THE SMASH PRODUCT OF SYMMETRIC FUNCTIONS. EXTENDED ABSTRACT

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Abstract. We construct a new operation among representations of the symmetric group that interpolates between the classical *internal* and *external* products, which are defined in terms of tensor product and induction of representations. Following Malvenuto and Reutenauer, we pass from symmetric functions to non-commutative symmetric functions and from there to the algebra of permutations in order to relate the internal and external products to the *composition* and *convolution* of linear endomorphisms of the tensor algebra. The new product we construct corresponds to the *smash* product of endomorphisms of the tensor algebra. For symmetric functions, the smash product is given by a construction which combines induction and restriction of representations. For non-commutative symmetric functions, the structure constants of the smash product are given by an explicit combinatorial rule which extends a well-known result of Garsia, Remmel, Reutenauer, and Solomon for the descent algebra. We describe the dual operation among quasi-symmetric functions in terms of alphabets.

Résumé. Nous construisons une nouvelle opération parmi les représentations du groupe symétrique qui interpole entre les produits *interne* et *externe*. Ces derniers sont définis en termes du produit tensoriel et de l'induction des représentations. D'après Malvenuto et Reutenauer, nous passons des fonctions symétriques aux fonctions symétriques non commutatives et à l'algèbre des permutations afin de rapporter les produits internes et externes à la composition et à la convolution d'endomorphismes linéaires de l'algèbre tensorielle. Le nouveau produit correspond au produit *smash* d'endomorphismes de l'algèbre tensorielle. Pour les fonctions symétriques, le produit smash est donné par une construction qui combine l'induction et la restriction de représentations. Pour les fonctions symétriques non commutatives, les constantes de structure du produit smash sont données par une règle combinatoire explicite qui prolonge un résultat bien connu de Garsia, Remmel, Reutenauer et Solomon pour l'algèbre de descentes. Nous décrivons l'opération duale au niveau des fonctions quasi-symétriques en termes d'alphabets.

Introduction

Our goal is to introduce a new operation among symmetric functions which interpolates between the classical internal and external products. This operation is best understood by considering not only symmetric functions but also three other algebras, related by

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means of the following fundamental commutative diagram:

$$\begin{array}{ccc} \Sigma & \hookrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ \Lambda & \hookrightarrow & \mathcal{Q} \end{array} \qquad (1)$$

Λ is the algebra of symmetric functions [9, 13, 16], $\mathcal{Q}$ is the algebra of quasi-symmetric functions [8], Σ is the algebra of non-commutative symmetric functions [7], and $\mathcal{S}$ is the algebra of permutations [11]. All four are graded connected Hopf algebras. The definitions of these Hopf algebras, as well as the maps that relate them, are reviewed in this paper.

We work over a field $\Bbbk$ of characteristic 0.

Let V be a representation of the symmetric group S_n. The *Frobenius characteristic* of V is the symmetric function

$$\sum_{\lambda \vdash n} \frac{\chi_V(\lambda)}{z_\lambda} \sum_{i_1,\ldots,i_r} x_{i_1}^{\ell_1} \cdots x_{i_r}^{\ell_r} .$$

The sum is over all partitions $\lambda = (\ell_1 \geq \cdots \geq \ell_r)$ of n, z_λ is the order of the stabilizer of the conjugacy class of permutations of cycle-type λ, and $\chi_V(\lambda)$ is the character of V evaluated on any such permutation. This association allows us to identify the algebra of symmetric functions with the direct sum of the Grothendieck groups of S_n:

$$\Lambda = \bigoplus_{n \geq 0} \operatorname{Rep}(S_n) .$$

Let V be a representation of S_p and W a representation of S_q. The *external* product of V and W is the representation

$$V * W := \operatorname{Ind}_{S_p \times S_q}^{S_{p+q}} (V \otimes W)$$

of S_{p+q}. This operation corresponds to the product of power series under the Frobenius characteristic. If V and W are representations of S_n, their *internal* product is the diagonal representation of S_n on their tensor product:

$$V \circ W := \operatorname{Res}_{S_n}^{S_n \times S_n} (V \otimes W) .$$

Explicit expressions for these products on the basis of irreducible representations (Schur functions) are of central interest in the theory of symmetric functions. While a complete solution for the case of the external product is known (the *Littlewood-Richardson rule*), only partial answers are known for the case of the internal product (the *Kronecker problem*).

