MATH 152H, FALL 2005, TEST II - SOLUTIONS

Disclaimer: This is only a sketch of solutions; some technical details are often omitted. You are expected to demonstrate all the details in your work.

Problem 1. (30 points) Let a be a positive real number, and let the curve $\mathcal{C}$ be given parametrically by

$$x(t) = a\ cos^2 t, \quad y(t) = a\ sin^2 t, \quad 0 \le t \le \frac{\pi}{4}.$$

a) (15 points) Find arc length of $\mathcal{C}$. Explain.
b) (15 points) Find the area of the surface obtained by rotating the curve $\mathcal{C}$ about the x-axis. Explain.

Solution. a) Notice that

$$x'(t) = -2a\ cos\ t\ sin\ t = -a\ sin\ 2t,$$

and

$$y'(t) = 2a\ sin\ t\ cos\ t = a\ sin\ 2t.$$

Therefore, the arc length is given by

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{(x'(t))^2 + (y'(t))^2}\ dt = \int_0^{\pi/4} \sqrt{2a^2\ sin^2 2t}\ dt \\ &= \sqrt{2}a \int_0^{\pi/4} sin\ 2t\ dt = -\left(\frac{\sqrt{2}a}{2}\right) cos\ 2t\ |_0^{\pi/4} = \frac{\sqrt{2}a}{2}. \end{aligned}$$

b) The surface area A is given by

$$\begin{aligned} A &= \int_0^{\pi/4} 2\pi y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\ dt \\ &= 2\sqrt{2}\pi a^2 \int_0^{\pi/4} sin^2 t\ sin\ 2t\ dt = 4\sqrt{2}\pi a^2 \int_0^{\pi/4} sin^3 t\ cos\ t\ dt. \end{aligned}$$

We now use substitution $u = sin\ t$, so $du = cos\ t\ dt$; also when $t = 0$, $u = 0$, and when $t = \pi/4$, $u = \sqrt{2}/2$. Therefore

$$A = 4\sqrt{2}\pi a^2 \int_0^{\sqrt{2}/2} u^3 du = \sqrt{2}\pi a^2\ u^4\ |_0^{\sqrt{2}/2} = \frac{\sqrt{2}\pi a^2}{4}.$$

Problem 2. (30 points) Consider a differential equation

$$y' + x^n y = x^m. \tag{1}$$

a) (10 points) Express general solution to (1) in terms of an indefinite integral. Explain.

b) (10 points) Use part a to find general solution in the case $n = m = 1$. Explain.

c) (10 points) Find the particular solution $f(x)$ for part b so that the tangent line to $f(x)$ at $x = 1$ has slope 1. Explain.

Solution. a) This is a linear first order differential equation, so pick integrating factor

$$I(x) = e^{\int x^n dx} = e^{\frac{x^{n+1}}{n+1}},$$

and multiply both sides of (1) by it. Then we have

$$y' e^{\frac{x^{n+1}}{n+1}} + x^n e^{\frac{x^{n+1}}{n+1}} y = x^m e^{\frac{x^{n+1}}{n+1}},$$

where the left hand side is derivative of $ye^{\frac{x^{n+1}}{n+1}}$. Therefore

$$y = e^{-\frac{x^{n+1}}{n+1}} \int x^m e^{\frac{x^{n+1}}{n+1}} dx. \tag{2}$$

b) Let $m = n = 1$, then from (2) we obtain

$$y = e^{-\frac{x^2}{2}} \int x e^{\frac{x^2}{2}} dx = e^{-\frac{x^2}{2}} \left(e^{\frac{x^2}{2}} + C \right) = 1 + Ce^{-\frac{x^2}{2}}. \tag{3}$$

c) From (3),

$$f(x) = 1 + Ce^{-\frac{x^2}{2}},$$

such that $f'(1) = 1$. Notice that

$$f'(x) = -Cxe^{-\frac{x^2}{2}},$$

and so

$$f'(1) = \frac{-C}{\sqrt{e}} = 1,$$

meaning that $C = -\sqrt{e}$. Hence the particular solution we are looking for is

$$f(x) = 1 - e^{\frac{1-x^2}{2}}.$$

Problem 3. (40 points) Let p be a real number, and $f(x) = \frac{1+e^{-x}}{x^p}$.

a) (15 points) For which values of p does the region

$$R_p = \{(x, y) \mid x \geq 1,\ 0 \leq y \leq f(x)\}$$

have finite area? Explain: you need to identify the values of p for which the area is finite, and show that for all others it is infinite.

b) (15 points) For values of p for which the area of R_p from part a is finite, prove that this area is $\leq \frac{e+1}{e(p-1)}$.

c) (10 points) Find the x-coordinate of the centroid of the region

$$S = \left\{(x, y)\ :\ \frac{-\pi}{2} \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq \frac{\cos\ x}{x^2}\right\}.$$

Solution. a) - b) Let A_p be the area of R_p, then

$$A_p = \int_1^\infty \frac{1+e^{-x}}{x^p}\,dx \geq \int_1^\infty \frac{1}{x^p}\,dx,$$

which is a divergent integral if $p \leq 1$ (p. 513 in the book, also did this in class). Hence A_p is infinite when $p \leq 1$, by Comparison Theorem. Now assume $p > 1$. Then

$$\begin{aligned} A_p &= \int_1^\infty \frac{1+e^{-x}}{x^p}\,dx \leq \int_1^\infty \frac{e+1}{ex^p}\,dx = \left(\frac{e+1}{e}\right)\lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx \\ &= \left(\frac{e+1}{e}\right)\lim_{t\to\infty}\left(-\frac{1}{(p-1)x^{p-1}}\right)_1^t = \frac{e+1}{e(p-1)}. \end{aligned}$$

Hence A_p is finite when $p > 1$, which finishes part a. Moreover, when $p > 1$, $A_p \leq \frac{e+1}{e(p-1)}$, which finishes part b.

c) Notice that $\frac{\cos\ x}{x^2}$ is an even function, which means that the region S is symmetric with respect to the y-axis. Hence its centroid must lie on the y-axis, so its x-coordinate must be equal to 0.