Baxter algebras

An overview

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Definition

Fix $\theta \in \mathbb{K}$, a scalar.

Let A be an associative algebra over $\mathbb{K}$, not necessarily unital.

A linear map $\beta : A \to A$ such that

$$\beta(a)\beta(b) = \beta\big(\beta(a)b + a\beta(b) - \theta\, ab\big)$$

is called a Baxter operator on A.

The pair (A, β) is called a Baxter algebra (of weight θ).

It suffices to consider $\theta = 0$ or 1 (since $\dfrac{1}{\theta}\beta$ is of weight 1).

Idempotent morphism

Let $\beta : A \to A$ be a morphism of algebras such that

$$\beta^2 = \beta.$$

Then β is a Baxter operator of weight 1:

$$\beta(a)\beta(b) = \underbrace{\beta\big(\beta(a)b\big)}_{\beta(a)\beta(b)} + \underbrace{\beta\big(a\beta(b)\big)}_{\beta(a)\beta(b)} + (-1)\ \underbrace{\beta(ab)}_{\beta(a)\beta(b)}$$

Baxter operators retain some properties of idempotent morphisms.

For instance, the image of β is a subalgebra.

Integral operator

Let $A = C(\mathbb{R})$, $\beta : A \longrightarrow A$ given by $\beta(f) := F$, where

$$F(x) := \int_0^x f(t)\, dt.$$

Then β is a Baxter operator of weight 0:

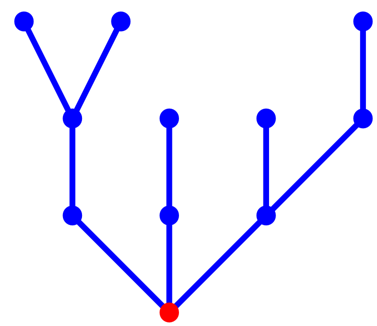
$$\int_0^x F'(t)G(t)\, dt = F(x)G(x) - \int_0^x F(t)G'(t)\, dt$$

$\Rightarrow$

$$\beta\big(f\beta(g)\big) = \beta(f)\beta(g) - \beta\big(\beta(f)g\big).$$

A discrete integral operator

Let P be a partially ordered set whose Hasse diagram is a rooted tree:



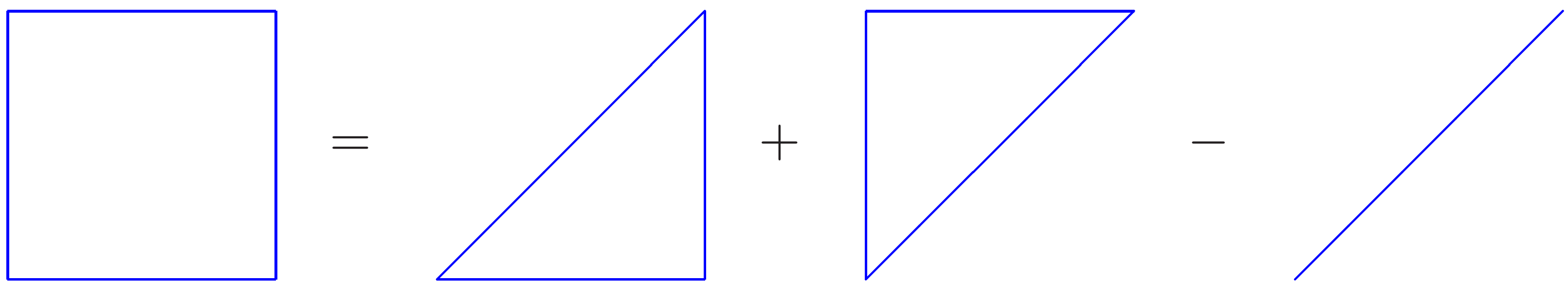
Let R be an algebra and $A := R^P$ (pointwise product).

Let $\beta : A \to A$ be

$$\beta(f)(x) := \sum_{t \leq x} f(t).$$

Then β is a Baxter operator of weight 1.

Proof:



$$\sum_{s,t\le x} f(s)g(t) = \sum_{s\le x}\sum_{t\le s} f(s)g(t) + \sum_{t\le x}\sum_{s\le t} f(s)g(t) - \sum_{s=t\le x} f(s)g(t)$$

Spitzer's identity

Theorem (Glen Baxter, 1960)

Let A be a commutative algebra, $a \in A$,

and β be a Baxter operator of weight θ on A.

