

Math 172, Exam 1A PRINT NAME _____
 October 1, 2009. There are 9 problems. Clearly indicate answers and show enough work to support them.

ONE (10 pts) Rotate the region in the first quadrant bounded by $y = x^2$ and $x = 2y$ about the x-axis. Use the disk/washer method to set up, but do not evaluate, an integral for the volume of that solid.

Solution: The curves intersect when $x^2 = y = x/2$, or for $x = 0$ and $x = 1/2$. On the interval $[0, 1/2]$ $y = x/2$ is the larger of the two functions, so a typical washer has outer radius $(x/2)$, inner radius x^2 , and cross sectional area $\pi[(x/2)^2 - (x^2)^2]$. Thus the required volume is $V = \int_0^{1/2} \pi[(x/2)^2 - (x^2)^2] dx$.

TWO (5 pts each) Just give the formulas. (a) $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$

(b) $\int \sec(x) dx = \ln|\sec(x) + \tan(x)| + c$ (c) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

(d) For $h(x) = \int_e^x \sqrt{\ln(t)} dt$, $h'(x) = \sqrt{\ln(x)}$

THREE (10 pts) The integral $\int_0^3 2\pi x(6 - 2x) dx$ represents the volume of a solid of rotation. Describe the solid, and give the largest cross sectional area by a plane perpendicular to the axis of rotation.

Solution: Rotate the triangular region below $y = 6 - 2x$ and above $[0, 3]$ about the y-axis. The resulting solid is a right circular cone with height 6 and base radius 3. The integral in the problem statement is the solid's volume found using the shell method. Cross sections perpendicular to the y-axis are disks. The cross section of largest area is the base which has area 9π .

FOUR (10 pts) A box-shaped tank 4 m long, 3 m wide and 2 m deep is full of ethyl alcohol. Set up, but do not evaluate, an integral expression giving the work needed to pump all the alcohol to point 6 m above the top of the tank. (The density of alcohol is 785 kg/m^3 .)

Solution: Consider a thin rectangular region in the tank that is x meters from the bottom and has height dx meters. The region's volume is $(3)(4) dx = 12 dx$ cubic meters and it holds $(785)(12 dx)$ kilograms of alcohol. The force of gravity on the region is $(9.8)(785)(12 dx)$ newtons and the alcohol in the region will be displaced $(8 - x)$ meters. The work done in displacing the alcohol in the region is

$(8 - x)(9.8)(785)(12 dx)$ joules. The work done to pump the tank is $\int_0^2 (8 - x)(9.8)(785)(12) dx$ joules.

FIVE (10 pts) $\int x^2 \ln(x) dx = ?$

Solution: Use parts to "differentiate away" the term $u = \ln(x)$; $dv = x^2 dx$, $du = x^{-1} dx$, $v = x^3/3$,
 $\int x^2 \ln(x) dx = (x^3/3)\ln(x) - \int (x^3/3)x^{-1} dx = (x^2/3)\ln(x) - (x^3/9) + c$

SIX (10 pts). Find the area of the region in quadrant one between the graphs of $y = x^2$ and $y = 2x$.

Solution: The curves intersect when $x^2 = y = 2x$, or for $x = 0$ and $x = 2$. On the interval $[0, 2]$ $y = 2x$

is the larger of the two functions, so the required area is $A = \int_0^2 [2x - x^2] dx = \left(x^2 - \frac{x^3}{3} \right) \Big|_{x=0}^{x=2} = \frac{4}{3}$.

SEVEN (10 pts) $\int \frac{x-2}{x(x-1)^2} dx = ?$

Solution: The partial fractions decomposition is $\frac{x-2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

or equivalently $x-2 = A(x-1)^2 + Bx(x-1) + Cx$. To find A, B and C:

taking $x = 0$, $A = -2$; taking $x = 1$, $C = -1$; equating the coefficients of x^2 , $0 = A + B$ and $B = 2$.

$$\int \frac{x-2}{x(x-1)^2} dx = \int \frac{-2}{x} dx + \int \frac{2}{x-1} dx + \int \frac{-1}{(x-1)^2} dx = -2 \ln|x| + 2 \ln|x-1| + \frac{1}{x-1} + D$$

EIGHT (10 pts) $\int_0^{\pi} \sin^2(3x) dx = ?$.

Solution: Use the double angle formula to integrate a product of sine and cosine to even powers (2 and 0, respectively, in this problem).

$$\int_0^{\pi} \sin^2(3x) dx = \int_0^{\pi} \frac{1}{2} [1 - \cos(6x)] dx = \frac{1}{2} \left[x - \frac{1}{6} \sin(6x) \right]_{x=0}^{x=\pi} = \frac{\pi}{2}$$

NINE (10 pts) Use an appropriate trig substitution to change $\int_0^2 (4+x^2)^2 dx$ to a trig integral. Be sure to change the limits of integration too, but don't evaluate the trig integral.

Solution: The expression $4+x^2 = a^2 + u^2$ calls for the substitution $u = a \tan(\theta)$ or $x = 2 \tan(\theta)$. With that substitution $dx = 2 \sec^2(\theta) d\theta$. To change the limits, $x = 0$ implies $\tan(\theta) = 0$ and $\theta = 0$, $x = 2$ implies $\tan(\theta) = 1$ and $\theta = \pi/4$. Substituting and then using $1 + \tan^2(\theta) = \sec^2(\theta)$,

$$\int_0^2 (4+x^2)^2 dx = \int_0^{\pi/4} (4 + 4 \tan^2(\theta))^2 2 \sec^2(\theta) d\theta = \int_0^{\pi/4} 32 \sec^6(\theta) d\theta$$

Math 172, Exam 1B PRINT NAME _____
 October 1, 2009. There are 9 problems. Clearly indicate answers and show enough work to support them.

