

MATH 152H, FALL 2005, FINAL EXAM - SOLUTIONS

Disclaimer: This is only a sketch of solutions; some technical details are often omitted. You are expected to demonstrate all the details in your work.

Problem 1. (30 points) Let $f(x) = \ln(1 - x)$.

a) (15 points) Use geometric series to find the Maclaurin series expansion for $f(x)$. What is the interval of convergence of this series? Explain.

b) (15 points) Use your result in part a to evaluate the integral

$$\int \frac{f(x)}{x^2} dx$$

as an infinite series. What is the interval of convergence of this series? Explain.

Solution. Part a). Recall that the geometric series is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

and its interval of convergence is $(-1, 1)$. Then for all $x \in (-1, 1)$,

$$\ln(1-x) = - \int \frac{dx}{1-x} = - \sum_{n=0}^{\infty} \int x^n dx = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1},$$

and the radius of convergence of this series is 1, since integration of a series term by term does not change the radius of convergence. Testing the endpoints of the interval $[-1, 1]$ for convergence, we see that when $x = -1$ the series

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

is convergent by the alternating series test. However when $x = 1$, the series

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which is divergent. Hence the interval of convergence of our series is $[-1, 1)$.

Part b). Notice that by our solution to part a

$$\frac{\ln(1-x)}{x^2} = -\frac{1}{x^2} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\frac{1}{x} - \sum_{n=1}^{\infty} \frac{x^{n-1}}{n+1},$$

and so

$$\int \frac{\ln(1-x)}{x^2} dx = -\ln|x| - \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}.$$

Once again, the radius of convergence did not change, so it is 1. Let us test the convergence at the endpoints of the interval $[-1, 1]$. When $x = -1$, our series becomes

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)},$$

which is absolutely convergent by the comparison test, since the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a p -series with $p = 2 > 1$, and so is convergent. Therefore our series converges at both endpoints, hence its interval of convergence is $[-1, 1]$.

Problem 2. (30 points) Let $f(x) = \frac{1}{x}$.

a) (15 points) Consider the region

$$\mathcal{R} = \{(x, y) : x \geq 1, f(x)^2 \leq y \leq f(x)\}.$$

Find the volume of the solid obtained by rotating \mathcal{R} about the x -axis. Explain.

b) (15 points) Solve the initial value problem

$$y' + f(x)y = f(x)^2, \quad y(1) = 1.$$

Explain.

Solution. Part a). Using the washers method, we see that this volume is equal to

$$\int_1^{\infty} A(x) \, dx,$$

where $A(x)$ is the area of a cross-section, which is an annulus with the outer radius $\frac{1}{x}$ and the inner radius $\frac{1}{x^2}$. Hence

$$A(x) = \pi \left(\frac{1}{x^2} - \frac{1}{x^4} \right),$$

and so the volume is

$$\pi \left(\int_1^{\infty} x^{-2} \, dx - \int_1^{\infty} x^{-4} \, dx \right) = \pi \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + \frac{1}{3t^3} + 1 - \frac{1}{3} \right) = \frac{2\pi}{3}.$$

Part b). We want to solve the linear first order differential equation

$$y' + \frac{y}{x} = \frac{1}{x^2}.$$

If we multiply both sides of this equation by the integrating factor

$$I(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x,$$

we obtain

$$xy' + y = \frac{1}{x},$$

which can be rewritten as

$$\frac{d}{dx}(xy) = \frac{1}{x}.$$

If we integrate both sides with respect to x , we get

$$xy = \int \frac{dx}{x} = \ln x + C,$$

and so the general solution is

$$y = \frac{\ln x + C}{x}.$$

Using the initial condition $y(1) = 1$, we see that $C = 1$, and so the solution to our initial value problem is

$$y = \frac{\ln x + 1}{x}.$$

Problem 3. (20 points) Consider the following three points in 3-dimensional space:

$$P = (a, 1, -1), \quad Q = (0, 1, 2), \quad R = (2, 0, -1),$$

where a is a real number.

a) (10 points) Describe all values of a for which the points P, Q, R are not colinear. Explain.

b) (10 points) Let $S = (0, 1, 1)$. Find all values of a for which the points P, Q, R, S are coplanar. Explain.

Solution. Part a). Consider the vectors

$$QP = \langle a, 0, -3 \rangle, \quad RP = \langle a - 2, 1, 0 \rangle.$$

The points P, Q, R are colinear if and only if these two vectors are parallel, that is if their cross product is equal to $\mathbf{0}$. Notice that

$$\begin{aligned} QP \times RP &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0 & -3 \\ a-2 & 1 & 0 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 0 & -3 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a & -3 \\ a-2 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a & 0 \\ a-2 & 1 \end{vmatrix} \\ &= 3\mathbf{i} + 3(a-2)\mathbf{j} + a\mathbf{k}, \end{aligned}$$

which is not equal to the zero vector for any values of a . Therefore the points P, Q, R are not colinear for any values of a .

Part b). Notice that the vector

$$SP = \langle a, 0, -2 \rangle.$$

Also, the points P, Q, R, S are coplanar if and only if the parallelepiped with vertices P, Q, R, S has volume equal to 0. Recall that this volume can be computed using the scalar triple product formula:

$$SP \cdot (QP \times RP) = (a\mathbf{i} - 2\mathbf{k}) \cdot (3\mathbf{i} + 3(a-2)\mathbf{j} + a\mathbf{k}) = 3a - 2a = a,$$

which is equal to 0 if and only if $a = 0$. Therefore P, Q, R, S are coplanar if and only if $a = 0$.

Problem 4. (20 points) Let $f(x) = \sin^2(x)$, and consider the region

$$\mathcal{R} = \{(x, y) : 0 \leq x \leq \pi, -f(x) \leq y \leq f(x)\}.$$

a) (10 points) Find the area of the region \mathcal{R} . Explain.

b) (10 points) Find the volume of the solid obtained by rotating the region \mathcal{R} about the line $x = -1$. Explain.

Solution. Part a). The region \mathcal{R} is symmetric with respect to the x -axis, so we can use the double angle formula for cosine to see that its area is

$$A = 2 \int_0^\pi \sin^2(x) dx = \int_0^\pi (1 - \cos(2x)) dx = \left(x - \frac{\sin(2x)}{2} \right)_0^\pi = \pi.$$

Part b). Notice that the centroid of \mathcal{R} is the point C with coordinates $(\frac{\pi}{2}, 0)$, therefore if \mathcal{R} is rotated about the line $x = -1$, C travels the distance

$$d = 2\pi \left(1 + \frac{\pi}{2} \right) = \pi(\pi + 2).$$

Then, by the Theorem of Pappus, this volume of revolution is equal to

$$Ad = \pi^2(\pi + 2).$$