

M251 Fall 2010 - practice for the final

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The exam will have around 15 problems.

1. Let $f(x, y) = x^2 + y^2 - \frac{1}{x^2y^2}$. Find all local maxima and minima of f .
2. Let

$$f(x, y) = x^4 - 2x^2 + y^2 + 1.$$

- (a) Find the critical points of f .
 - (b) Compute the second partials $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$.
 - (c) For each critical point, determine whether it is a local maxima, a local minima or a saddle point.
3. Let

$$f(x, y) = \frac{x^3}{3} - x + \frac{y^3}{3} - y.$$

- (a) Find the critical points of f .
 - (b) Compute the second partials $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$.
 - (c) For each critical point, determine whether it is a local maxima, a local minima or a saddle point.
4. Find all maximum and minimum values of $f(x, y, z) = x + y + z$ subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.
 5. Use Lagrange multipliers to find the maximum value of f subject to the given constraints:

$$f(x, y, z) = x + 2y + 3z; \quad x^2 + 2y^2 + 3z^2 = 54.$$

6. Compute the iterated integrals:

- (a) $\int_0^1 \int_0^3 xy \, dx dy$.

- (b) $\int_0^1 \int_{y^2}^y x^3 + y \, dx dy$.

7. Find the limits of integration in the given integral.

- (a) $\iint_D f(x, y) \, dA$ where D is bounded by $y = x - 6$ and $y^2 = x$.

- (b) $\iint_D f(x, y) \, dA$ where D is bounded by the parabolas $x = 1 - y^2$ and $x = y^2 - 1$.

- (c) $\iint_D f(x, y) \, dA$ where D is the triangle with vertices $(0, 0)$, $(1, 1)$ and $(-2, 1)$.

- (d) $\iiint_E f(x, y, z) dV$ where E lies under the plane $z = x + 2y$ and above the region in the xy -plane bounded by the curves $y = x^2$, $y = 0$ and $x = 1$.

8. Change the order of integration in the integrals below.

(a) $\int_1^2 \int_0^{\ln(x)} f(x, y) dy dx.$

(b) $\int_0^1 \int_{x^4}^x f(x, y) dy dx.$

(c) Rewrite the integral

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) dz dy dx$$

as an iterated integral in the order $dx dy dz$.

(d) Compute $\int_0^1 \int_x^1 e^{x/y} dy dx$. Hint: switch the order of integration.

(e) $\int_{-2}^2 \int_0^{4-y^2} f(x, y) dx dy.$

9. Set up the integral for the volume of the solid.

(a) The solid lies under the paraboloid $z = 25 - x^2 - y^2$ and above the triangle in the xy -plane with vertices $(-3, -4)$, $(-3, 4)$ and $(5, 0)$.

(b) The solid lies inside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

(c) The solid is bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

(d) The solid is bounded by the elliptic paraboloid

$$z = 1 + 2x^2 + 3y^2$$

and the planes $x + 2y = 4$ and the coordinate planes.

(e) The solid is the tetrahedra bounded by the planes $x + y + z = 1$, $y = x$, $x = 0$ and $z = 0$.

(f) The solid is under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0, 2] \times [1, 4]$.

(g) The solid is above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$.

(h) The solid is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 0$.

(i) Set up the integral for the volume of the region bounded by the paraboloid $x = y^2 + 2z^2$ and the parabolic cylinder $x = 2 - y^2$.

10. Set up the integral for the area of the region using polar coordinates.

- (a) Find the area enclosed by one loop of the three-leaved rose $r = \cos(3\theta)$. Hint: when $\theta = \pi/6$, $r = 0$.
- (b) The region is one loop of $r = 2\cos(2\theta)$.
- (c) The region is the cardioid $r = 1 + \cos(\theta)$.
- (d) The region lies inside $r = 2\cos(\theta)$ and outside the circle $r = 2$.
- (e) The region lies inside the circle $r = 2$ and outside the circle $x^2 + (y - 1)^2 = 1$ (which in polar coordinates is $r = 2\sin(\theta)$).
- (f) The region lies inside the circle $r = 3\cos(\theta)$ and outside the cardioid $r = 1 + \cos(\theta)$.
11. Convert the integral to polar coordinates. You do not have to evaluate them.