Then, in $A[[t]]$,

$$\sum_{n\geq 0} \beta\Big(a\beta\big(a\beta(\dots a\beta(a))\dots\big)\Big)\, t^n =$$

$$\exp\left(\sum_{j\geq 1} \frac{\theta^{j-1}}{j}\beta(a^j)\, t^j\right).$$

When $\theta = 0$, RHS reduces to

$$\exp\big(\beta(a)\, t\big).$$

Spitzer's identity in low degrees

Degree 2:

$$\beta\big(a\beta(a)\big) \;=\; \frac{\theta}{2}\,\beta(a^2) + \frac{1}{2}\,\beta(a)^2.$$

Degree 3:

$$\beta\Big(a\beta\big(a\beta(a)\big)\Big) \;=\; \frac{\theta^2}{3}\,\beta(a^3) + \frac{\theta}{2}\,\beta(a)\beta(a^2) + \frac{1}{6}\,\beta(a)^3.$$

Example: Spitzer for an idempotent morphism.

Recall in this case $\theta = 1$.

LHS:

$$\sum_{n\geq 0} \beta\Big(a\beta\big(a\beta(\ldots a\beta(a))\ldots\big)\Big)\, t^n = \sum_{n\geq 0} \beta(a)^n\, t^n = \frac{1}{1-\beta(a)t}.$$

RHS:

$$\exp\left(\sum_{j\geq 1} \frac{1}{j}\beta(a^j)\, t^j\right) = \exp\left(\sum_{j\geq 1} \frac{1}{j}\big(\beta(a)t\big)^j\right) = \exp\Big(-\log\big(1-\beta(a)t\big)\Big).$$

Example: Spitzer for the integral operator.

LHS:

$$\sum_{n\geq 0} \beta\Bigg(f\beta\Big(f\beta(\ldots f\beta(f))\ldots\Big)\Bigg)(x)\ t^n$$

$$= \sum_{n\geq 0} \int_{\Delta_x^n} f(s_1)\cdots f(s_n)\,ds\ t^n,$$

where

$$\Delta_x^n := \{(s_1,\ldots,s_n) \mid 0 \leq s_1 \leq \cdots \leq s_n \leq x\} \text{ (simplex)}.$$

RHS:

$$\exp\Big(\beta(f)(x)\,t\Big) = \sum_{n\geq 0} \frac{1}{n!} \int_{\mathrm{I}_x^n} f(s_1)\cdots f(s_n)\,ds\ t^n,$$

where

$$\mathrm{I}_x^n := \{(s_1,\ldots,s_n) \mid 0 \leq s_1,\cdots,s_n \leq x\} \text{ (cube)}.$$

Example: Spitzer for a discrete integral operator.

Let P be the poset $\{0 < 1\}$. Let $f \in R^P$ be

$$f(0) = x, \quad f(1) = y.$$

LHS:

$$\sum_{n \geq 0} \beta\Big(f\beta\big(f\beta(\ldots f\beta(f)) \ldots\big)\Big)(1)\, t^n = \sum_{n\geq 0} \frac{x^{n+1} - y^{n+1}}{x - y}\, t^n.$$

RHS:

$$\exp\left(\sum_{j\geq 1} \frac{1}{j} \beta(f^j)(1)\, t^j\right) = \exp\Big(\sum_{j\geq 1} \frac{x^j + y^j}{j}\, t^j\Big).$$

Example: (close to) Jackson's q-integral.

Let $A = \mathbb{K}[x]$,

$$\beta(x^n) := \frac{x^n}{1-q^n}.$$

Then β is a Baxter operator of weight 1.

LHS:

$$\sum_{n\geq 0} \beta\Bigg(x\beta\Big(x\beta(\ldots x\beta(x))\ldots\Big)\Bigg)\, t^n$$

$$= \sum_{n\geq 0} \frac{x^n}{(1-q)(1-q^2)\cdots(1-q^n)}\, t^n.$$

LHS:

$$\sum_{n\geq 0} \frac{x^n}{(1-q)(1-q^2)\cdots(1-q^n)}\ t^n$$

RHS:

$$\exp\left(\sum_{j\geq 1} \frac{1}{j}\beta(x^j)\, t^j\right) = \exp\Bigl(\sum_{j\geq 1} \frac{1}{j} \frac{x^j}{1-q^j}\ t^j\Bigr) = \exp\Bigl(\sum_{k\geq 0}\sum_{j\geq 1} \frac{1}{j} q^{jk} x^j\ t^j\Bigr)$$

$$= \prod_{k\geq 0} \exp\Bigl(\sum_{j\geq 1} \frac{1}{j}(q^k x t)^j\Bigr) = \prod_{k\geq 0} \frac{1}{1-q^k x t}$$

Conclusion:

$$\sum_{n\geq 0} \frac{x^n}{(1-q)(1-q^2)\cdots(1-q^n)}\ t^n = \prod_{k\geq 0} \frac{1}{1-q^k x t}$$

(Euler).

