

THE HOPF ALGEBRA OF UNIFORM BLOCK PERMUTATIONS. EXTENDED ABSTRACT

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ABSTRACT. We introduce the Hopf algebra of uniform block permutations and show that it is self-dual, free, and cofree. These results are closely related to the fact that uniform block permutations form a factorizable inverse monoid. This Hopf algebra contains the Hopf algebra of permutations of Malvenuto and Reutenauer and the Hopf algebra of symmetric functions in non-commuting variables of Gebhard, Rosas, and Sagan.

RÉSUMÉ. Nous présentons l'algèbre de Hopf des permutations de blocs uniformes et démontrons qu'elle est auto duale, libre et colibre. Ces résultats sont liés au fait que les permutations de blocs uniformes constituent un monoïde inverse factorisable. Cette algèbre de Hopf contient l'algèbre de Hopf des permutations de Malvenuto et Reutenauer et l'algèbre de Hopf des fonctions symétriques à variables non commutatives de Gebhard, Rosas, et Sagan.

1. UNIFORM BLOCK PERMUTATIONS

1.1. Set partitions. Let n be a non-negative integer and let $[n] := \{1, 2, \dots, n\}$. A *set partition* of $[n]$ is a collection of non-empty disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. For example, $\mathcal{A} = \{\{2, 5, 7\}\{1, 3\}\{6, 8\}\{4\}\}$, is a set partition of $[8]$ with 4 blocks. We often specify a set partition by listing the blocks from left to right so that the sequence formed by the minima of the blocks is increasing, and by listing the elements within each block in increasing order. For instance, the set partition above will be denoted $\mathcal{A} = \{1, 3\}\{2, 5, 7\}\{4\}\{6, 8\}$. We use $\mathcal{A} \vdash [n]$ to indicate that \mathcal{A} is a set partition of $[n]$.

The *type* of a set partition \mathcal{A} of $[n]$ is the partition of n formed by the sizes of the blocks of \mathcal{A} . The symmetric group S_n acts on the set of set partitions of $[n]$: given $\sigma \in S_n$ and $\mathcal{A} \vdash [n]$, $\sigma(\mathcal{A})$ is the set partition whose blocks are $\sigma(A)$ for $A \in \mathcal{A}$. The orbit of \mathcal{A} consists of those set partitions of the same type as \mathcal{A} . The stabilizer of \mathcal{A} consists of those permutations that preserve the blocks, or that permute blocks of the same size. Therefore, the number of set

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partitions of type $1^{m_1}2^{m_2} \dots n^{m_n}$ (m_i blocks of size i) is

$$(1) \quad \frac{n!}{m_1! \dots m_n! (1!)^{m_1} \dots (n!)^{m_n}}.$$

1.2. The monoid of uniform block permutations. The monoid (and the monoid algebra) of uniform block permutations has been studied by FitzGerald [10] and Kosuda [14, 15] in analogy to the partition algebra of Jones and Martin [13, 18].

A *block permutation* of $[n]$ consists of two set partitions \mathcal{A} and \mathcal{B} of $[n]$ with the same number of blocks and a bijection $f : \mathcal{A} \rightarrow \mathcal{B}$. For example, if $n = 3$, $f(\{1, 3\}) = \{3\}$ and $f(\{2\}) = \{1, 2\}$ then f is a block permutation. A block permutation is called *uniform* if it maps each block of \mathcal{A} to a block of \mathcal{B} of the same cardinality. For example, $f(\{1, 3\}) = \{1, 2\}$, $f(\{2\}) = \{3\}$ is uniform. Each permutation may be viewed as a uniform block permutation for which all blocks have cardinality 1. In this paper we only consider block permutations that are uniform.

To specify a uniform block permutation $f : \mathcal{A} \rightarrow \mathcal{B}$ we must choose two set partitions \mathcal{A} and \mathcal{B} of the same type $1^{m_1} \dots n^{m_n}$ and for each i a bijection between the m_i blocks of size i of \mathcal{A} and those of \mathcal{B} . We deduce from (1) that the total number of uniform block permutations of $[n]$ is

$$(2) \quad u_n := \sum_{1^{m_1} \dots n^{m_n} \vdash n} \left(\frac{n!}{(1!)^{m_1} \dots (n!)^{m_n}} \right)^2 \frac{1}{m_1! \dots m_n!}$$

where the sum runs over all partitions of n . Starting at $n = 0$, the first values are

$$1, 1, 3, 16, 131, 1496, 22482, \dots$$

This is sequence A023998 in [21]. These numbers and generalizations are studied in [20]; in particular, the following recursion is given in [20, equation (11)]:

$$u_{n+1} = \sum_{k=0}^n \binom{n}{k} \binom{n+1}{k} u_k, \quad u_0 = 1.$$

We represent uniform block permutations by means of graphs. For instance, either one of the two graphs in Figure 1 represents the uniform block permutation f given by

$$\{1, 3, 4\} \rightarrow \{3, 5, 6\}, \quad \{2\} \rightarrow \{4\}, \quad \{5, 7\} \rightarrow \{1, 2\}, \quad \{6\} \rightarrow \{8\}, \quad \text{and} \quad \{8\} \rightarrow \{7\}.$$

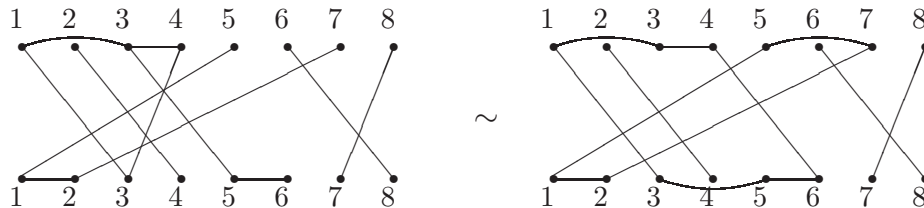


FIGURE 1. Two graphs representing the same uniform block permutation

Different graphs may represent the same uniform block permutation. For a graph to represent a uniform block permutation $f : \mathcal{A} \rightarrow \mathcal{B}$ of $[n]$ the vertex set must consist of two copies of $[n]$ (top and bottom) and each connected component must contain the same number of vertices on the top as on the bottom. The set partition \mathcal{A} is read off from the adjacencies on the top, \mathcal{B} from those on the bottom, and f from those in between.