In Section 1 we introduce a new product $V \# W$ between representations of S_p and S_q which contains the internal and external products as the terms of extreme degrees, as well as additional terms, and which is still associative. We say that this *smash product* of representations interpolates between the internal and external products. For example, the smash product of the complete symmetric functions $h_{(2,1)}$ and h_3 is

$$h_{(2,1)} \# h_3 = h_{(2,1)} + h_{(1,1,1,1)} + h_{(2,1,1)} + h_{(2,2,1)} + h_{(2,1,1,1)} + h_{(3,2,1)},$$

where the external product is recognized in the last term and the internal product in the first one, together with additional terms of degrees four and five.

The existence of this operation poses the problem of finding an explicit description for its structure constants on the basis of Schur functions. The answer would contain as extreme cases the Littlewood-Richardson rule and (a still unknown) rule for the Kronecker coefficients.

The smash product arises from a construction in the theory of Hopf algebras which is a simple generalization of the notion of semidirect product of groups. Let H be a Hopf algebra. Considering the action by translations of H on its dual H^* leads to an associative operation on the space $\mathrm{End}(H)$ of linear endomorphisms of H (Section 2). Let $H = \bigoplus_{n\geq 0} H_n$ be a graded connected Hopf algebra. In this general setting, the smash product of two endomorphisms $f : H_p \to H_p$ and $g : H_q \to H_q$ is a sum of various endomorphisms $H_n \to H_n$, with $\max(p, q) \leq n \leq p+q$. The endomorphism corresponding to $n = p+q$ is the familiar *convolution* of f and g, while that one corresponding to $n = p = q$ is simply the composition of g and f.

The connection to symmetric functions is made through diagram (1), starting from the opposite vertex. We take $H = T(V)$, the tensor algebra of a vector space. First of all, Schur-Weyl duality allows us to restrict the smash product of endomorphisms of $T(V)$ to the direct sum of the symmetric group algebras

$$\mathcal{S} = \bigoplus_{n\geq 0} \Bbbk S_n$$

(Corollary 2.3). Here S_n is viewed as the endomorphisms of $V^{\otimes n}$ which permute the coordinates. The convolution product on $\mathcal{S}$ is the product of Malvenuto and Reutenauer, while the composition product is simply the group algebra product. Let

$$\Sigma = \bigoplus_{n\geq 0} \Sigma_n$$

be the direct sum of *Solomon's descent algebras*. A result of Garsia and Reutenauer which characterizes the elements of Σ in terms of the action on the tensor algebra allows us to show that the smash product also restricts to Σ (Theorem 3.1). In particular, we recover the classical result of Solomon that each Σ_n is a subalgebra of the symmetric group algebra $\Bbbk S_n$, and the fact that Σ is closed under the convolution of permutations. Endowed with the latter, Σ is the algebra of non-commutative symmetric functions.

In Theorem 3.3 we provide a combinatorial rule which describes the structure constants of the smash product of Σ on the basis of shuffles (the complete non-commutative symmetric functions). This rule interpolates nicely between the well-known rule of Garsia, Remmel, Reutenauer, and Solomon for the descent algebras and the concatenation rule for the product of complete non-commutative symmetric functions.

We arrive at the smash product of symmetric functions by showing that the smash product of Σ corresponds to the smash product of Λ via Solomon's epimorphism $\Sigma \twoheadrightarrow \Lambda$ (Theorem 4.1).

In Section 5 we provide a description for the coproduct of quasi-symmetric functions corresponding to the smash product of non-commutative symmetric functions by duality. The description is in terms of alphabets (Theorem 5.1).

We also discuss the Hopf algebra structure that accompanies the smash product on Σ, Λ and $\mathcal{Q}$.

1. The smash product of representations of the symmetric group

Let p, q, n be non-negative integers with $\max(p, q) \leq n \leq p + q$. Given permutations $\sigma \in S_p$ and $\tau \in S_q$, let $\sigma \times \tau \in S_{p+q}$ be

$$(\sigma \times \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq p, \\ \tau(i-p)+p & \text{if } p+1 \leq i \leq p+q. \end{cases} \tag{2}$$

Let $S_p \times_n S_q := S_{n-q} \times S_{p+q-n} \times S_{n-p}$. Consider the embeddings

$$S_p \times_n S_q \hookrightarrow S_n \quad (\sigma, \rho, \tau) \mapsto \sigma \times \rho \times \tau\,, \tag{3}$$

$$S_p \times_n S_q \hookrightarrow S_p \times S_q \quad (\sigma, \rho, \tau) \mapsto (\sigma \times \rho, \rho \times \tau)\,. \tag{4}$$

The *smash product* of a representation V of S_p and a representation W of S_q is

$$V \# W := \bigoplus_{n=\max(p,q)}^{p+q} \mathrm{Ind}_{S_p \times_n S_q}^{S_n} \mathrm{Res}_{S_p \times_n S_q}^{S_p \times S_q} (V \otimes W)\,. \tag{5}$$

This is an element of the direct sum of the Grothendieck groups of the symmetric groups

$$\Lambda = \bigoplus_{n \geq 0} \mathrm{Rep}(S_n)\,.$$

We work over a field $\Bbbk$ of characteristic 0; Λ is a graded vector space over $\Bbbk$.