- (a) $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2} dx dy.$
- (b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}} dy dx.$
- (c) $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x dy dx.$
- (d) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} dy dx.$
- (e) $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)^{3/2} dy dx.$

12. Cylindrical and Spherical Coordinates.

- (a) Use cylindrical coordinates to find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.
- (b) Set up the integral for the volume of the region bounded above by the surface $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$. Use cylindrical coordinates.
- (c) Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV$ where B is the ball $x^2 + y^2 + z^2 \leq 1$.
- (d) Change the integral to spherical coordinates:

$$\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} z \sqrt{x^2 + y^2 + z^2} dz dy dx.$$

- (e) Evaluate $\iiint_E y^2 dV$ where E is the part of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.
- (f) Use spherical coordinates to set up the triple-integral for the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and the cone $z = \sqrt{x^2 + y^2}$.

13. Change of Variables.

- (a) Use the transformation $x = \frac{r \cos(\theta)}{t}$, $y = \frac{r \sin(\theta)}{t}$, $z = r^2$ to find the volume of the region R that lies between the paraboloids $z = x^2 + y^2$, $z = 4(x^2 + y^2)$ and the planes $z = 1$, $z = 4$.
- (b) Find the area of the region R bounded by the curves $xy = 1$, $xy = 3$, $xy^{1.4} = 1$, $xy^{1.4} = 2$. Hint: use the transformation $u = xy$, $v = xy^{1.4}$.
- (c) Set up the integral for the area of the region R bounded by the curves $\sqrt{xy} = 1$, $\sqrt{xy} = 2$, $\sqrt{\frac{x}{y}} = 2$, $\sqrt{\frac{x}{y}} = 3$. Include the limits of integration. Hint: use the transformation $u = \sqrt{xy}$, $v = \sqrt{\frac{x}{y}}$.
14. Find a Cartesian equation for the curve described by the polar equation $r = 2 \sin(\theta)$.
15. A lamina occupies that part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Its density function is $\rho(x, y) = \sqrt{x^2 + y^2}$.
- (a) Write down the integral for the mass of the lamina. You do not have to evaluate it.
- (b) Write down the integrals for M_x and M_y , the moments about the x and y axes. You do not have to evaluate them.
- (c) Explain how to obtain the center of mass from the mass of the lamina and M_x and M_y .
16. A lamina occupies the region D bounded by the parabola $x = 1 - y^2$ and the coordinate axes and lies in the first quadrant. The density function is $\rho(x, y) = y$.
- (a) Set up the integral for the mass m of the lamina.
- (b) Let (\bar{x}, \bar{y}) be the center of mass of the lamina. Set up the integral for \bar{x} .
17. Set up the integrals for the mass and center of mass of the lamina D bounded by the parabola $x = y^2$ and the line $y = x - 2$ if the density function is $\rho(x, y) = 3$.
18. Parametrize the given curve C .
- (a) C is the line segment from $(4, -9, 1)$ to $(0, 2, 5)$.
- (b) C is the part of the circle $x^2 + y^2 = 4$ with $y > 0$.
- (c) C is the part of the graph of $x = \cos(y)$ from $(1, 0)$ to $(0, \pi/2)$.
19. Compute the given line integral with respect to arc length.
- (a) $\int_C y ds$ where C is the arc of the parabola $y^2 = 2x$ from $(0, 0)$ to $(2, 2)$.