The *diagram* of f is the unique representing graph in which all connected components are cycles and the elements in each cycle are joined in order, as in the second graph of Figure 1.

The set P_n of block permutations of $[n]$ is a monoid. The product $g \cdot f$ of two uniform block permutations f and g of $[n]$ is obtained by gluing the bottom of a graph representing f to the top of a graph representing g . The resulting graph represents a uniform block permutation which does not depend on the graphs chosen. An example is given in Figure 2. Note that gluing the diagram of f to the diagram of g may not result in the diagram of $g \cdot f$.

The identity is the uniform block permutation that maps $\{i\}$ to $\{i\}$ for all i . Viewing permutations as uniform block permutations as above, we get that the symmetric group S_n is a submonoid of P_n .

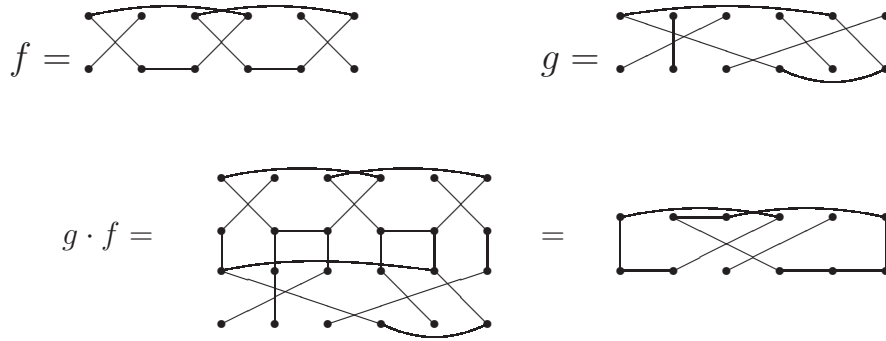
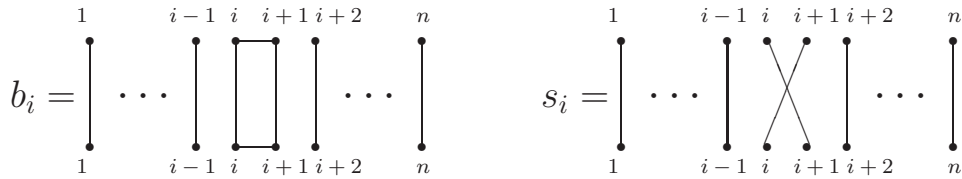


FIGURE 2. Product of uniform block permutations

We recall a presentation of the monoid P_n given in [10, 14, 15]. Consider the uniform block permutations b_i and s_i with diagrams



The monoid P_n is generated by the elements $\{b_i, s_i \mid 1 \leq i \leq n-1\}$ subject to the following relations:

- (1) $s_i^2 = 1, \quad b_i^2 = b_i, \quad 1 \leq i \leq n-1;$
- (2) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i b_{i+1} s_i = s_{i+1} b_i s_{i+1}, \quad 1 \leq i \leq n-2;$

- (3) $s_i s_j = s_j s_i, \quad b_i s_j = s_j b_i, \quad |i - j| > 1;$
(4) $b_i s_i = s_i b_i = b_i, \quad 1 \leq i \leq n - 1;$
(5) $b_i b_j = b_j b_i, \quad 1 \leq i, j \leq n - 1.$

The submonoid generated by the elements $s_i, 1 \leq i \leq n - 1$ is the symmetric group S_n , viewed as a submonoid of P_n as above.

We will see in Sections 2.3 and 2.4 that P_n is a factorizable inverse monoid. Therefore, a presentation for P_n may also be derived from the results of [9].

1.3. An ideal indexed by set partitions. Let $\mathbb{k}P_n$ be the monoid algebra of P_n over a commutative ring \mathbb{k} .

Given a set partition $\mathcal{A} \vdash [n]$, let $Z_{\mathcal{A}} \in \mathbb{k}P_n$ denote the sum of all uniform block permutations $f : \mathcal{A} \rightarrow \mathcal{B}$, where \mathcal{B} varies:

$$Z_{\mathcal{A}} := \sum_{f: \mathcal{A} \rightarrow \mathcal{B}} f.$$

For instance,

$$Z_{\{1,3\}\{2,4\}} = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \end{array} + \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \\ \text{Diagram 9} \\ \text{Diagram 10} \\ \text{Diagram 11} \\ \text{Diagram 12} \end{array} + \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \\ \text{Diagram 15} \\ \text{Diagram 16} \\ \text{Diagram 17} \\ \text{Diagram 18} \end{array} + \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \\ \text{Diagram 21} \\ \text{Diagram 22} \\ \text{Diagram 23} \\ \text{Diagram 24} \end{array} + \begin{array}{c} \text{Diagram 25} \\ \text{Diagram 26} \\ \text{Diagram 27} \\ \text{Diagram 28} \\ \text{Diagram 29} \\ \text{Diagram 30} \end{array} + \begin{array}{c} \text{Diagram 31} \\ \text{Diagram 32} \\ \text{Diagram 33} \\ \text{Diagram 34} \\ \text{Diagram 35} \\ \text{Diagram 36} \end{array}$$

Lemma 1.1. *Let \mathcal{A} be a set partition of $[n]$ and σ a permutation of $[n]$. Then*

$$\sigma \cdot Z_{\mathcal{A}} = Z_{\mathcal{A}} \quad \text{and} \quad Z_{\mathcal{A}} \cdot \sigma = Z_{\sigma^{-1}(\mathcal{A})}.$$

In addition,

$$Z_{\mathcal{A}} \cdot b_i = \begin{cases} Z_{\mathcal{A}} & \text{if } i \text{ and } i+1 \text{ belong to the same block of } \mathcal{A} \\ \binom{|A|+|A'|}{|A|} Z_{\mathcal{B}} & \text{if } i \text{ and } i+1 \text{ belong to different blocks } A \text{ and } A' \text{ of } \mathcal{A} \end{cases}$$

where the set partition \mathcal{B} is obtained by merging the blocks A and A' of \mathcal{A} and keeping the others.

