MATH 152H, FALL 2005, TEST I - SOLUTIONS

Disclaimer: This is only a sketch of solutions; some technical details are often omitted. You are expected to demonstrate all the details in your work.

Problem 1. (55 points) Let C be a circle of radius 2 centered at the origin in the xy-plane.

a) (10 points) Sketch the region S enclosed between C, the x-axis, and the line $y = x$, labeling all of its boundaries.
b) (25 points) Find the area of S using integration. Show all work, and present the answer in the most simplified form possible.
c) (20 points) Find the volume of the solid obtained by rotating S about the y-axis. Use integration techniques.

Solution. **a)** The circle C intersects with the line $y = x$ when $x = \pm\sqrt{2}$, so the region S consists of two equal sectors, one in the first and one in the third quadrant.

b) To find the area of S it is enough to find the area of the sector in the first quadrant and multiply it by two. This sector can be separated into two subregions: the right triangle T_1 with vertices $(0,0), (\sqrt{2},0), (\sqrt{2},\sqrt{2})$, and the region T_2 bounded by the line $x = \sqrt{2}$, the x-axis, and the circle C. Since we are in the first quadrant, equation of C is $y = \sqrt{4-x^2}$. Area of T_1 is 1 by the formula for the area of a right triangle. Area of T_2 is given by the integral

$$\int_{\sqrt{2}}^{2} \sqrt{4-x^2}\, dx,$$

which can be evaluated by the use of trigonometric substitution $x = 2sin(\theta)$ with $\theta \in [-\pi/2, \pi/2]$, that is:

$$\begin{aligned}
\int \sqrt{4-x^2}\ dx &= \int \sqrt{4-4sin^2(\theta)}\ 2cos(\theta)\ d\theta \\
&= 4\int cos^2(\theta)\ d\theta = 2\int (1+cos(2\theta))\ d\theta \\
&= 2\left(\theta + \frac{1}{2}sin(2\theta)\right) + c = 2\left(\theta + sin(\theta)\ cos(\theta)\right) \\
&= 2\left(sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{4}\sqrt{4-x^2}\right) + c \qquad (1)
\end{aligned}$$

Combining (1) with the Fundamental Theorem of Calculus, we obtain:

$$\begin{aligned}
\int_{\sqrt{2}}^{2} \sqrt{4-x^2}\ dx &= 2\left(sin^{-1}1 - sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) - 1 \\
&= 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - 1 = \frac{\pi}{2} - 1.
\end{aligned}$$

Therefore:

$$\begin{aligned}
Area(S) &= 2(Area(T_1) + Area(T_2)) \\
&= 2\left(1 + \frac{\pi}{2} - 1\right) = \pi.
\end{aligned}$$

c) To obtain the volume of this solid it is sufficient to find the volume of the solid obtained by rotating the sector of S which lies in the first quadrant and multiply it by two. To find this volume, we integrate with respect to y from 0 to $\sqrt{2}$ using the washers method. At each y from 0 to $\sqrt{2}$, area of the corresponding annulus (washer) is equal to

$$A(y) = \pi\left(\sqrt{4-y^2}\right)^2 - \pi y^2 = 2\pi(2-y^2).$$

Therefore the volume is

$$\begin{aligned}
\int_0^{\sqrt{2}} A(y)dy &= 2\pi \int_0^{\sqrt{2}} (2-y^2)dy \\
&= 2\pi\left(2y - \frac{y^3}{3}\right)_0^{\sqrt{2}} = \frac{8\pi\sqrt{2}}{3}.
\end{aligned}$$

Therefore the volume of our solid becomes

$$\frac{16\pi\sqrt{2}}{3}.$$

Problem 2. **(15 points)** Let $f(t) = sin^{97}(t)\ cos^{74}(t)$. Find the average value of $f(t)$ on the interval $[-\pi, \pi]$.