Let $(V \# W)_n$ denote the component of degree n in (5). Consider the top component ($n = p+q$). In this case embedding (4) is the identity and (3) is the standard parabolic embedding $S_p \times S_q \hookrightarrow S_{p+q}$. We thus get

$$(V \# W)_{p+q} = \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} (V \otimes W),$$

which is the usual external product of representations [6, 17].

On the other hand, when $n = p = q$, embedding (3) is the identity and (3) is the diagonal embedding $S_n \hookrightarrow S_n \times S_n$. Therefore,

$$(V \# W)_n = \mathrm{Res}_{S_n}^{S_n \times S_n} (V \otimes W),$$

the internal product of representations (also known as Kronecker's product) [6, 17].

The smash product contains terms of intermediate degrees between $\max(p, q)$ to $p+q$; in this sense it "interpolates" between the internal and external products. It is a remarkable fact that, as the internal and external products, the smash product is associative, and can be lifted to other settings (non-commutative symmetric functions, permutations, and dually, quasi-symmetric functions).

Let $\alpha = (a_1, \ldots, a_r)$ be a composition of n and

$$S_\alpha := S_{a_1} \times \cdots \times S_{a_r}\,.$$

We view S_α as a subgroup of S_n by iterating (2). Let h_α denote the permutation representation of S_n corresponding to the action by multiplication on the quotient S_n/S_α. The isomorphism class of h_α does not depend on the order of the parts of α. As α runs over the set of partitions of n, the representations h_α form a linear basis of $\mathrm{Rep}(S_n)$.

We provide an explicit description for the smash product on this basis. Let $\alpha = (a_1, \ldots, a_r) \vDash p$ and $\beta = (b_1, \ldots, b_s) \vDash q$ be two compositions and n an integer with

$\max(p,q) \le n \le p+q$. Let $a_0 = n-p$, $b_0 = n-q$, and let $\mathcal{M}^n_{\alpha,\beta}$ be the set of all $(s+1)\times(r+1)$-matrices

$$M = (m_{ij})_{0\le i\le s,\, 0\le j\le r}$$

with non-negative integer entries and such that

- the sequence of column sums is $(a_0, a_1, \dots, a_r)$,
- the sequence of row sums is $(b_0, b_1, \dots, b_s)$,
- the first entry is $m_{00} = 0$.

We illustrate these conditions as follows.

$$\begin{array}{cccc|c} 0 & m_{01} & \cdots & m_{0r} & n-q \\ m_{10} & m_{11} & \cdots & m_{1r} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{s0} & m_{s1} & \cdots & m_{sr} & b_s \\ \hline n-p & a_1 & \cdots & a_r & \end{array}$$

Let $p(M)$ be the partition of n whose parts are the non-zero m_{ij}.

Theorem 1.1.

$$h_\alpha \# h_\beta = \bigoplus_{n=\max(p,q)}^{p+q} \bigoplus_{M\in\mathcal{M}^n_{\alpha,\beta}} h_{p(M)}\,.$$

The proof of Theorem 1.1 follows from the Mackey rule from representation theory.

The space Λ can be equipped with a coproduct [6, 17]. View

$$\mathrm{Rep}(S_p\times S_q) = \mathrm{Rep}(S_p)\otimes \mathrm{Rep}(S_q)\,.$$

Given $V\in \mathrm{Rep}(S_n)$ define

$$\Delta(V) = \sum_{p+q=n} \mathrm{Res}^{S_n}_{S_p\times S_q}(V)\,, \tag{6}$$

where the restriction is along the parabolic embedding (2).

Theorem 1.2. *The space* Λ *endowed with the smash product* (5) *and the coproduct* (6) *is a connected Hopf algebra. It is commutative and cocommutative.*

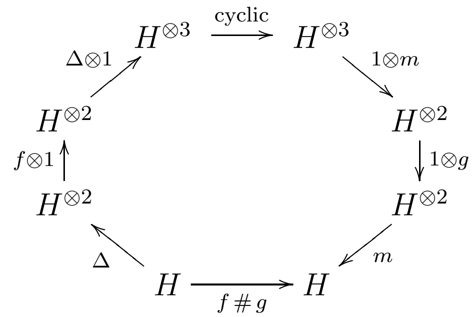
This is deduced from a similar statement for the space of non-commutative symmetric functions Σ (Theorem 3.4) via the canonical surjection $\Sigma \twoheadrightarrow \Lambda$. This is done in Section 4.