Let \mathcal{Z}_n denote the subspace of $\mathbb{k}P_n$ linearly spanned by the elements $Z_{\mathcal{A}}$ as \mathcal{A} runs over all set partitions of $[n]$.

Corollary 1.2. \mathcal{Z}_n is a right ideal of the monoid algebra $\mathbb{k}P_n$.

2. THE HOPF ALGEBRA OF UNIFORM BLOCK PERMUTATIONS

In this section we define the Hopf algebra of uniform block permutations. It contains the Hopf algebra of permutations of Malvenuto and Reutenauer as a Hopf subalgebra.

2.1. Schur-Weyl duality for uniform block permutations. Let r and m be positive integers. Let C_r denote the cyclic group of order r with generator t :

$$C_r := \langle t \mid t^r = 1 \rangle.$$

Consider the complex reflection group

$$G(r, 1, m) := C_r \wr S_m.$$

Let V be the *monomial* representation of $G(r, 1, m)$. Thus, V is an m -dimensional vector space with a basis $\{e_1, e_2, \dots, e_m\}$ on which $G(r, 1, m)$ acts as follows:

$$t \cdot e_1 = e^{2\pi i/r} e_1, \quad t \cdot e_i = e_i \text{ for } i > 1, \text{ and } \sigma \cdot e_i = e_{\sigma(i)} \text{ for } \sigma \in S_m.$$

Consider now the diagonal action of $G(r, 1, m)$ on the tensor powers $V^{\otimes n}$,

$$g \cdot (e_{i_1} e_{i_2} \cdots e_{i_n}) = (g \cdot e_{i_1})(g \cdot e_{i_2}) \cdots (g \cdot e_{i_n}).$$

The centralizer of this representation has been calculated by Tanabe.

Proposition 2.1. [22] *There is a right action of the monoid P_n on $V^{\otimes n}$ determined by*

$$(e_{i_1} \cdots e_{i_n}) \cdot b_j = \delta(i_j, i_{j+1}) e_{i_1} \cdots e_{i_n} \quad \text{and} \quad (e_{i_1} \cdots e_{i_n}) \cdot \sigma = e_{i_{\sigma(1)}} \cdots e_{i_{\sigma(n)}}$$

for $1 \leq i \leq n-1$ and $\sigma \in S_n$. This action commutes with the left action of $G(r, 1, m)$ on $V^{\otimes n}$. Moreover, if $m \geq 2n$ and $r > n$ then the resulting map

$$(3) \quad \mathbb{C}P_n \rightarrow \text{End}_{G(r,1,m)}(V^{\otimes n})$$

is an isomorphism of algebras.

Classical Schur-Weyl duality states that the symmetric group algebra can be similarly recovered from the diagonal action of $GL(V)$ on $V^{\otimes n}$: if $\dim V \geq n$ then

$$(4) \quad \mathbb{C}S_n \cong \text{End}_{GL(V)}(V^{\otimes n}).$$

Malvenuto and Reutenauer [17] deduce from here the existence of a multiplication among permutations as follows. Given $\sigma \in S_p$ and $\tau \in S_q$, view them as linear endomorphisms of the tensor algebra

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n}$$

by means of (4) (σ acts as 0 on $V^{\otimes n}$ if $n \neq p$, similarly for τ). The tensor algebra is a Hopf algebra, so we can form the *convolution* product of any two linear endomorphisms:

$$T(V) \xrightarrow{\Delta} T(V) \otimes T(V) \xrightarrow{\sigma \otimes \tau} T(V) \otimes T(V) \xrightarrow{m} T(V),$$

where Δ and m are the coproduct and product of the tensor algebra. Since these two maps commute with the action of $GL(V)$, the convolution of σ and τ belongs to $\text{End}_{GL(V)}(V^{\otimes n})$, where $n = p + q$. Therefore, there exists an element $\sigma * \tau \in \mathbb{C}S_n$ whose right action equals the convolution of σ and τ . This is the product of Malvenuto and Reutenauer.

The same argument applies to uniform block permutations, in view of Proposition 2.1. We proceed to describe the resulting operation in explicit terms. As for permutations, this structure can be enlarged to that of a graded Hopf algebra.

2.2. Product and coproduct of uniform block permutations. Consider the graded vector space

$$\mathcal{P} := \bigoplus_{n \geq 0} \mathbb{k}P_n.$$

P_0 consists of the unique uniform block permutation of $[n]$, represented by the empty diagram, which we denote by \emptyset .

Let f and g be uniform block permutations of $[n]$ and $[m]$ respectively. Adding n to every entry in the diagram of g and placing it to the right of the diagram of f we obtain the diagram of a uniform block permutation of $[n+m]$, called the concatenation of f and g and denoted $f \times g$. Figure 3 shows an example.