Solution. Notice that for each real number t,

$$\begin{aligned} f(-t) &= sin^{97}(-t)\ cos^{74}(-t) = (-1)^{97} sin^{97}(t)\ cos^{74}(t) \\ &= -sin^{97}(t)\ cos^{74}(t) = -f(t), \end{aligned}$$

hence $f(t)$ is an odd function. Therefore the graph of $f(t)$ is symmetric with respect to the origin, and so

$$\int_0^{\pi} f(t)\ dt = -\int_{-\pi}^{0} f(t)\ dt.$$

Hence

$$\int_{-\pi}^{\pi} f(t)\ dt = \int_{-\pi}^{0} f(t)\ dt + \int_0^{\pi} f(t)\ dt = 0.$$

On the other hand,

$$f_{ave}[-\pi, \pi] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ dt,$$

and so $f_{ave}[-\pi, \pi] = 0$.

Problem 3. (30 points) Let S be the region in the xy-plane enclosed by the curve $f(x) = \frac{1}{\sqrt{x(x-1)}}$, the x-axis, and the lines $x = 2$ and $x = 3$. Let $g(x) = \sqrt{(x^2 - x)(x^2 + 4)}$.

a) (15 points) Let V_1 be a solid whose base is S so that cross-sections perpendicular to the x-axis are disks. Find the volume of V_1.

b) (15 points) Let V_2 be a solid whose base is S so that cross-sections perpendicular to the x-axis are triangles with base lying in S and height $g(x)$. Find the volume of V_2.

Solution. a) Since cross-sections of V_1 perpendicular to the x-axis are disks whose diameters are precisely equal to $f(x)$ for each $x \in [2, 3]$, area of each such cross-section is

$$\pi \left(\frac{f(x)}{2}\right)^2 = \frac{\pi}{4} \frac{1}{x(x-1)} = \frac{\pi}{4}\left(\frac{A}{x} + \frac{B}{x-1}\right) = \frac{\pi}{4}\left(\frac{(A+B) - A}{x(x-1)}\right),$$

by the method of partial fractions. Hence $A + B = 0$, and $A = -1$, so $B = 1$. Therefore

$$\begin{aligned} Volume(V_1) &= \int_2^3 \frac{\pi}{4} f(x)^2\, dx = \frac{\pi}{4}\int_2^3 \left(\frac{1}{x-1} - \frac{1}{x}\right)\, dx \\ &= \frac{\pi}{4}\left(\ln|x-1| - \ln|x|\right)_2^3 = \frac{\pi}{4}\ln\left(\frac{4}{3}\right). \end{aligned}$$

b) Since cross-sections of V_2 perpendicular to the x-axis are triangles with base $f(x)$ and height $g(x)$ for each $x \in [2, 3]$, area of each such cross-section is

$$\frac{f(x)g(x)}{2} = \frac{\sqrt{x(x-1)(x^2+4)}}{2\sqrt{x(x-1)}} = \frac{\sqrt{x^2+4}}{2}.$$

Therefore,

$$Volume(V_2) = \frac{1}{2}\int_2^3 \sqrt{x^2+4}\, dx.$$

Making a substitution $x = 2tan(\theta)$ with $\theta \in (-\pi/2, \pi/2)$, we obtain

$$\begin{aligned} \int \sqrt{x^2+4}\, dx &= 4\int sec^3(\theta)\, d\theta \\ &= 2sec(\theta)tan(\theta) + 2\ln|sec(\theta) + tan(\theta)| \\ &= \frac{x}{2}\sqrt{x^2+4} + 2\ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + c, \end{aligned}$$

where the evaluation of the integral $\int sec^3(\theta)\ d\theta$ can be found in Example 9 in Section 8.2 on p. 469 in the book. Combining this with the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} Volume(V_2) &= \left(\frac{x}{4}\sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right|\right)_2^3 \\ &= \frac{3\sqrt{13}}{4} - \sqrt{2} + \ln\left(\frac{\sqrt{13}+3}{2\sqrt{2}+2}\right). \end{aligned}$$