- (b) $\int_C xyz \, ds$ where C is the line segment from $(1, 2, 3)$ to $(3, 2, 1)$.
- (c) $\int_C (z - x) \, ds$ where C is the part of the curve (t, t^2, t^3) from $(1, 1, 1)$ to $(2, 4, 8)$.
- (d) Compute $\int_C xyz \, ds$ where C is the line segment from $(0, 1, 2)$ to $(1, 2, 3)$.
- (e) Set up the integral for the mass of a thin wire in the shape of a quarter-circle $x^2 + y^2 = 1$, $x \geq 0, y \geq 0$ if the density function is $\rho(x, y) = x + y$.
20. (14.2) Work and force fields.
- (a) Compute the work done by the field $\mathbf{F} = z\mathbf{i} + x\mathbf{k}$ on a particle that moves along the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 2$.
- (b) Compute the work done by the field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} - y\mathbf{k}$ on a particle that moves along the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$.
- (c) Compute the work done by the field $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ on a particle that moves in a straight line from $(2, -1, 3)$ to $(4, 2, -1)$.
- (d) Compute the work done by the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ on a particle that moves along the curve $\vec{r}(t) = \langle \sin(t), \cos(t), 2t \rangle$, $0 \leq t \leq \pi$.
21. (14.2) Find the mass or centroid of a thin wire with given density function.
- (a) Find the centroid of a thin wire shaped like the part of the helix $(\cos(t), \sin(t), t)$ for $0 \leq t \leq 2\pi$.
- (b) Find the mass of a wire shaped like the first-quadrant portion of the circle $x^2 + y^2 = 1$ with density $\rho = x + y$.
- (c) Set up the integral for the mass of a thin wire shaped like the part of the curve $(x, \sin(x))$ for $0 \leq x \leq \pi$ with density function $\rho(x, y) = y$.
22. (14.3) Determine whether the given vector field is conservative. If it is, find a potential function for it.
- (a) $\mathbf{F} = (6xy - y^3)\mathbf{i} + (4y + 3x^2 - 2xy^2)\mathbf{j}$.
- (b) $\mathbf{F} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$.
- (c) $\mathbf{F} = \cos(y)\mathbf{i} + \sin(x)\mathbf{j}$.
23. (14.3) Show that the given line integral is independent of path, then calculate its value.
- (a) $\int_C \cos(x)dx + \sin(y)dy$ where C is the path parametrized by $\vec{r}(t) = (\sqrt{t^3 + 1}, \cos(\pi t))$ for $0 \leq t \leq 2$.
- (b) $\int_C x^3 dx + y^3 dy$ where C is the path parametrized by $\vec{r}(t) = (\cos(t) \sin(t), t)$ for $0 \leq t \leq 2\pi$.

24. (14.4) Apply Green's theorem to compute the line integral around the given positively oriented curve.

(a) $\int_C (x + y^2)dx + (y + x^2)dy$, where C is the square with vertices $(\pm 1, \pm 1)$.

(b) $\int_C y^2 dx + (2x - 3y)dy$ where C is the circle $x^2 + y^2 = 9$.

(c) $\int_C (x^2 - y^2)dx + (xy)dy$ where C is the boundary of the region bounded by the line $y = x$ and the parabola $y = x^2$.

(d) $\int_C x dx + y dy$ where C is the perimeter of the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

(e) $\int_C y dx - x dy$ where C is the perimeter of the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

25. (14.5) Compute the divergence and curl of the given vector field.

(a) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$,

(b) $\mathbf{F}(x, y, z) = x\mathbf{i} + 2x\mathbf{j} + 3x\mathbf{k}$.

(c) $\mathbf{F}(x, y, z) = x\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$.

26. (14.5) Determine whether or not the given vector field is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F} = \langle e^{xz}, -2e^{yz}, 3xe^y \rangle$.

(b) $\mathbf{F} = xy\mathbf{j} + xyz\mathbf{k}$.

(c) $\mathbf{F} = y^2z\mathbf{i} - x^2yz\mathbf{k}$.

(d) $\mathbf{F} = x^2y\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$.

(e) $\mathbf{F} = xe^y\mathbf{i} - ze^{-y}\mathbf{j} + y \ln(z)\mathbf{k}$.

(f) $\mathbf{F} = 2xy\mathbf{i} + (x^2 - y^2)\mathbf{j}$.

(g) $\mathbf{F} = (x \cos(y) + \sin(y))\mathbf{i} + (y \cos(x) + \sin(x))\mathbf{j}$.

(h) $\mathbf{F} = e^{x-y}(xy + y)\mathbf{i} + e^{x-y}(xy + x)\mathbf{j}$.

(i) $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$.

(j) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

27. Show that the given line integral is independent of path, then calculate its value.

$$\int_C x dx + y dy$$

C is parametrized by $\vec{r}(t) = \langle te^t, \sin(t^2 - 2t + 1) \rangle$ for $0 \leq t \leq 1$.

