MATH 152H, FALL 2005, TEST III - SOLUTIONS

Disclaimer: This is only a sketch of solutions; some technical details are often omitted. You are expected to demonstrate all the details in your work.

Problem 1. (30 points) Let $f(x) = \sqrt{x}$.

a) (20 points) Find the Taylor series expansion for $f(x)$ centered at 1. Explain.

b) (10 points) Find the radius of convergence for the series you obtained in part a. Explain.

Solution. a) Notice that $f(1) = 1$, and

$$f'(x) = \frac{1}{2x^{1/2}}, \quad f'(1) = \frac{1}{2},$$
$$f''(x) = -\frac{1}{2^2 x^{3/2}}, \quad f''(1) = -\frac{1}{2^2},$$
$$f'''(x) = \frac{3}{2^3 x^{5/2}}, \quad f'''(1) = \frac{3}{2^3},$$
$$f^{(4)}(x) = -\frac{3 \times 5}{2^4 x^{5/2}}, \quad f^{(4)}(1) = -\frac{3 \times 5}{2^4},$$

therefore, more generally, for each $n \geq 2$,

$$f^{(n)}(1) = (-1)^{n+1} \frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2^n}.$$

Hence, the Taylor series for $f(x)$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= 1 + \frac{x-1}{2} + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2^n n!}(x-1)^n. \end{aligned}$$

b) To determine the radius of convergence, use Ratio Test:

$$\lim_{n\to\infty} \left| \frac{1 \times 3 \times 5 \times \cdots \times (2n-3) \times (2(n+1)-3) 2^n n! (x-1)^{n+1}}{1 \times 3 \times 5 \times \cdots \times (2n-3) 2^{n+1} (n+1)! (x-1)^n} \right|$$
$$= \frac{|x-1|}{2} \lim_{n\to\infty} \frac{2n-1}{n+1} = |x-1|,$$

and we need this limit to be < 1 for the series to converge by Ratio Test, that is we need $|x-1| < 1$. Thus the radius of convergence is 1.

Problem 2. **(40 points)** For the series in parts a and b, determine whether they are absolutely convergent, convergent, or divergent. Explain.

a) (20 points) $\sum_{n=2}^{\infty} \sin\left(\frac{(2n-1)\pi}{2}\right) \ln\left(\frac{n}{n-1}\right)$.

b) (10 points) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

c) (10 points) For which integer values of p is the series

$$\sum_{n=1}^{\infty} e^{(-1)^p p \ln(n)}$$

convergent? Explain.

Solution. **a)** For this series, notice that

$$\sum_{n=2}^{\infty} \sin\left(\frac{(2n-1)\pi}{2}\right) \ln\left(\frac{n}{n-1}\right) = \sum_{n=2}^{\infty} (-1)^{n-1} \ln\left(\frac{n}{n-1}\right),$$

and

$$\lim_{n\to\infty} \ln\left(\frac{n}{n-1}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{1-1/n}\right) = \ln 1 = 0,$$

hence it is a convergent alternating series by the Alternating Series Test. On the other hand,

$$\begin{aligned} \sum_{n=2}^{\infty} \left|\sin\left(\frac{(2n-1)\pi}{2}\right) \ln\left(\frac{n}{n-1}\right)\right| &= \sum_{n=2}^{\infty} \ln\left(\frac{n}{n-1}\right) \\ &= \sum_{n=2}^{\infty} \left(\ln(n) - \ln(n-1)\right). \end{aligned}$$

Notice that, by L'Hospital's Rule:

$$\lim_{n\to\infty} \frac{\ln(n) - \ln(n-1)}{1/n} = \lim_{n\to\infty} \frac{1/n - 1/(n-1)}{-1/n^2} = \lim_{n\to\infty} \frac{n^2}{n(n-1)} = 1,$$

and hence, by Limit Comparison Test, either both series,

$$\sum_{n=2}^{\infty} \left(\ln(n) - \ln(n-1)\right) \text{ and } \sum_{n=2} \frac{1}{n},$$

converge, or both diverge, but we know that the harmonic series is divergent, therefore $\sum_{n=2}^{\infty} (\ln(n) - \ln(n-1))$ must also diverge. Another possible way to demonstrate this is by using the integral test.

Conclusion: this series is convergent, but not absolutely convergent.

b) For this series, use the Limit Comparison Test, comparing it to $\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Therefore, since the harmonic series is divergent, so is our series.

c) Notice that

$$\sum_{n=1}^{\infty} e^{(-1)^p p \ln(n)} = \sum_{n=1}^{\infty} e^{\ln(n^{(-1)^p p})} = \sum_{n=1}^{\infty} n^{(-1)^p p}.$$

So if $p > 0$ is even, or if $p < 0$ is odd, the series becomes

$$\sum_{n=1}^{\infty} n^{|p|},$$

which is clearly divergent by the Divergence Test. On the other hand, if $p > 0$ is odd, or if $p < 0$ is even, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^{|p|}},$$

which is convergent if $|p| > 1$ by the p-series test. Hence the series is convergent if p is either positive odd, or negative even, and $|p| > 1$.

Problem 3. (30 points) Let $a \geq 0$ be a real number, and let T be a triangle with vertices $A = (a, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$ in three-dimensional coordinate system.

a) (15 points) Find all possible values for a so that T is a right triangle. What are the measures of all angles of T in this case? Explain.

b) (10 points) Recall that centroid of T is at the point D with coordinates $\left(\frac{a}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Find the value for a so that the distance from the origin to D is 5. Explain.

c) (5 points) Suppose that a is such that T is an equilateral triangle. Write down an equation for the sphere centered at the origin that passes through the vertices of T. Explain.

Solution. a) We can write down the vectors corresponding to sides:

$$AB = (-a, 1, 0),\ AC = (-a, 0, 1),\ BC = (0, -1, 1).$$

Let the angle between AB and AC be θ_1, between AB and BC θ_2, and between AC and BC θ_3. Therefore

$$\cos(\theta_1) = \frac{|AB \cdot AC|}{\|AB\|\|AC\|} = \frac{a^2}{a^2+1},$$
$$\cos(\theta_2) = \frac{|AB \cdot BC|}{\|AB\|\|BC\|} = \frac{1}{\sqrt{2(a^2+1)}},$$
$$\cos(\theta_3) = \frac{|AC \cdot BC|}{\|AC\|\|BC\|} = \frac{1}{\sqrt{2(a^2+1)}}.$$

In order for T to be right triangle, one of these cosines must be 0, which is only possible for $\cos(\theta_1)$ in case if $a = 0$. In this case $\theta_1 = 90°$, $\theta_2 = \theta_3 = 45°$.

b) Write d for the distance from the origin to D, then

$$d = \sqrt{\frac{a^2+2}{9}} = \frac{\sqrt{a^2+2}}{3} = 5,$$

therefore

$$a^2 + 2 = 225,$$

thus $a = \sqrt{223}$.

c) In this case $a = 1$, and the equation for this sphere is

$$x^2 + y^2 + z^2 = 1.$$