2. The smash product of endomorphisms and permutations

Let H be an arbitrary Hopf algebra, let $m : H\otimes H \to H$ be the product and $\Delta : H\otimes H\otimes H$ be the coproduct. The space $\mathrm{End}(H)$ of linear endomorphisms of H carries several associative products. Let $f, g\in \mathrm{End}(H)$. Composition and convolution are respectively defined by the diagrams

$$\begin{array}{ccc} & H & \\ {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} \\ H & \xrightarrow[f\circ g]{} & H \end{array} \qquad \begin{array}{ccc} H\otimes H & \xrightarrow{f\otimes g} & H\otimes H \\ {\scriptstyle\Delta}\uparrow & & \downarrow{\scriptstyle m} \\ H & \xrightarrow[f*g]{} & H \end{array} \tag{7}$$

The smash product of endomorphisms is defined by the diagram



(8)

where the map cyclic : $H^{\otimes 3} \to H^{\otimes 3}$ is $x \otimes y \otimes z \mapsto y \otimes z \otimes x$. Associativity follows from the Hopf algebra axioms.

The smash product is often defined in a different setting [12]: given a Hopf algebra H and an H-module-algebra A, the smash product is an operation on the space $A \otimes H$ defined by

$$(a \otimes h)(b \otimes k) := \sum a(h_1 \cdot b) \otimes h_2 k\,. \tag{9}$$

If $A = H^*$ and H acts on A by translation then (9) corresponds to (8) via the canonical inclusion $H^* \otimes H \hookrightarrow \mathrm{End}(H)$. Note that we have not made any finite-dimensionality assumptions.

Assume that H is a graded connected Hopf algebra. Thus $H = \bigoplus_{n\geq 0} H_n$ and m and Δ are degree-preserving maps. We are interested in linear endomorphisms of H which preserve the grading and are zero except on finitely many components:

$$\mathrm{end}(H) := \bigoplus_{n\geq 0} \mathrm{End}(H_n)\,.$$

The following result is central to our constructions.

Proposition 2.1. *The composition, convolution, and smash products of* $\mathrm{End}(H)$ *restrict to* $\mathrm{end}(H)$*. Moreover, if* $f \in \mathrm{End}(H_p)$*,* $g \in \mathrm{End}(H_q)$ *then*

$$f \# g \;\in \bigoplus_{n=\max(p,q)}^{p+q} \mathrm{End}(H_n) \tag{10}$$

and the top and bottom components of $f \# g$ *are*

$$(f \# g)_{p+q} = f * g \quad \textit{and, if } p = q,\ (f \# g)_p = g \circ f\,.$$

Thus the smash product interpolates between the composition and convolution products. The analogous interpolation property at all other levels (permutations, non-commutative symmetric functions, symmetric functions) is a consequence of this general result.

In order to specialize this construction we let

$$H = T(V) := \bigoplus_{n\geq 0} V^{\otimes n}$$

be the tensor algebra of a vector space V. It is a graded connected Hopf algebra with coproduct uniquely determined by

$$v \mapsto 1 \otimes v + v \otimes 1 \quad \text{for} \quad v \in V\,.$$

The general linear group $\mathrm{GL}(V)$ acts on V and hence on each $V^{\otimes n}$, diagonally. Schur-Weyl duality states that the only endomorphisms of $T(V)$ which commute with the action of $GL(V)$ are (linear combinations of) permutations. Let

$$\mathcal{S} := \bigoplus_{n\geq 0} \Bbbk S_n$$

be the direct sum of all symmetric group algebras.

Lemma 2.2. *(Schur-Weyl duality). Suppose* $\dim V = \infty$. *Then*

$$\mathcal{S} = \mathrm{end}_{GL(V)}\big(T(V)\big)\,.$$

Here each $\sigma \in S_n$ is viewed as an endomorphism of $V^{\otimes n}$ by means of its right action:

$$v_1 \cdots v_n \mapsto v_{\sigma(1)} \cdots v_{\sigma(n)}\,.$$

Malvenuto and Reutenauer [11] deduce from here that $\mathcal{S}$ is closed under convolution: since the product and coproduct of $T(V)$ commute with the action of $GL(V)$, the convolution of two permutations must be a linear combination of permutations, by Schur-Weyl duality. The same argument gives us:

Corollary 2.3. *The space $\mathcal{S}$ is closed under the smash product of endomorphisms.*

This conceptual argument is important because it can be applied to other dualities than Schur-Weyl's, i.e., to centralizer algebras of groups (or even Hopf algebras) acting on the tensor algebra other than the general linear group. It can also be applied to other products of endomorphisms, a remarkable case being that of *Drinfeld* product, which we intend to study in future work.