28. Find a parametric representation for the sphere $x^2 + y^2 + z^2 = 1$.
29. Find a parametric representation for the part of the elliptic paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.
30. Evaluate $\iint_S z^2 dS$ where S is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 1$.
31. Evaluate $\iint_S x^2 y^2 z dS$ where S is the upper hemi-sphere parametrized by $\vec{r}(x, y) = \langle x, y, \sqrt{1 - x^2 - y^2} \rangle$ for x, y in the disk $x^2 + y^2 \leq 1$. You do not have to compute it all the way through.
32. Evaluate $\iint_S \mathbf{F} \cdot d\vec{S}$ where S is as in the previous problem and $\mathbf{F} = \langle z, z, x^2 \rangle$.
33. The function $\vec{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), \theta \rangle$ for $0 < r < 1$ and $0 < \theta < 2\pi$ parametrizes part of a helicoid.
- Compute $\vec{r}_r \times \vec{r}_\theta$.
 - Find an equation for the tangent plane at $\vec{r}(1/2, \pi) = (-1/2, 0, \pi)$.
 - Find the area of the part of the helicoid parametrized by \vec{r} .
34. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ and S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy -plane.
35. Use Stokes' theorem to evaluate $\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} dS$ where $\mathbf{F} = \langle -2y, 2x, z\sqrt{x^2 + 1} \rangle$ and S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$ with the upward unit normal vector.
36. Use Stoke's theorem to calculate the amount of work done by the force field

$$\mathbf{F} = \langle x^x + z^2, y^y + x^2, z^z + y^2 \rangle$$

when a particle under its influence moves around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant oriented counterclockwise as viewed from above.

37. Use Stoke's theorem to evaluate $\int_C \mathbf{F} \cdot d\vec{r}$ where

$$\mathbf{F} = \langle 2z, 4x, 5y \rangle$$

and C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$ oriented counterclockwise as viewed from above.

38. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (xy^2)\mathbf{i} + (yz)\mathbf{j} + (zx^2)\mathbf{k}$ and S is the surface of the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the planes $z = 1$ and $z = 3$.

39. True/False Questions.

- (a) If $\operatorname{div}(\nabla f) = 0$ then $f = 0$.
- (b) If f has continuous derivatives of all orders and S is a closed surface bounding a region E then

$$\iint_S \nabla f \cdot \mathbf{n} dS = \iiint_E \nabla^2(f) dV.$$

Here $\nabla^2(f) = \operatorname{div}(\nabla f)$.

- (c) If \mathbf{F} is an irrotational vector field defined on all of \mathbb{R}^3 then \mathbf{F} is conservative.
- (d) If $\operatorname{curl}(\mathbf{F}) = 0$ then \mathbf{F} is irrotational.
- (e) The outward unit normal to the sphere $x^2 + y^2 + z^2 = 1$ at the point $(-1, 0, 0)$ is the vector $\langle +1, 0, 0 \rangle$.
- (f) If \mathbf{F} is any vector field and S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$ then $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0$.
- (g) There is a vector field \mathbf{F} such that $\operatorname{curl}(\mathbf{F}) = x\mathbf{i} + x\mathbf{j} + x\mathbf{k}$.
- (h) If f and g are continuous one-variable functions then

$$\int_1^2 \int_5^6 [f(x) + g(y)] dx dy = \int_1^2 f(x) dx + \int_5^6 g(y) dy.$$

- (i) If f is a function of two variables with continuous partial derivatives and f has a local maximum at the point (x_0, y_0) then $\nabla f(x_0, y_0) = 0$.
- (j) For any two vector fields, \mathbf{F} and \mathbf{G} , $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$.
- (k) There is a vector field \mathbf{F} such that $\operatorname{curl}(\mathbf{F}) = x\mathbf{i} + x\mathbf{j} + x\mathbf{k}$.

40. Answer the questions in the captions of Figures 1, 2, 3 and 4. (4 points each)

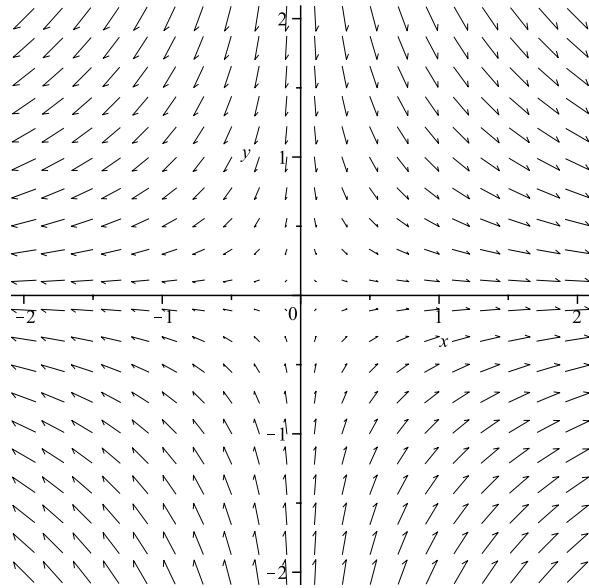


Figure 1: Does $\text{curl}(\mathbf{F})(1, 1)$ point in the positive z -direction, in the negative z -direction or is it equal to zero?

