MATH 304, SECTIONS 502 - 503, FALL 2005

TAKE-HOME SHORT ANSWER TEST 5

This take-home test is to be completed by the student alone, with no aid from anyone else. This test is due on Wednesday, November 30, 2005, in class.

Aggie Honor Code:

"An Aggie does not lie, cheat, or steal, or tolerate those who do."

Please read and sign the following statement:

"On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work."

Signature of student: SOLUTIONS

Student's name printed: SOLUTIONS

Question 1: (20 points)
Find the best linear least squares fit to the following set of points:

$$(0, 1),\ (1, 0),\ (2, 2).$$

Answer to Question 1:
The solution we are looking for is a linear polynomial

$$p(x) = c_0 + c_1 x,$$

which is the closest fit to the above data points. To find the coefficients c_0 and c_1 of this polynomial, consider the overdetermined linear system

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

If we let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$, and $\boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, then $A^t = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, and

$$A^t A = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}.$$

In particular, notice that A has rank 2, and so $A^t A$ is invertible, and

$$\left(A^t A\right)^{-1} = \begin{pmatrix} 5/6 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Also, $A^t \boldsymbol{y} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, therefore

$$\hat{\boldsymbol{c}} = \left(A^t A\right)^{-1} A^t \boldsymbol{y} = \begin{pmatrix} 5/6 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

is the unique least squares solution to the system $A\boldsymbol{c} = \boldsymbol{y}$. Hence the unique solution to our linear approximation problem is the polynomial

$$p(x) = \frac{1}{2} + \frac{1}{2} x.$$

Question 2: (20 points)
Recall that $\mathbb{P}_3$ is the space of all polynomials of degree less than 3 with real coefficients. Define an inner product $< \, , \, >$ on $\mathbb{P}_3$, given by

$$< p, q >= \sum_{i=1}^{3} p(i-1)q(i-1),$$

and let $\| \ \|$ be the norm with respect to this inner product.

(a) Let

$$p(x) = 1 + x - x^2, \ q(x) = 2 - x.$$

Find a polynomial $f(x)$ in $\mathbb{P}_3$ which is orthogonal to both, $p(x)$ and $q(x)$ with respect to $< \, , \, >$. (15 points)

(b) Find the distance between $p(x) + q(x)$ and $f(x)$. (5 points)

Answer to Question 2:

(a) We want to find a polynomial

$$f(x) = c_0 + c_1 x + c_2 x^2,$$

such that

$$\begin{aligned} < p, f > &= \sum_{i=1}^{3} p(i-1)f(i-1) = f(0) + f(1) - f(2) \\ &= c_0 + (c_0 + c_1 + c_2) - (c_0 + 2c_1 + 4c_2) \\ &= c_0 - c_1 - 3c_2 = 0, \end{aligned}$$

and

$$\begin{aligned} < q, f > &= \sum_{i=1}^{3} q(i-1)f(i-1) = 2f(0) + f(1) \\ &= 2c_0 + (c_0 + c_1 + c_2) = 3c_0 + c_1 + c_2 = 0. \end{aligned}$$

For instance, we can take

$$f(x) = 1 - 5x + 2x^2,$$

or any scalar multiple of it.

Answer to Question 2:

(b) The distance between $p(x) + q(x)$ and $f(x)$ is $\|g(x)\|$, where

$$g(x) = p(x) + q(x) - f(x) = 2 + 5x - 3x^2.$$

Then

$$\|g(x)\| = \sqrt{< g, g >} = \sqrt{\sum_{i=1}^{3} g(i-1)^2} = \sqrt{4 + 16 + 0} = 2\sqrt{5},$$

hence the distance between $p(x) + q(x)$ and $f(x)$ is $2\sqrt{5}$.

Question 3: (20 points)
Recall that $C[-\pi, \pi]$ is the vector space of all continuous functions on the interval $[-\pi, \pi]$. Define an inner product $< \, , \, >$ on this space by

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Then $\cos(x)$ and $\cos(2x)$ are orthogonal unit vectors in $C[-\pi, \pi]$ with respect to $< \, , \, >$.

(a) Let

$$g(x) = 3\cos(x) - 5\sin^2(x) + 5\cos^2(x).$$

Express $g(x)$ as a linear combination of $\cos(x)$ and $\cos(2x)$. (5 points)

(b) Use part a and theorems from section 5.5 to evaluate the following integrals:

$$\int_{-\pi}^{\pi} g(x)^2 dx \qquad \int_{-\pi}^{\pi} g(x)\cos(2x)dx.$$

(15 points)

Answer to Question 3:

(a) One of the double angle formulas states that

$$\cos(2x) = \cos^2(x) - \sin^2(x),$$

therefore

$$g(x) = 3\cos(x) + 5\cos(2x). \tag{1}$$

(b) 1. Notice that

$$\int_{-\pi}^{\pi} g(x)^2 dx = \pi \|g(x)\|^2,$$

and therefore using (1) along with Parseval's Formula (p. 257, section 5.5), we obtain

$$\int_{-\pi}^{\pi} g(x)^2 dx = \pi(3^2 + 5^2) = 34\pi.$$

Answer to Question 3:

2. Notice that by (1)

$$\begin{aligned}\int_{-\pi}^{\pi} g(x)\cos(2x)dx &= 3\int_{-\pi}^{\pi}\cos(x)\cos(2x)dx + 5\int_{-\pi}^{\pi}\cos^2(2x) \\ &= 3\pi < \cos(x), \cos(2x) > +5\pi\|\cos(2x)\|^2 = 5\pi,\end{aligned}$$

since $\cos(x)$ and $\cos(2x)$ are orthogonal unit vectors with respect to this inner product and the associated norm.

Question 4: (15 points)
Let V be a two-dimensional subspace of $\mathbb{R}^3$ spanned by the vectors

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \boldsymbol{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Use Gram-Schmidt process to find an orthonormal basis for V with respect to the usual dot product on $\mathbb{R}^3$.

Answer to Question 4:

First let

$$\boldsymbol{u}_1 = \frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

Next define

$$\boldsymbol{p}_1 = (\boldsymbol{x}_2 \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$\boldsymbol{x}_2 - \boldsymbol{p}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Finally let

$$\boldsymbol{u}_2 = \frac{1}{\|\boldsymbol{x}_2 - \boldsymbol{p}_1\|}(\boldsymbol{x}_2 - \boldsymbol{p}_1) = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}.$$

By Gram-Schmidt process, $\boldsymbol{u}_1, \boldsymbol{u}_2$ found here form an orthonormal basis for V.