Malvenuto and Reutenauer give the following explicit formula for the convolution of permutations $\sigma \in S_p$ and $\tau \in S_q$:

$$\sigma * \tau = \sum_{\xi\in \mathrm{Sh}(p,q)} \xi \circ (\sigma \times \tau)\,, \tag{11}$$

where $\mathrm{Sh}(p,q) = \{\xi \in S_{p+q} \mid \xi(1) < \cdots < \xi(p),\ \xi(p+1) < \cdots < \xi(p+q)\}$ is the set of (p,q)-shuffles. Similarly, we find:

$$\sigma \,\#\, \tau = \sum_{\substack{\max(p,q)\leq n\leq p+q\\ \xi\in \mathrm{Sh}(p,n-p)\\ \eta\in \mathrm{Sh}(p+q-n,n-q)}} \xi \circ \big((\sigma\circ\eta)\times 1_{n-p}\big) \circ \beta_{2n-p-q,p+q-n} \circ (1_{n-q}\times \tau) \tag{12}$$

where $\beta_{u,v}$ is the shuffle of maximum length in $\mathrm{Sh}(u,v)$. According to Proposition 2.1, we must have

$$(\sigma \,\#\, \tau)_{p+q} = \sigma * \tau \ \text{ and, if } p = q,\ (\sigma \,\#\, \tau)_p = \sigma \circ \tau\,,$$

results which may be directly verified from (11) and (12). Note that since the action of S_n on $V^{\otimes n}$ is from the right, composition of permutations corresponds to composition of endomorphisms in the opposite order.

We thus have three associative products on $\mathcal{S}$: composition (the usual product in each symmetric group algebra $\Bbbk S_n$), convolution (which produces an element of degree $p+q$ out of two elements of degrees p and q), and the smash product, which produces elements of various degrees interpolating between the previous two, and is still associative.

3. The smash product of non-commutative symmetric functions

The descent set of a permutation $\sigma \in S_n$ is

$$\mathrm{Des}(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}\,.$$

Given $J \subseteq [n-1]$, let

$$X_J := \sum_{\substack{\sigma\in S_n\\ \mathrm{Des}(\sigma)\subseteq I}} \sigma \in \Bbbk S_n\,. \tag{13}$$

It is convenient to index basis elements of Σ_n by compositions of n by means of the bijection

$$(a_1, a_2, \dots, a_r) \leftrightarrow \{a_1, a_1 + a_2, \dots, a_1 + \cdots + a_{r-1}\}\,.$$

For instance, if $n = 9$, $X_{(1,2,4,2)} = X_{\{1,3,7\}}$.

Let Σ_n be the subspace of $\Bbbk S_n$ linearly spanned by the elements X_α as α runs over all compositions of n and

$$\Sigma := \bigoplus_{n\geq 0} \Sigma_n\,.$$

A fundamental result of Solomon [15] states that Σ_n is a subalgebra of the symmetric group algebra $\Bbbk S_n$. $(\Sigma_n, \circ)$ is Solomon's descent algebra. Thus Σ is closed under the composition product. It is also well-known that Σ is closed under the convolution product [7, 8, 11]; in fact,

$$X_{(a_1,\dots,a_r)} * X_{(b_1,\dots,b_s)} = X_{(a_1,\dots,a_r,b_1,\dots,b_s)}\,. \tag{14}$$

$(\Sigma, *)$ is the algebra of non-commutative symmetric functions.

The following theorem generalizes the two previous results in view of the interpolation property of the smash product.

Theorem 3.1. *Σ is closed under the smash product.*

We proceed to sketch four different proofs of this result, each of which is interesting in its own right.

The most conceptual, but less explicit, proof is based on an important result of Garsia and Reutenauer [5] which charactertizes the elements of $\mathcal{S}$ which belong to Σ in terms of the action on the tensor algebra. Let $L(V)$ be the smallest subspace of $T(V)$ containing V and closed under $[x, y] := xy - yx$. $T(V)$ is the free associative algebra on V, $L(V)$ is the free Lie algebra on V.

Theorem 3.2. [5] *Let $\varphi \in \mathcal{S}$. Then $\varphi \in \Sigma$ if and only if for every $P_1, \dots, P_k \in L(V)$, the subspace spanned by*

$$\{P_{\tau(1)} \cdots P_{\tau(k)} \mid \tau \in S_k\}$$

is invariant under the right action of φ.