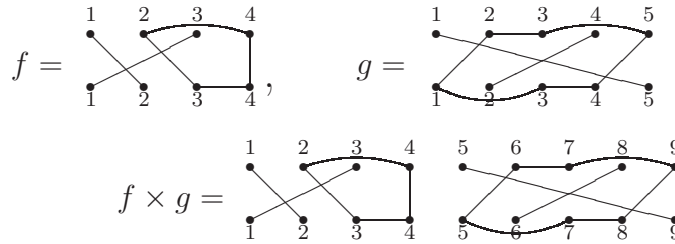


FIGURE 3. Concatenation of diagrams

Let $\text{Sh}(n, m)$ denote the set of (n, m) -shuffles, that is, those permutations $\xi \in S_{n+m}$ such that

$$\xi(1) < \xi(2) < \cdots < \xi(n) \quad \text{and} \quad \xi(n+1) < \xi(n+2) < \cdots < \xi(n+m).$$

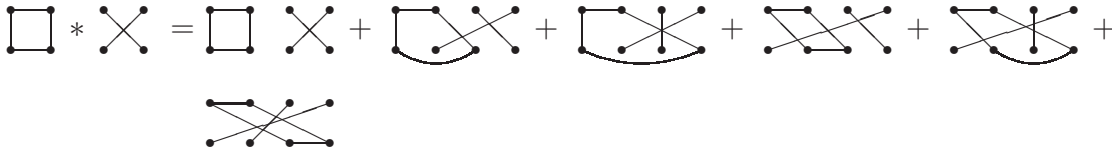
Let $sh_{n,m} \in \mathbb{k}S_{n+m}$ denote the sum of all (n, m) -shuffles.

The product $*$ on \mathcal{P} is defined by

$$f * g := sh_{n,m} \cdot (f \times g) \in \mathbb{k}P_{n+m}$$

for all $f \in P_n$ and $g \in P_m$, and extended by linearity. It is easy to see that this product corresponds to convolution of endomorphisms of the tensor algebra via the map (3), when $\mathbb{k} = \mathbb{C}$.

For example,

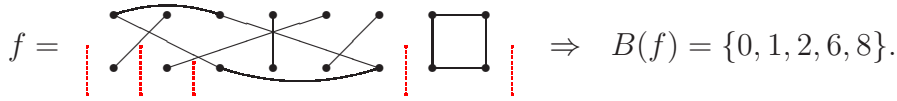


A *breaking point* of a set partition \mathcal{B} is an integer $i \in \{0, 1, \dots, n\}$ for which there exists a subset $S \subseteq \mathcal{B}$ such that

$$\bigcup_{B \in S} B = \{1, \dots, i\} \quad (\text{and hence}) \quad \bigcup_{B \in \mathcal{B} \setminus S} B = \{i + 1, \dots, n\}.$$

Given a uniform block permutation $f : \mathcal{A} \rightarrow \mathcal{B}$, let $B(f)$ denote the set of breaking points of \mathcal{B} . Note that $i = 0$ and $i = n$ are breaking points of any f . If f is a permutation, that is if all blocks of f are of size 1, then $B(f) = \{0, 1, \dots, n\}$.

In terms of the diagram of a uniform block permutation, if it is possible to put a vertical line between the first i and the last $n - i$ vertices in the bottom row without intersecting an edge between the two sets of vertices, then i is a breaking point.

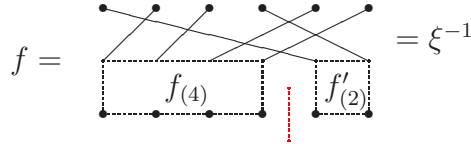


Lemma 2.2. *If i is a breaking point of f , then there exists a unique $(i, n - i)$ -shuffle $\xi \in S_n$ and unique uniform block permutations $f_{(i)} \in P_i$ and $f'_{(n-i)} \in P_{n-i}$ such that*

$$f = (f_{(i)} \times f'_{(n-i)}) \cdot \xi^{-1}.$$

Conversely, if such a decomposition exists, i is a breaking point of f .

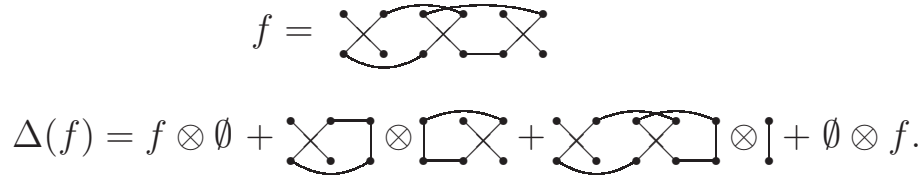
We illustrate this statement with an example where $i = 4$ and $\xi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$:



We are now ready to define the coproduct on \mathcal{P} . Given $f \in P_n$ set

$$\Delta(f) := \sum_{i \in B(f)} f_{(i)} \otimes f'_{(n-i)},$$

where $f_{(i)}$ and $f'_{(n-i)}$ are as in Lemma 2.2. An example follows.



Recall that an element $x \in \mathcal{P}$ is called primitive if $\Delta(x) = x \otimes \emptyset + \emptyset \otimes x$. Every uniform block permutation with breaking set $\{0, n\}$ is primitive, but there are other primitive elements in \mathcal{P} . For example, the following element of $\mathbb{k}P_3$ is primitive:



Recall that \emptyset denotes the empty uniform block permutation. Let $\varepsilon : \mathcal{P} \rightarrow \mathbb{k}$ be

$$\varepsilon(f) = \begin{cases} 1 & \text{if } f = \emptyset \in P_0, \\ 0 & \text{if } f \in P_n, n \geq 1. \end{cases}$$

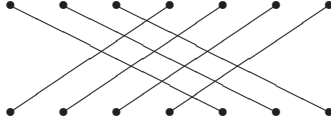
Theorem 2.3. *The graded vector space \mathcal{P} , equipped with the product $*$, coproduct Δ , unit \emptyset and counit ε , is a graded connected Hopf algebra.*

Associativity and coassociativity follow from basic properties of shuffles (for the product one may also appeal to (3) and associativity of the convolution product). The existence of the antipode is guaranteed in any graded connected bialgebra. Compatibility between Δ and $*$ requires a special argument. We sketch part of it.

