Math 172, Spring 09
Review exercises. Version of May 4, 2009.

1. Find the area of the region bounded by

$$y = \frac{1-x}{x}, \; 2x + 3y = 2, \quad \text{and} \quad y = 2$$

   by two methods:

   (a) Integrating with respect to x.

   (b) Integrating with respect to y.

2. In each case, find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

   (a) $y = -\frac{1}{x}, \; y = 0, \; x = 1, \; x = 3.$

   (b) $y = e^x, \; y = 0, \; x = 0, \; x = 1.$

3. Calculate:

$$\int_3^7 \frac{1}{x^2 - x - 2}\, dx, \qquad \int \frac{x+2}{x^2 + 2x - 3}\, dx.$$

4. Calculate:

$$\int_0^\infty x^2 e^{-x^3}\, dx, \qquad \int_{-\infty}^3 \frac{1}{x^2+9}\, dx, \qquad \int_1^\infty \frac{\ln x}{x^3}\, dx.$$

5. The projected population y of the world (in billions of people) over the next few decades approximately satisfies the differential equation

$$\frac{dy}{dt} = .0015y(12 - y), \quad y(0) = 6,$$

   where t is time in years *since* 2000.

   (a) Solve the differential equation by separating variables.

   (b) According to this model, what is the limiting population size?

6. (a) List the first four terms of the sequence defined by $a_n = \cos(2\pi n)$.
   Does $\lim_{n\to\infty} a_n$ exist?

   (b) Consider the function $f(x) = \cos(2\pi x)$, for all real numbers x.
   Does $\lim_{x\to\infty} f(x)$ exist?

7. Find the limit of the following sequences.

$$\frac{2n}{\sqrt{n^2+1}}, \qquad \frac{\ln(2+e^n)}{3n}, \qquad \frac{\sin(n\pi/6)}{n^2}.$$

8. Find the following sums.

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^n}, \qquad \sum_{n=1}^{\infty} \frac{1+(-1)^n}{2^{2n+1}}.$$

9. For each positive integer i, let a_i be one of the integers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

   (a) Explain why the series $\dfrac{a_1}{10} + \dfrac{a_2}{100} + \dfrac{a_3}{1000} + \cdots$ is convergent.

   (b) Let S be the sum of this series. Explain why the decimal representation of S is

$$S = 0.a_1 a_2 a_3 ...$$

10. Classify the following series.

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}, \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n^2}.$$

11. Find the following sums.

$$\sum_{k=0}^{\infty} \frac{2^{k+1}}{5^{k-1}}, \qquad \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}} \right).$$

12. Classify the following series.

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}, \qquad \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7}, \qquad \sum_{n=2}^{\infty} \frac{n}{3^n(n-1)}.$$

13. Determine if the following series are absolutely convergent, convergent but not absolutely, or divergent.

$$\sum_{i=1}^{\infty} (-1)^i \frac{2^i}{i!}, \qquad \sum_{i=2}^{\infty} (-1)^i \frac{\ln i}{\sqrt{i}}.$$

14. For the following functions, find a power series representation for the function and determine the interval of convergence.

$$\frac{1}{1+2x}, \qquad 10^x.$$

15. Determine the interval of convergence of the following power series,

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}, \qquad \sum_{n=1}^{\infty} \frac{(x-4)^n}{n 5^n}.$$

16. Indicate whether each of the following statements is true or false and explain why.

   (a) If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} \cos(a_n)$ diverges.

   (b) If $\displaystyle\sum_{n=1}^{\infty} \sin(a_n)$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

17. Let f and g be analytic functions. Consider their power series representations

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n.$$

(a) Write the first few terms ($n \leq 3$) of the power series representation for the product fg of the two functions.

(b) Explain how the following formula can be deduced:

$$(fg)'(0) \; = \; f'(0)g(0) + f(0)g'(0)$$

(the product rule for derivatives).

18. Let

$$f(x) \; = \; \sum_{n=1}^{\infty} \frac{2^n}{n} x^n.$$

(a) Find the interval of convergence of this power series.

(b) Show that $y = f(x)$ is a solution to the differential equation

$$(1-2x)\frac{dy}{dx} \; = \; 2, \qquad -\frac{1}{2} < x < \frac{1}{2},$$

with initial condition

$$y(0) = 0.$$

(c) Deduce that

$$f(x) \; = \; -\ln(1-2x).$$

19. Let α be a real number and n a natural number.

(a) Write the definition of the binomial coefficient $\binom{\alpha}{n}$.

(b) Show that

$$\binom{-\alpha}{n} = (-1)^n \binom{\alpha+n-1}{n}.$$

(c) Deduce that

$$\frac{1}{(1-x)^\alpha} = \sum_{n=0}^{\infty} \binom{\alpha+n-1}{n} x^n$$

for $-1 < x < 1$.

20. (a) Find the area below the graph of the ellipse $y = 2\sqrt{h^2 - x^2}$ (Figure 1).

(b) Find the volume of the following solid, whose vertical cross-sections are ellipses of height $2h = \sqrt{x}$, as indicated.

21. (a) Let f be a continuous function on $[0, a]$. Use a (very simple) substitution to show that
$$\int_0^a f(x)\,dx \;=\; \int_0^a f(a-x)\,dx.$$
(b) Use part (a) and the trigonometric identities
$$\sin(\pi/2 - x) = \cos x \qquad \text{and} \qquad \cos(\pi/2 - x) = \sin x$$
to show that
$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx \;=\; \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx.$$
(c) Use part (b) to show that
$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx \;=\; \frac{\pi}{4}.$$

22. (a) Represent $\int e^{-x^3}\,dx$ as a power series.
(b) Approximate $\int_0^1 e^{-x^3}\,dx$ to within 0.001.

23. Calculate the antiderivative using power series:
$$\int \frac{1 + x - e^x}{x^2}\,dx.$$
(Leave the answer expressed as a power series.)

24. Find the radius and interval of convergence for the power series
$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

25. Consider the solid of Figure 2. The base is the region between the x-axis, $0 \le x \le \pi$, and the curve $y = 2\sin x$. The vertical cross sections are semicircles with diameters running from the axis to the curve.

(a) Find the volume of the solid.
(b) Find the area of the base.

26. A solid tank is shown in Figure 3, together with its horizontal cross-section at height y, which is a right triangle of the indicated dimensions. Find the volume of the tank.

27. Calculate the following antiderivatives.
$$\text{(a)} \int \frac{dx}{\sqrt[3]{3x+4}} \qquad \text{(b)} \int \frac{\tan(\ln x)}{x}\,dx \qquad \text{(c)} \int \sqrt{x^2 - x^4}\,dx$$

28. Classify the following series.
$$\text{(a)} \sum_{n=1}^{\infty} \frac{2n^2}{n^3+1}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{\ln n}{n}, \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{\ln n}{n^2}, \qquad \text{(d)} \sum_{n=1}^{\infty} (-1)^n \frac{n}{4^n}.$$

29. (a) Find the sum of the series
$$\sum_{n=1}^{\infty} \frac{1}{4^n}.$$
(b) Explain (briefly but precisely) how to derive the result in part (a) from the picture in Figure 5.