This implies Σ is closed under the smash product. The classical fact that $L(V)$ can also be described as the primitive elements of $T(V)$ allows us to handle the smash product of endomorphisms (8) easily, and to verify that the invariance property is preserved. This proof is valuable because it can be extended to other situations. For instance, the same argument shows that Σ is closed under the Drinfeld product.

A more laborious but also more informative proof consists in obtaining an explicit formula for the smash product of two basis elements of Σ, by a direct combinatorial analysis of (12) when applied to (13). The result is expressed in terms of the same matrices as in Theorem 1.1.

Theorem 3.3. *Let $\alpha \vDash p$ and $\beta \vDash q$ be two compositions. Then*

$$X_\alpha \# X_\beta = \sum_{n=\max(p,q)}^{p+q} \sum_{M\in\mathcal{M}^n_{\alpha,\beta}} X_{c(M)} \tag{15}$$

where $c(M)$ is the composition whose parts are the non-zero entries of M, read from left to right and from top to bottom.

As an example we have the following formula

$$X_{(1^p)} \# X_{(1^q)} = \sum_{n=\max(p,q)}^{p+q} \binom{p}{n-q}\binom{q}{n-p}(p+q-n)!\, X_{(1^n)}.$$

where (1^p) is the composition of p with p parts equal to 1. The coefficients arise from the possible ways to fill the matrix

$$\begin{array}{cccc|c} 0 & * & \cdots & * & n-q \\ * & * & \cdots & * & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 1 \\ \hline n-p & 1 & \cdots & 1 & \end{array}$$

with row and column sums as prescribed.

Similarly, one verifies

$$X_{(1)}^{\#(n)} = \sum_{k=1}^{n} S(n,k) X_{(1^k)}\,,$$

where the $S(n,k)$ are the Stirling numbers of the second kind.

By the interpolation property of the smash product, Theorem 3.3 contains as special cases rules for the product in Solomon's descent algebra and for the convolution product of two basis elements of Σ. One readily verifies that the former is precisely the well-known rule of Garsia, Remmel, Reutenauer, and Solomon as given in [5, Proposition 1.1], while the latter is (14).

This proof is important because it allows us to make the connection with the smash product of representations of the symmetric group. This point is taken up in Section 4

There is a third proof which is analogous to the proof Blessenohl and Laue [3] that each Σ_n is closed under the composition product. This is of a purely combinatorial nature and is based on a description of descent classes as equivalence classes for a certain relation.

The fourth proof is geometric and is based on an extension of the smash product to the Coxeter complex of the symmetric group (that is, to the faces of the permutahedron). This makes a connection with recent work of Brown, Mahajan, Schocker, and others on this aspect of the theory of descent algebras [4, 1, 14].

There is a coproduct Δ on the space $\mathcal{S}$ which is compatible with the convolution product, in the sense that $(\mathcal{S}, *, \Delta)$ is a graded connected Hopf algebra [2, 11]. Δ is *not* compatible with the composition product of permutations [10, Remarque 5.15]. The space Σ is closed under Δ; in fact

$$\Delta(X_{(a_1,\ldots,a_r)}) = \sum_{\substack{a_i=b_i+c_i \\ 0\leq b_i,c_i}} X_{(b_1,\ldots,b_r)^\frown} \otimes X_{(c_1,\ldots,c_r)^\frown}\,, \tag{16}$$

where $^\frown$ indicates that zero parts are ommited. Moreover, it is known that Δ is compatible with *both* the convolution product and the composition product of Σ.

Theorem 3.4. *The space Σ endowed with the smash product* (15) *and the coproduct* (16) *is a connected Hopf algebra. It is cocommutative but not commutative.*

To prove compatibility between Δ and $\#$ on Σ we make use of the explicit rule (15). The existence of the antipode is automatic in a connected Hopf algebra. An explicit formula is given in Theorem 5.2.

Theorem 3.5. *The map $(\Sigma, *, \Delta) \to (\Sigma, \#, \Delta)$ given by*

$$X_{(a_1,\ldots,a_r)} \mapsto X_{(a_1)} \# \cdots \# X_{(a_r)} \tag{17}$$

is an isomorphism of Hopf algebras (which does not preserve gradings).

For this reason $(\Sigma, \#, \Delta)$ may be seen as a deformation of the Hopf algebra of non-commutative symmetric functions.