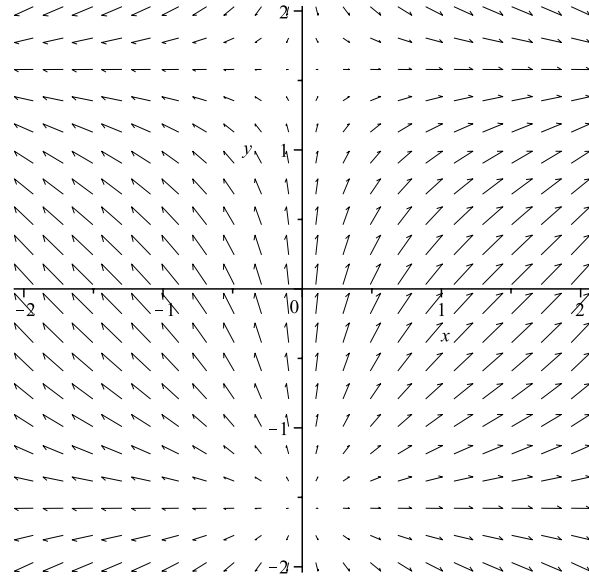


Figure 2: Let \mathbf{F} be the vector field sketched above and let C be the straight line from $(1, -1.5)$ to $(1, 1.5)$. Is the line integral $\int_C \mathbf{F} \cdot d\vec{r}$ positive, negative or zero?

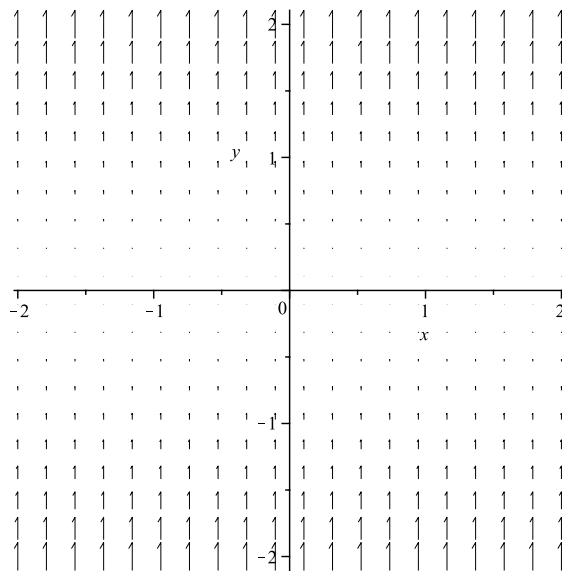


Figure 3: Is $\text{div } \mathbf{F}$ positive, negative or zero at the point $(1, -1.5)$?

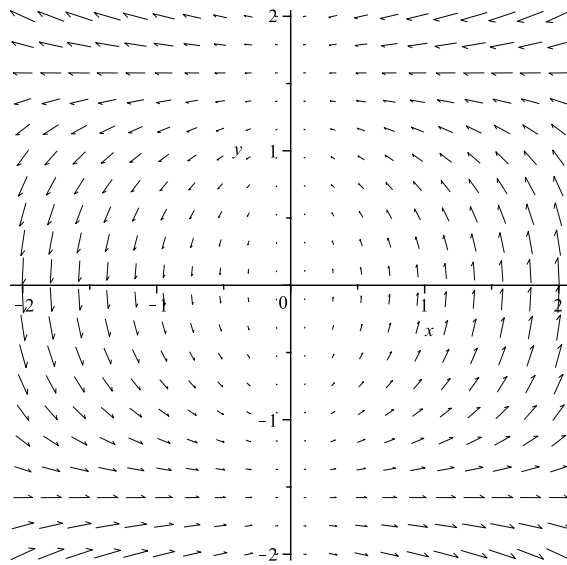


Figure 4: Is this vector field conservative?