Let $\beta_{n,m}$ be the (n, m) -shuffle such that

$$\beta_{n,m}(i) = \begin{cases} m + i & \text{if } 1 \leq i \leq n, \\ i - n & \text{if } n + 1 \leq i \leq n + m. \end{cases}$$

The diagram of $\beta_{3,4}$ is shown below.



The inverse of $\beta_{n,m}$ is $\beta_{m,n}$.

Let $f \in P_n$, $g \in P_m$. A summand in $\Delta(f) * \Delta(g)$ is of the form

$$\xi_1 \cdot (f' \times g') \otimes \xi_2 \cdot (f'' \times g'')$$

where $p \in B(f)$, $f' \in P_p$, $f'' \in P_{n-p}$, $q \in B(g)$, $g' \in P_q$, $g'' \in P_{m-q}$, $\xi_1 \in \text{Sh}(p, q)$, $\xi_2 \in \text{Sh}(n-p, m-q)$, and there exist unique $\eta_1 \in \text{Sh}(p, n-p)$ and $\eta_2 \in \text{Sh}(q, m-q)$ such that $f \cdot \eta_1 = f' \times f''$ and $g \cdot \eta_2 = g' \times g''$.

Let $\beta := 1_p \times \beta_{n-p,q} \times 1_{m-q}$. Then

$$\begin{aligned} \xi_1 \cdot (f' \times g') \otimes \xi_2 \cdot (f'' \times g'') &= (\xi_1 \times \xi_2) \cdot ((f' \times g') \times (f'' \times g'')) \\ &= (\xi_1 \times \xi_2) \cdot \beta \cdot ((f' \times f'') \times (g' \times g'')) \cdot \beta^{-1} \\ &= (\xi_1 \times \xi_2) \cdot \beta \cdot (f \cdot \eta_1 \times g \cdot \eta_2) \cdot \beta^{-1} \\ &= (\xi_1 \times \xi_2) \cdot \beta \cdot (f \times g) \cdot (\eta_1 \times \eta_2) \cdot \beta^{-1}. \end{aligned}$$

Let $\xi := (\xi_1 \times \xi_2) \cdot \beta$ and $\eta := (\eta_1 \times \eta_2) \cdot \beta^{-1}$. One verifies that $\xi \in \text{Sh}(n, m)$ and $\eta \in \text{Sh}(p + q, n + m - p - q)$. Therefore,

$$\xi_1 \cdot (f' \times g') \times \xi_2 \cdot (f'' \times g'') = \xi \cdot (f \times g) \cdot \eta$$

is a summand in $\Delta(f * g)$. \square

Consider the following graded subspace of \mathcal{P} :

$$\mathcal{S} := \bigoplus_{n \geq 0} \mathbb{k}S_n.$$

Proposition 2.4. \mathcal{S} is a Hopf subalgebra of \mathcal{P} .

\mathcal{S} is the Hopf algebra of permutations of Malvenuto and Reutenauer [17]. Let σ be a permutation. In the notation of [2], the element $\sigma \in \mathcal{S}$ corresponds to the basis element F_σ^* of \mathcal{SSym}^* , or equivalently the element $F_{\sigma^{-1}}$ of \mathcal{SSym} .

2.3. Inverse monoid structure and self-duality. As \mathcal{S} , the Hopf algebra \mathcal{P} is self-dual. To see this, recall that a block permutation is a bijection $f : \mathcal{A} \rightarrow \mathcal{B}$ between two set partitions of $[n]$. Let $\tilde{f} : \mathcal{B} \rightarrow \mathcal{A}$ denote the inverse bijection. If f is uniform then so is \tilde{f} . The diagram of $\tilde{f} \in P_n$ is obtained by reflecting the diagram of f across a horizontal line. Note that for $\sigma \in S_n \subseteq P_n$ we have $\tilde{\sigma} = \sigma^{-1}$.

Let \mathcal{P}^* be the graded dual space of \mathcal{P} :

$$\mathcal{P}^* = \bigoplus_{n \geq 0} (\mathbb{k}P_n)^*.$$

Let $\{f^* \mid f \in P_n\}$ be the basis of $(\mathbb{k}P_n)^*$ dual to the basis P_n of $\mathbb{k}P_n$.

Proposition 2.5. The map $\mathcal{P}^* \rightarrow \mathcal{P}$, $f^* \mapsto \tilde{f}$, is an isomorphism of graded Hopf algebras.

The operation $f \mapsto \tilde{f}$ is also relevant to the monoid structure of P_n . Indeed, the following properties are satisfied

$$f = f\tilde{f}f \quad \text{and} \quad \tilde{f} = \tilde{f}f\tilde{f}.$$

Together with (5) below, these properties imply that P_n is an *inverse monoid* [7, Theorem 1.17]. The following properties are consequences of this fact [7, Lemma 1.18]:

$$\widetilde{fg} = \tilde{g}\tilde{f}, \quad \tilde{\tilde{f}} = f$$

(they can also be verified directly).

2.4. Factorizable monoid structure and the weak order. Let E_n denote the poset of set partitions of $[n]$: we say that $\mathcal{A} \leq \mathcal{B}$ if every block of \mathcal{B} is contained in a block of \mathcal{A} . This poset is a lattice, and this structure is related to the monoid structure of uniform block permutations as follows. If $id_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ denotes the uniform block permutation which is the identity map on the set of blocks of \mathcal{A} , then

$$(5) \quad id_{\mathcal{A}} \cdot id_{\mathcal{B}} = id_{\mathcal{A} \wedge \mathcal{B}}.$$

In other words, viewing E_n as a monoid under the meet operation \wedge , the map

$$E_n \rightarrow P_n, \quad \mathcal{A} \mapsto id_{\mathcal{A}},$$

is a morphism of monoids.