4. The smash product of symmetric functions

In this section we connect the constructions of Sections 1 and 3. Consider the map $\phi : \Sigma \to \Lambda$ defined by

$$X_\alpha \mapsto h_\alpha$$

for any composition α. In the original work of Solomon on the descent algebra, it is shown that the composition product of Σ_n corresponds to the internal product of Λ_n via ϕ. It is also known that the convolution product of Σ corresponds to the external product of Λ [7, 11] and that coproduct (16) corresponds to coproduct (6) under ϕ. This generalizes as follows.

Theorem 4.1. *The map $\phi : (\Sigma, \#, \Delta) \to (\Lambda, \#, \Delta)$ is a morphism of Hopf algebras.*

This follows from Theorems 1.1 and 3.3. In particular this shows that the smash product of representations (5) and the coproduct of representations (6) are compatible.

From Theorem 3.5 we deduce that $(\Lambda, *, \Delta)$ (the Hopf algebra of symmetric functions) and $(\Lambda, \#, \Delta)$ are isomorphic under a non-degree-preserving isomorphism.

5. The smash coproduct of quasi-symmetric functions

Let $\mathbf{X} = \{x_1, x_2, \ldots\}$ be a countable set, totally ordered by $x_1 < x_2 < \cdots$. We say that $\mathbf{X}$ is an *alphabet*. Let $\Bbbk[[\mathbf{X}]]$ be the algebra of formal power series on $\mathbf{X}$ and $\mathcal{Q} := \mathcal{Q}(\mathbf{X})$ the subspace linearly spanned by the elements

$$M_\alpha := \sum_{i_1<\cdots<i_r} x_{i_1}^{a_1} \cdots x_{i_r}^{a_r} \tag{18}$$

as $\alpha = (a_1, \ldots, a_r)$ runs over all compositions of n, $n \geq 0$. $\mathcal{Q}$ is a graded subalgebra of $\Bbbk[[\mathbf{X}]]$ known as the algebra of quasi-symmetric functions [8]. This algebra carries two coproducts $\Delta_\circ$ and Δ_* which are defined via evaluation of quasi-symmetric functions on alphabets. Let $\mathbf{Y}$ be another alphabet. We can view the disjoint union $\mathbf{X}+\mathbf{Y}$ and the Cartesian product $\mathbf{X}\times\mathbf{Y}$ as alphabets as follows: on $\mathbf{X}+\mathbf{Y}$ we keep the ordering among the variables of $\mathbf{X}$ and among the variables of $\mathbf{Y}$, and we require that every variable of $\mathbf{X}$ precede every variable of $\mathbf{Y}$. On $\mathbf{X}\times\mathbf{Y}$ we impose the reverse lexicographic order:

$$(x_h, y_i) \le (x_j, y_k) \Leftrightarrow y_i < y_k \text{ or } (y_i = y_k \text{ and } x_h < x_j)\,.$$

The coproducts are defined by the formulas

$$\Delta_\circ\big(f(\mathbf{X})\big) := f(\mathbf{X}\times\mathbf{Y}) \;\text{ and }\; \Delta_*\big(f(\mathbf{X})\big) := f(\mathbf{X}+\mathbf{Y})\,,$$

together with the identification $\mathcal{Q}(\mathbf{X},\mathbf{Y}) \cong \mathcal{Q}(\mathbf{X})\otimes\mathcal{Q}(\mathbf{X})$ (separation of variables).

Consider the following pairing between the homogeneous components of degree n of $\mathcal{Q}$ and Σ:

$$\langle M_\alpha, X_\beta\rangle = \delta_{\alpha,\beta}\,. \tag{19}$$

It is known [7, 8, 11] that this pairing identifies the product of quasi-symmetric functions with the coproduct (16) of Σ, and the coproducts $\Delta_\circ$ and Δ_* with the composition and convolution products of Σ. In other words,

$$\langle fg, u\rangle = \langle f\otimes g, \Delta(u)\rangle, \quad \langle \Delta_\circ f, u\otimes v\rangle = \langle f, u\circ v\rangle, \quad \langle \Delta_* f, u\otimes v\rangle = \langle f, u*v\rangle,$$

for any $f, g\in\mathcal{Q}$ and $u, v\in\Sigma$. Here we set

$$\langle f\otimes g, u\otimes v\rangle = \langle f, u\rangle\langle g, v\rangle\,.$$

Let $\Delta_\#$ be the coproduct of $\mathcal{Q}$ dual to the smash product of Σ:

$$\langle \Delta_\# f, u\otimes v\rangle = \langle f, u\,\#\,v\rangle\,.$$

Since the smash product is a sum of terms of various degrees (10), the smash coproduct is a finite sum of the form

$$\Delta(f) = \sum_i f_i\otimes f_i' \;\text{ with }\; 0\le \deg(f_i) \text{ and } \deg(f_i') \le \deg(f) \le \deg(f_i)+\deg(f_i')\,.$$

The summands corresponding to $\deg(f)=\deg(f_i)=\deg(f_i')$ and to $\deg(f)=\deg(f_i)+\deg(f_i')$ are the coproducts $\Delta_\circ(f)$ and $\Delta_*(f)$, respectively.

