. As before, using (2.2), (2.6), (4.16) and (4.17) with $\bar{f} = 0$, we first note that

$$\frac{d}{ds}\{(u,v) - (u_h, v_h)\} = (f, v - v_h) = (f, \bar{e}).$$

Further, integrate the above equation from 0 to t. Then, with $u_h(0) = P_h u_0$ and $g_h = P_h g$, we obtain

$$\begin{aligned} (e(t), g) &= (u_0, v(0)) - (u_h(0), v_h(0)) + \int_0^t (f, \bar{e})ds \\ &= (u_0, \bar{e}(0)) + \int_0^t (f, \bar{e})ds. \end{aligned}$$

Since $u_0 = 0$ and $\tilde{\bar{e}}_s = \bar{e}$, we rewrite the above equation as

$$(e(t), g) = \int_0^t (f, \tilde{\bar{e}}_s)ds = -(f(0), \tilde{\bar{e}}(0)) - \int_0^t (f_s, \tilde{\bar{e}})ds.$$

Apply Cauchy-Schwarz's inequality and (4.26) to have

$$\begin{aligned} (e(t), g) &\le \|f(0)\|\|\tilde{\bar{e}}(0)\| + \int_0^t \|f_s(s)\|\|\tilde{\bar{e}}(s)\|ds \\ &\le Ch^2\|g\| \left(\|f(0)\| + \max_{[0,t]} \|f_s(s)\| \right), \end{aligned}$$

which yields the desired result. □

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References

[1] G. Baker, J. Bramble and V. Thomée, *Single step Galerkin approximations for parabolic problems*, Math. Comp., 31 (1977), pp. 818–847.

[2] J. H. Bramble, A. H. Schatz, V. Thomée and L. B. Wahlbin, *Some convergence estimates for Galerkin type approximations for parabolic equations*, SIAM J. Numer. Anal. 14 (1977), pp. 218–241.

[3] P. G. Ciarlet, *The finite element method for elliptic problems*, North Holland, Amsterdam NY, Oxford, 1978.

[4] M. Huang and V. Thomée, *Some convergence estimates for semidiscrete type schemes for time-dependent nonselfadjoint parabolic equations*, Math. Comp. 37 (1981), pp. 327–346.

[5] M. Huang and V. Thomée, *On the backward Euler method for parabolic equations with rough initial data*, SIAM J. Numer. Anal. 19 (1982), 599–603.

[6] M. Luskin and R. Rannacher, *On the smoothing property of the Galerkin method for parabolic equations*, SIAM J. Numer. Anal. 19 (1981), pp. 93–113.

[7] P. A. Raviart, *The use of numerical integration in finite element methods for solving parabolic equations*, In J.J.H. Miller, editor, Topics in Numerical Analysis, pages 233–264. Academic Press, 1973.

[8] P. Sammon, *Convergence estimates for semidiscrete parabolic equation approximations*, SIAM J. Numer. Anal. 19 (1982), pp. 68–92.

[9] M. F. Wheeler, *A priori* L^2 *error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal. 10 (1973), pp. 723–759.

[10] V. Thomée, *Negative norm estimates and superconvergence in Galerkin methods for parabolic problems*, Math. Comp., 34 (1980), pp. 93–113.

[11] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Lecture Notes in Math. 1054 (1984), Springer-Verlag.