Any uniform block permutation $f \in P_n$ decomposes (non-uniquely) as

$$(6) \quad f = \sigma \cdot id_{\mathcal{A}}$$

for some $\sigma \in S_n$ and $\mathcal{A} \in E_n$. Note that σ is invertible and $id_{\mathcal{A}}$ is idempotent, by (5). It follows that P_n is a *factorizable inverse monoid* [6, Section 2], [16, Chapter 2.2]. Moreover, by Lemma 2.1 in [6], any invertible element in P_n belongs to S_n and any idempotent element in P_n belongs to (the image of) E_n . This lemma also guarantees that in (6), the idempotent $id_{\mathcal{A}}$ is uniquely determined by f (which is clear since \mathcal{A} is the domain of f). On the other hand, σ is not unique, and we will make a suitable choice of this factor to define a partial order on P_n .

Consider the action of S_n on P_n by left multiplication. Given $\mathcal{A} \in E_n$, the orbit of $id_{\mathcal{A}}$ consists of all uniform block permutations $f : \mathcal{A} \rightarrow \mathcal{B}$ with domain \mathcal{A} , and the stabilizer is the *parabolic* subgroup

$$S_{\mathcal{A}} := \{\sigma \in S_n \mid \sigma(A) = A \ \forall A \in \mathcal{A}\}.$$

Consider the set of \mathcal{A} -shuffles:

$$\text{Sh}(\mathcal{A}) := \{\xi \in S_n \mid \text{if } i < j \text{ are in the same block of } \mathcal{A} \text{ then } \xi(i) < \xi(j)\}.$$

It is well-known that these permutations form a set of representatives for the left cosets of the subgroup $S_{\mathcal{A}}$. Therefore, given a uniform block permutation $f : \mathcal{A} \rightarrow \mathcal{B}$ there is a unique \mathcal{A} -shuffle ξ_f such that

$$f = \xi_f \cdot id_{\mathcal{A}}.$$

We use this decomposition to define a partial order on P_n as follows:

$$f \leq g \iff \xi_f \leq \xi_g,$$

where the partial order on the right hand side is the left weak order on S_n (see for instance [2]). We refer to this partial order as the *weak order* on P_n . Thus, P_n is the disjoint union of certain subposets of the weak order on S_n :

$$P_n \cong \bigsqcup_{\mathcal{A} \in [n]} \text{Sh}(\mathcal{A})$$

(in fact, each $\text{Sh}(\mathcal{A})$ is a lower order ideal S_n). Figures 4-8 show 5 of the 15 components of P_4 . Note that even when \mathcal{A} and \mathcal{B} are set partitions of the same type the posets $\text{Sh}(\mathcal{A})$ and $\text{Sh}(\mathcal{B})$ need not be isomorphic.

The partial order we have defined on P_n should not be confused with the *natural partial order* which is defined on any inverse semigroup [8, Chapter 7.1], [16, Chapter 1.4].

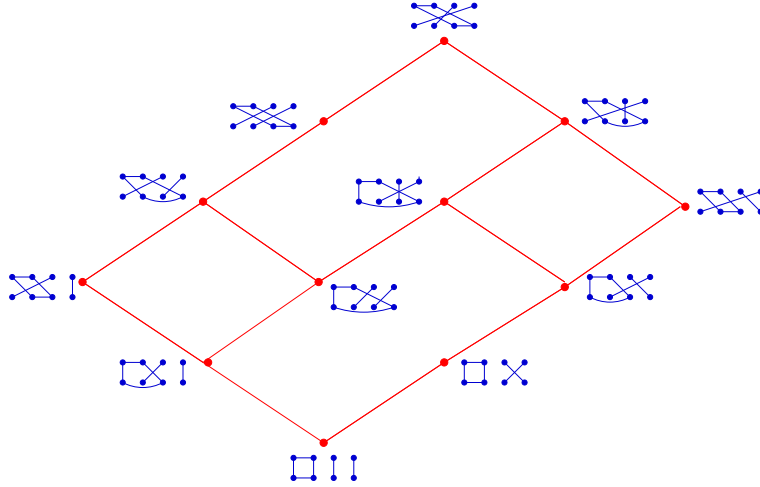


FIGURE 4. The component of P_4 corresponding to $\mathcal{A} = \{1, 2\}\{3\}\{4\}$

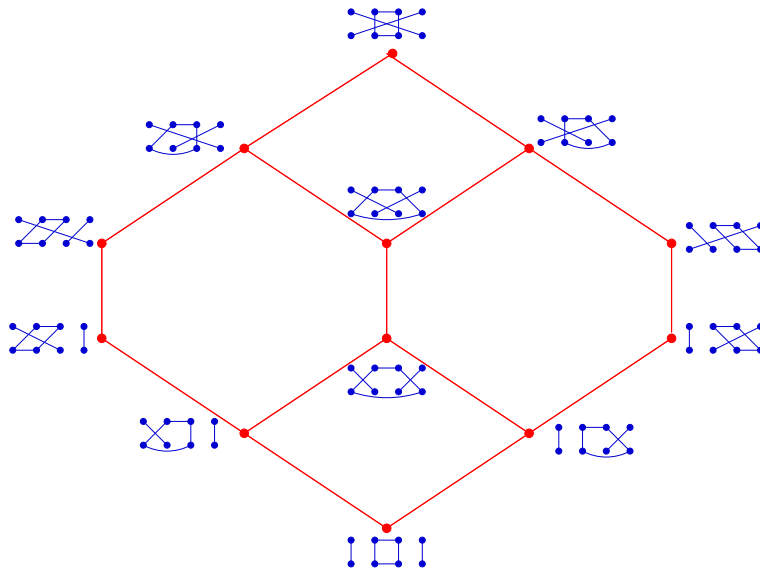


FIGURE 5. The component of P_4 corresponding to $\mathcal{A} = \{1\}\{2, 3\}\{4\}$