Let $\mathbf{1}+\mathbf{X}$ denote the alphabet $\mathbf{X}$ together with a new variable x_0 smaller than all the others. Let

$$(\mathbf{1}+\mathbf{X})\times(\mathbf{1}+\mathbf{Y})-\mathbf{1}$$

be the Cartesian product of the alphabets $\mathbf{1}+\mathbf{X}$ and $\mathbf{1}+\mathbf{X}$ with reverse lexicographic ordering and with the variable (x_0, y_0) removed.

The following result was obtained in conversation with Arun Ram.

Theorem 5.1. *For any $f\in\mathcal{Q}$,*

$$\Delta_\#\big(f(\mathbf{X})\big) = f\big((\mathbf{1}+\mathbf{X})\times(\mathbf{1}+\mathbf{Y})-\mathbf{1}\big)\,.$$

The set $(\mathbf{1}+\mathbf{X})\times(\mathbf{1}+\mathbf{Y})-\mathbf{1}$ can be identified with the disjoint union of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{X}\times\mathbf{Y}$, so the evaluation of f on this alphabet produces an element of $\mathcal{Q}(\mathbf{X},\mathbf{Y})\cong\mathcal{Q}(\mathbf{X})\otimes\mathcal{Q}(\mathbf{Y})$.

Endowed with the coproduct $\Delta_\#$, the algebra $\mathcal{Q}$ is a Hopf algebra, in duality with the connected Hopf algebra $(\Sigma, \#, \Delta)$ by means of (19). We turn to the antipode of this Hopf algebra.

First, define the evaluation of quasi-symmetric functions on the the opposite of an alphabet $\mathbf{X}$ by the equation

$$M_\alpha(-\mathbf{X}) = (-1)^r \sum_{i_1 \geq \cdots \geq i_r} x_{i_1}^{a_1} \cdots x_{i_r}^{a_r},$$

for any composition $\alpha = (a_1, \ldots, a_r)$. Compare with (18). Second, define the alphabet

$$\mathbf{X}^* := \mathbf{X} + \mathbf{X}^2 + \mathbf{X}^3 + \cdots$$

as the disjoint union of the Cartesian powers $\mathbf{X}^n$ under reverse lexicographic order. For instance $(x_3, x_1, x_2) < (x_2, x_2) < (x_1, x_3, x_2)$.

Theorem 5.2. *The antipode of the Hopf algebra of quasi-symmetric functions $\mathcal{Q}$ endowed with the smash coproduct is*

$$S_\#(f) = f(-\mathbf{X}^*),$$

We define the *exponential* of an alphabet $\mathbf{X}$ as

$$\mathbf{e}(\mathbf{X}) = \mathbf{X} + \mathbf{X}^{(2)} + \mathbf{X}^{(3)} + \cdots,$$

where the *divided power* $\mathbf{X}^{(n)}$ is the set

$$\mathbf{X}^{(n)} = \{(x_{i_1}, x_{i_2}, \cdots x_{i_n}) \in \mathbf{X}^n \mid x_{i_1} < x_{i_2} < \cdots < x_{i_n}\}.$$

We endow $\mathbf{e}(\mathbf{X})$ with the reverse lexicographic order, so that $\mathbf{e}(\mathbf{X})$ is a subalphabet of $\mathbf{X}^*$. Let $\widehat{\mathcal{Q}} := \prod_{n \geq 0} \mathcal{Q}_n$. Given $f \in \widehat{\mathcal{Q}}$, define

$$\varphi(f) := f\big(\mathbf{e}(\mathbf{X})\big).$$

If $f = \sum_{n \geq 0} f_n$ with f_n of degree n then the only the terms in $\varphi(f)$ which contribute to the component of degree n are those $\varphi(f_i)$ for which $i \leq n$. Thus $\varphi(f)$ is a well-defined element of $\widehat{\mathcal{Q}}$.

Under the pairing (19), $\widehat{\mathcal{Q}}$ identifies with the full linear dual of the space Σ.

Theorem 5.3. *The map $\varphi : \widehat{\mathcal{Q}} \to \widehat{\mathcal{Q}}$ is dual to the isomorphism $(\Sigma, *, \Delta) \cong (\Sigma, \#, \Delta)$ of Theorem 3.5.*

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