ONE (10 pts) Use an appropriate trig substitution to change $\int_0^3 \sqrt{9+x^2} dx$ to a trig integral. Be sure to change the limits of integration too, but don't evaluate the trig integral.

Solution: The expression $9+x^2 = a^2 + u^2$ calls for the substitution $u = a \tan(\theta)$, or $x = 3 \tan(\theta)$. With that substitution $dx = 3 \sec^2(\theta) d\theta$. To change the limits, $x = 0$ implies $\tan(\theta) = 0$ and $\theta = 0$, $x = 3$ implies $\tan(\theta) = 1$ and $\theta = \pi/4$. Substituting and then using $1 + \tan^2(\theta) = \sec^2(\theta)$,

$$\int_0^3 \sqrt{9+x^2} dx = \int_0^{\pi/4} \sqrt{9+9 \tan^2(\theta)} 3 \sec^2(\theta) d\theta = \int_0^{\pi/4} 9 \sec^3(\theta) d\theta.$$

TWO (5 pts each) Just give the formulas. (a) $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha)$

(b) $\int \ln(x) dx = x \ln|x| - x + c$ (c) $\int \sec(x) dx = \ln|\sec(x) + \tan(x)| + c$

(d) For $h(x) = \int_{\pi}^x \sqrt{\sin(t)} dt$, $h'(x) = \sqrt{\sin(x)}$

THREE (10 pts) $\int x \cos(x) dx = ?$

Solution: Use parts to "differentiate away" the term $u = x$; $dv = \cos(x) dx$, $du = dx$, $v = \sin(x)$,

$$\int x \cos(x) dx = x \sin(x) - \int \sin(x) dx = x \sin(x) + \cos(x) + c$$

FOUR (10 pts) $\int_0^{\pi} \cos^2(2x) dx = ?$

Solution: Use the double angle formula to integrate a product of sine and cosine to even powers (0 and 2, respectively, in this problem).

$$\int_0^{\pi} \cos^2(2x) dx = \int_0^{\pi} \frac{1}{2} [1 + \cos(4x)] dx = \frac{1}{2} \left[x + \frac{1}{4} \sin(4x) \right]_{x=0}^{x=\pi} = \frac{\pi}{2}$$

FIVE (10 pts) Rotate the region in the first quadrant bounded by $y = x^2$ and $x = 2y$ about the y-axis. Use the shell method to set up, but do not evaluate, an integral for the volume of that solid.

Solution: The curves intersect when $x^2 = y = x/2$, or for $x = 0$ and $x = 1/2$. On the interval $[0, 1/2]$ $y = x/2$ is the larger of the two functions, so a cylindrical shell of thickness dx has height $[(x/2) - x^2]$ and volume $2\pi x [(x/2) - x^2] dx$. Thus the required volume is $V = \int_0^{1/2} 2\pi x [(x/2) - x^2] dx$.

SIX (10 pts) The integral $\int_0^3 \pi (6 - 2x)^2 dx$ represents the volume of a solid of rotation. Describe the solid, and give the largest cross sectional area by a plane perpendicular to the axis of rotation.

Solution: Rotate the triangular region below $y = 6 - 2x$ and above $[0, 3]$ about the x -axis. The resulting solid is a right circular cone with height 3 and base radius 6. The integral in the problem statement is the solid's volume found using the disk method. Cross sections perpendicular to the x -axis are disks. The cross section of largest area is the base, which has area 36π .

SEVEN (10 pts). Find the area of the region in quadrant one between the graphs of $y = x^2$ and $y = 3x$.

Solution: The curves intersect when $x^2 = y = 3x$, or for $x = 0$ and $x = 3$. On the interval $[0, 3]$ $y = 3x$ is the larger of the two functions, so the required area is $A = \int_0^3 [3x - x^2] dx = \left(\frac{3x^2}{2} - \frac{x^3}{3} \right) \Big|_{x=0}^{x=3} = \frac{9}{2}$.

EIGHT (10 pts) A box-shaped tank 6 m long, 2 m wide and 3 m deep is full of gasoline. Set up, but do not evaluate, an integral expression giving the work needed to pump all the gasoline to a point 4 m above the top of the tank. (The density of gasoline is 737 kg/m^3 .)

Solution: Consider a thin rectangular region in the tank that is x meters from the bottom and has height dx meters. The region's volume is $(6)(2) dx = 12 dx$ cubic meters and it holds $(737)(12 dx)$ kilograms of gasoline. The force of gravity on the region is $(9.8)(737)(12 dx)$ newtons and the gasoline in the region will be displaced $(7 - x)$ meters. The work done in displacing the gasoline in the region is

$(7 - x)(9.8)(737)(12 dx)$ joules. The work done to pump the tank is $\int_0^3 (7 - x)(9.8)(737)(12) dx$ joules.

NINE (10 pts) $\int \frac{x - 4}{x(x - 2)^2} dx = ?$

Solution: The partial fractions decomposition is $\frac{x - 4}{x(x - 2)^2} = \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2}$

or equivalently $x - 4 = A(x - 2)^2 + Bx(x - 2) + Cx$. To find A, B and C:

taking $x = 0$, $A = -1$; taking $x = 2$, $C = -1$; equating the coefficients of x^2 , $0 = A + B$ and $B = 1$.

$$\int \frac{x - 4}{x(x - 2)^2} dx = \int \frac{-1}{x} dx + \int \frac{1}{x - 2} dx + \int \frac{-1}{(x - 2)^2} dx = -\ln|x| + \ln|x - 2| + \frac{1}{x - 2} + D$$