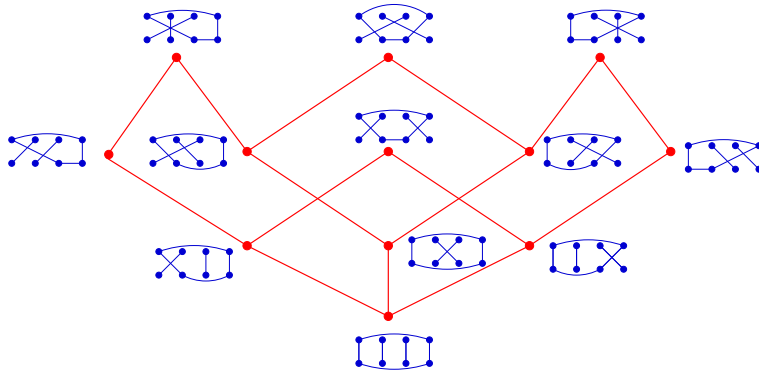


FIGURE 6. The component of P_4 corresponding to $\mathcal{A} = \{1, 4\}\{2\}\{3\}$

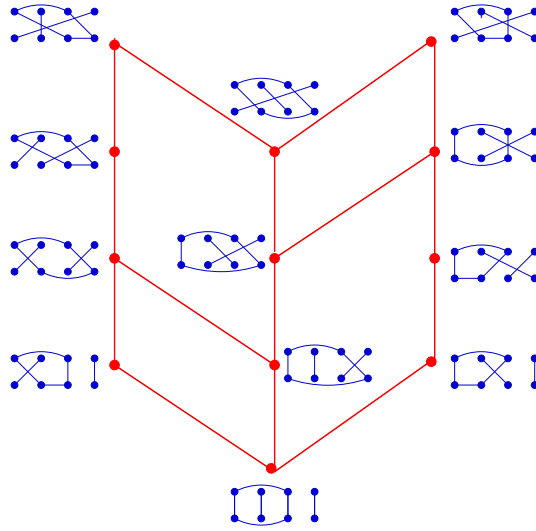


FIGURE 7. The component of P_4 corresponding to $\mathcal{A} = \{1, 3\}\{2\}\{4\}$

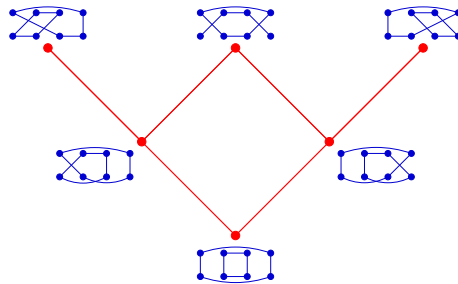


FIGURE 8. The component of P_4 corresponding to $\mathcal{A} = \{1, 4\}\{2, 3\}$

Remark 2.6. As observed by Sloane [21], there is a connection between uniform block permutations and the *patience games* of Aldous and Diaconis [3]. Starting from a deck of cards a patience game produces a number of card piles according to certain simple rules (the output is not unique). If the cards are numbered $1, \dots, n$, the initial deck is a permutation of $[n]$ and the resulting piles form a set partition of $[n]$. Suppose $\sigma \in S_n$. The set partitions \mathcal{A} such that $\sigma \in \text{Sh}(\mathcal{A})$ are precisely the possible outputs of patience games played from a deck of cards with $\sigma^{-1}(1)$ in the bottom, followed by $\sigma^{-1}(2)$, up to $\sigma^{-1}(n)$ on the top. Thus, uniform block permutations are in bijection with the pairs consisting of the input and the output of a patience game via $(\sigma, \mathcal{A}) \leftrightarrow \sigma \cdot \text{id}_{\mathcal{A}}$.

2.5. The second basis and the Hopf algebra structure. Following the ideas of [2], we use the weak order on P_n to define a new linear basis of the spaces $\mathbb{k}P_n$, on which the algebra structure of \mathcal{P} is simple.

For each element $g \in P_n$ let

$$X_g := \sum_{f \leq g} f.$$

By Möbius inversion, the set $\{X_g \mid g \in P_n\}$ is a linear basis of \mathcal{P}_n .

Given $p, q \geq 0$, let $\xi_{p,q} \in S_{p+q}$ be the permutation

$$\xi_{p,q} := \begin{pmatrix} 1 & 2 & \dots & p & p+1 & p+2 & \dots & p+q \\ q+1 & q+2 & \dots & q+p & 1 & 2 & \dots & q \end{pmatrix}.$$

This is the maximum element of $\text{Sh}(p, q)$ under the weak order. The product of \mathcal{P} takes the following simple form on the X -basis.

Proposition 2.7. *Let $g_1 \in P_p$ and $g_2 \in P_q$ be uniform block permutations. Then*

$$X_{g_1} * X_{g_2} = X_{\xi_{p,q} \cdot (g_1 \times g_2)}.$$

Corollary 2.8. *The Hopf algebra \mathcal{P} is free as an algebra and cofree as a graded coalgebra.*

Let V denote the space of primitive elements of \mathcal{P} . It follows that the generating series of \mathcal{P} and V are related by

$$\mathcal{P}(x) = \frac{1}{1 - V(x)}.$$

Since

$$\mathcal{P}(x) = 1 + x + 3x^2 + 16x^3 + 131x^4 + 1496x^5 + 22482x^6 + \dots$$

we deduce that

$$V(x) = x + 2x^2 + 11x^3 + 98x^4 + 1202x^5 + 19052x^6 + \dots$$

Remark 2.9. The same conclusion may be derived by introducing another basis

$$Z_g := \sum_{f \geq g} f.$$

This has the property that

$$Z_{g_1} * Z_{g_2} = Z_{g_1 \times g_2}.$$

Note that $Z_{id_{\mathcal{A}}}$ is the element denoted $Z_{\mathcal{A}}$ in Section 1.3.

