MATH 304, SECTIONS 502 - 503, FALL 2005

TAKE-HOME SHORT ANSWER TEST 4

This take-home test is to be completed by the student alone, with no aid from any other student. This test is due on Wednesday, November 9, 2005, in class.

Aggie Honor Code:

"An Aggie does not lie, cheat, or steal, or tolerate those who do."

Please read and sign the following statement:

"On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work."

Signature of student: SOLUTIONS

Student's name printed: SOLUTIONS

Question 1: (15 points)
(a) Is the following a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$:

$$L(x_1, x_2, x_3) = (x_3, x_1 + 2x_2 - 5x_3, x_1 - x_2) \ ?$$

Briefly justify your answer. (5 points)
(b) Is the following a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$:

$$L(x_1, x_2, x_3) = (x_2 - x_3, x_1 + 2) \ ?$$

Briefly justify your answer. (5 points)
(c) Is the following a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$:

$$L(x_1, x_2, x_3) = (x_2 - x_3, x_1 x_2, 2x_2 - x_1, x_3 + x_2) \ ?$$

Briefly justify your answer. (5 points)

Answer to Question 1:

(a) Yes; $L(x_1, x_2, x_3) = \begin{pmatrix} 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

(b) No; the property $L(\boldsymbol{x} + \boldsymbol{y}) = L(\boldsymbol{x}) + L(\boldsymbol{y})$ fails because of the way the second coordinate of $L(x_1, x_2, x_3)$ is defined.

(c) No; the property $L(\boldsymbol{x} + \boldsymbol{y}) = L(\boldsymbol{x}) + L(\boldsymbol{y})$ fails because of the way the second coordinate of $L(x_1, x_2, x_3)$ is defined.

Question 2: (15 points)
Find characteristic polynomial, eigenvalues, and corresponding eigenspaces for the following matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

Answer to Question 2:

For the matrix A, the characteristic polynomial is:

$$\begin{aligned} p_A(t) &= \det(A - tI_3) = \det \begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 2 & 1 & -(2+t) \end{pmatrix} \\ &= -t^3 - 2t^2 + t + 2 = -(t-1)(t+1)(t+2), \end{aligned}$$

therefore the eigenvalues of A are $1, -1, -2$. Next we find the eigenspaces.

1. Let $\lambda_1 = 1$, then the correpsonding eigenspace is:

$$\begin{aligned} V_1 &= \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix} \boldsymbol{x} = \mathbf{0} \right\} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

2. Let $\lambda_2 = -1$, then the correpsonding eigenspace is:

$$\begin{aligned} V_2 &= \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \boldsymbol{x} = \mathbf{0} \right\} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Answer to Question 2:

3. Let $\lambda_3 = -2$, then the correpsonding eigenspace is:

$$\begin{aligned} V_3 &= \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \boldsymbol{x} = \mathbf{0} \right\} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \right\}. \end{aligned}$$

Question 3: (15 points)
(a) Suppose that A is a 157×157 matrix with real entries. Prove that A must have at least one real eigenvalue. (5 points)

(b) Suppose A is a 2×2 matrix with real entries so that $\det(A) = 9$, and

$$a_{11} + a_{22} = 6.$$

Find eigenvalues of A. (5 points)

(c) Let $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$. Factor the matrix A into a product XDX^{-1}, where D is diagonal. (5 points)

Answer to Question 3:

(a) The total number of eigenvalues, counting with multiplicity, must be 157. Complex eigenvalues come in conjugate pairs, so there must be an even number of them, but 157 is an odd number. Therefore there has to be at least one real eigenvalue.

(b) The eigenvalues λ_1, λ_2 must satisfy

$$\lambda_1\lambda_2 = \det(A) = 9, \quad \lambda_1 + \lambda_2 = a_{11} + a_{22} = 6,$$

which means that $\lambda_1 = 6 - \lambda_2$, and so

$$\lambda_2(6 - \lambda_2) = -\lambda_2^2 + 6\lambda_2 = 9,$$

hence $\lambda_2^2 - 6\lambda_2 + 9 = (\lambda_2 - 3)^2 = 0$, therefore

$$\lambda_1 = \lambda_2 = 3.$$

(c) Characteristic polynomial of A is

$$p_A(t) = \det \begin{pmatrix} 5-t & 6 \\ -2 & -2-t \end{pmatrix} = (t-1)(t-2),$$

so the eigenvalues are $1, 2$. We can also find eigenvectors corresponding to these eigenvalues, for instance $\boldsymbol{x}_1 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, corresponding to $\lambda_1 = 1$, and $\boldsymbol{x}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, corresponding to $\lambda_2 = 2$. These are linearly independent vectors, and so $A = XDX^{-1}$, where

$$X = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Question 4: (15 points)
Solve the following initial value problem.

$$\begin{aligned} y_1' &= y_1 \\ y_2' &= y_1 + 2y_2 \\ y_3' &= y_2 - 3y_3 \end{aligned} \qquad y_1(0) = 1,\ y_2(0) = 2,\ y_3(0) = 3$$

Answer to Question 4:

Our system of linear first order differential equations can be rewritten as $\boldsymbol{Y}' = A\boldsymbol{Y}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{pmatrix}, \quad \boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \boldsymbol{Y}' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix}.$$

Characteristic polynomial of the matrix A is

$$p_A(t) = \det \begin{pmatrix} 1-t & 0 & 0 \\ 1 & 2-t & 0 \\ 0 & 1 & -3-t \end{pmatrix} = -(1-t)(2-t)(3+t),$$

and so the eigenvalues of A are $1, 2, -3$. We can also find corresponding eigenvectors, for instance $\boldsymbol{x}_1 = \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix}$ corresponding to $\lambda_1 = 1$, also $\boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$ corresponding to $\lambda_2 = 2$, and $\boldsymbol{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ corresponding to $\lambda_3 = -3$. These three vectors are easily seen to be linearly independent, and so the general solution to our system will be of the form

$$\boldsymbol{Y} = C_1 e^{\lambda_1 t}\boldsymbol{x}_1 + C_2 e^{\lambda_2 t}\boldsymbol{x}_2 + C_3 e^{\lambda_3 t}\boldsymbol{x}_3 = \begin{pmatrix} -4C_1 e^t \\ 4C_1 e^t + 5C_2 e^{2t} \\ C_1 e^t + C_2 e^{2t} + C_3 e^{-3t} \end{pmatrix}.$$

Combining this with the initial conditions, we find that
$y_1(0) = -4C_1 = 1,\ y_2(0) = 4C_1 + 5C_2 = 2,\ y_3(0) = C_1 + C_2 + C_3 = 3,$
therefore $C_1 = \frac{-1}{4}$, $C_2 = \frac{3}{5}$, and $C_3 = \frac{53}{20}$. Hence the solution to our initial value problem is:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} e^t \\ -e^t + 3e^{2t} \\ -\frac{1}{4}e^t + \frac{3}{5}e^{2t} + \frac{53}{20}e^{-3t} \end{pmatrix}.$$

Question 5: (15 points)
For each of the following subspaces V of $\mathbb{R}^3$ find its orthogonal complement.

(a) $V = \text{span}\{(1,2,3),(0,0,1)\}$. (5 points)

(b) V is the rowspace of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. (5 points)

(c) V is the nullspace of the matrix $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}$. (5 points)

Answer to Question 5:

(a) $V^{\perp} = \text{span}\left\{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right\}$.

(b) $V^{\perp} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$.

(c) $V^{\perp} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}\right\}$.