3. THE HOPF ALGEBRA OF SYMMETRIC FUNCTIONS IN NON-COMMUTING VARIABLES

Let X be a countable set, the *alphabet*. A *word of length n* is a function $w : [n] \rightarrow X$. Let $\mathbb{k}\langle\langle X \rangle\rangle$ be the algebra of non-commutative power series on the set of variables X . Its elements are infinite linear combinations of words, and the product is concatenation of words.

The *kernel* of a word w of length n is the set partition $\mathcal{K}(w)$ of $[n]$ whose blocks are the non-empty fibers of w . Order the set of set partitions of $[n]$ by refinement, as in Section 2.4. For each set partition \mathcal{A} of $[n]$, let

$$p_{\mathcal{A}} := \sum_{\mathcal{K}(w) \leq \mathcal{A}} w \in \mathbb{k}\langle\langle X \rangle\rangle.$$

This is the sum of all words w such that if i and j are in the same block of \mathcal{A} then $w(i) = w(j)$. For instance

$$p_{\{1,3\}\{2,4\}} = xyxy + xzxz + yxyx + \cdots + x^4 + y^4 + z^4 + \cdots.$$

The subspace of $\mathbb{k}\langle\langle X \rangle\rangle$ linearly spanned by the elements $p_{\mathcal{A}}$ as \mathcal{A} runs over all set partitions of $[n]$, $n \geq 0$, is a subalgebra Π of $\mathbb{k}\langle\langle X \rangle\rangle$, graded by length. The elements of Π can be characterized as those power series of finite degree that are invariant under any permutation of the variables. Π is the algebra of symmetric functions in non-commuting variables introduced by Wolf [23] and recently studied by Gebhard, Rosas, and Sagan [11, 12, 19] in connection to Stanley's chromatic symmetric function.

Π is in fact a graded Hopf algebra [4, 1]. The coproduct is defined via evaluation of symmetric functions on two copies of the alphabet X . In order to describe the product and coproduct of Π on the basis elements $p_{\mathcal{A}}$ we introduce some notation.

Given set partitions $\mathcal{A} \vdash [n]$ and $\mathcal{B} \vdash [m]$ let $\mathcal{A} \times \mathcal{B}$ be the set partition of $[n+m]$ whose blocks are the blocks of \mathcal{A} and the sets $\{b+n \mid b \in B\}$ where B is a block of \mathcal{B} . This corresponds to the operation \times on uniform block permutations in the sense that $id_{\mathcal{A}} \times id_{\mathcal{B}} = id_{\mathcal{A} \times \mathcal{B}}$. For example, if $\mathcal{A} = \{1, 3, 4\}\{2, 5\}\{6\} \vdash [6]$ and $\mathcal{B} = \{1, 4\}\{2\}\{3, 5\} \vdash [5]$, then $\mathcal{A} \times \mathcal{B} = \{1, 3, 4\}\{2, 5\}\{6\}\{7, 10\}\{8\}\{9, 11\} \vdash [11]$.

To a set partition $\mathcal{A} \vdash [n]$ and a subset $S \subseteq \mathcal{A}$ we associate a new set partition \mathcal{A}_S as follows. Write

$$\bigcup_{A \in S} A = \{j_1, \dots, j_m\} \subseteq [n]$$

with $j_1 < j_2 < \cdots < j_m$. \mathcal{A}_S is the set partition of $[m]$ whose blocks are obtained from the blocks $A \in S$ by replacing each j_t by t , for $1 \leq t \leq m$. For instance, if $S = \{1, 5\}\{2, 7\}$ then $\mathcal{A}_S = \{1, 3\}\{2, 4\} \vdash [4]$.

The product and coproduct of Π are given by

$$(7) \quad p_{\mathcal{A}p_{\mathcal{B}}} = p_{\mathcal{A} \times \mathcal{B}},$$

$$(8) \quad \Delta(p_{\mathcal{A}}) = \sum_{S \sqcup T = \mathcal{A}} p_{\mathcal{A}_S} \otimes p_{\mathcal{A}_T},$$

the sum over all decompositions of \mathcal{A} into disjoint sets of blocks S and T . For example, if $\mathcal{A} = \{1, 2, 6\}\{3, 5\}\{4\}$, then

$$\begin{aligned} \Delta(p_{\mathcal{A}}) = & p_{\mathcal{A}} \otimes 1 + p_{\{1,2,5\}\{3,4\}} \otimes p_{\{1\}} + p_{\{1,2,4\}\{3\}} \otimes p_{\{1,2\}} + p_{\{1,3\}\{2\}} \otimes p_{\{1,2,3\}} + \\ & p_{\{1,2,3\}} \otimes p_{\{1,3\}\{2\}} + p_{\{1,2\}} \otimes p_{\{1,2,4\}\{3\}} + p_{\{1\}} \otimes p_{\{1,2,5\}\{3,4\}} + 1 \otimes p_{\mathcal{A}}. \end{aligned}$$

Consider now the direct sum of the subspaces \mathcal{Z}_n of $\mathbb{k}P_n$ introduced in Section 1.3:

$$\mathcal{Z} := \bigoplus_{n \geq 0} \mathcal{Z}_n \subset \mathcal{P}.$$

Theorem 3.1. \mathcal{Z} is a Hopf subalgebra of \mathcal{P} . Moreover, the map

$$\mathcal{Z} \rightarrow \Pi, \quad Z_{\mathcal{A}} \mapsto p_{\mathcal{A}}$$

is an isomorphism of graded Hopf algebras.

Thus the Hopf algebra of uniform block permutations \mathcal{P} contains the Hopf algebra Π of symmetric functions in non-commuting variables. Note also that this reveals the existence of a second operation on Π : according to Corollary 1.2, each homogeneous component Π_n carries an associative non-unital product that turns it into a right ideal of the monoid algebra $\mathbb{k}P_n$. Connections between Π and other combinatorial Hopf algebras are studied in [1].

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