A classical problem in probability

Let X be a random variable. The k-th moment of X is

$$E(X^k).$$

Given a sequence of i.i.d. random variables

$$X_1, X_2, \ldots, X_n, \ldots \sim X$$

let

$$S_n := X_1 + \cdots + X_n$$

and

$$M_n := \max(S_0, S_1, \ldots, S_n).$$

Spitzer's Problem.

Express the moments of M_n in terms of the moments of X.

The positive part as a Baxter operator

Let $C(\mathbb{R})$ be a suitable algebra of functions on $\mathbb{R}$.

Given a random variable X, let $\varphi_X \in C(\mathbb{R})$ be its characteristic function (the Fourier transform of its distribution measure). Then

$$\varphi_X^{(k)}(0) := E(X^k).$$

The positive part of X is

$$X^+ := \max(X, 0).$$

Lemma.

There is a Baxter operator β of weight 1 on $C(\mathbb{R})$ such that

$$\beta(\varphi_X) = \varphi_{X^+}$$

for any random variable X.

Proof.

Let X and Y be independent random variables: $\varphi_X \varphi_Y = \varphi_{X+Y}$.

We have

$$\beta(\varphi_X)\beta(\varphi_Y) \stackrel{?}{=} \beta\big(\beta(\varphi_X)\varphi_Y\big) + \beta\big((\varphi_X \beta(\varphi_Y)\big) - \beta(\varphi_X \varphi_Y)$$

$$\iff \varphi_{X^+ + Y^+} \stackrel{?}{=} \varphi_{(X^+ + Y)^+} + \varphi_{(X + Y^+)^+} - \varphi_{(X+Y)^+}$$

This follows from

$$\{x^+ + y^+, \, (x+y)^+\} = \{(x^+ + y)^+, \, (x + y^+)^+\}$$

(equality of multisets).

Original Spitzer's identity

Theorem (Spitzer, 1956).

$$\sum_{n\geq 0} \varphi_{M_n} t^n = \exp\Big(\sum_{j\geq 1} \frac{1}{j}\,\varphi_{S_j^+}\, t^j\Big)$$

Proof.

RHS:

$$\beta(\varphi_X^j) = \beta(\varphi_{X_1} \cdots \varphi_{X_j}) = \beta(\varphi_{X_1+\cdots+X_j}) = \beta(\varphi_{S_j}) = \varphi_{S_j^+}.$$

LHS:

$$\beta\big(\varphi_X \beta(\varphi_X)\big) = \varphi_{(X_1+X_2^+)^+} = \varphi_{\max(0,X_1,X_1+X_2)}$$

and by induction

$$\beta\Bigg(\varphi_X \beta\Big(\varphi_X \beta\big(\ldots \varphi_X \beta(\varphi_X)\big)\ldots\Big)\Bigg) = \varphi_{\max(S_0,S_1,\ldots,S_n)} = \varphi_{M_n}.$$

Rota's proof of Spitzer's identity (1968) ($\theta \neq 0$).

It suffices to prove the identity in the free commutative Baxter algebra A of weight 1 on one generator a.

Let $R = \mathbb{K}[x_1, x_2, x_3, \ldots]$ and $P = \{1 < 2 < 3 < \cdots\}$. Consider the Baxter algebra R^P. Recall

$$\beta\big(f_1(\mathbf{x}),\ f_2(\mathbf{x}),\ f_3(\mathbf{x}), \ldots\big) = \big(f_1(\mathbf{x}),\ f_1(\mathbf{x}) + f_2(\mathbf{x}),\ f_1(\mathbf{x}) + f_2(\mathbf{x}) + f_3(\mathbf{x}), \ldots\big).$$

Let $\Phi : A \to R^P$ be the unique morphism of Baxter algebras such that

$$\Phi(a) := (x_1, x_2, x_3, \ldots).$$

Rota proves Φ is injective. So it suffices to check the identity in R^P.

LHS:

$$\sum_{n\geq 0} \beta\Big(a\beta\big(a\beta(\ldots a\beta(a))\ldots\big)\Big)\, t^n = \sum_{n\geq 0} \big(h_n(x_1),\, h_n(x_1,x_2),\ldots\big)\, t^n$$

where

$$h_n(\mathbf{x}) := \sum_{i_1\leq i_2\leq\cdots\leq i_n} x_{i_1} x_{i_2}\cdots x_{i_n}.$$

RHS:

$$\exp\left(\sum_{j\geq 1}\frac{1}{j}\beta(a^j)\, t^j\right) = \exp\Big(\sum_{j\geq 1}\frac{1}{j}\big(p_j(x_1),\, p_j(x_1,x_2),\ldots\big)\, t^j\Big)$$

where

$$p_n(\mathbf{x}) := x_1^n + x_2^n + x_3^n + \cdots.$$

It only remains to show that

$$\sum_{n\geq 0} h_n(\mathbf{x})\, t^n = \exp\Big(\sum_{j\geq 1} \frac{1}{j} p_j(\mathbf{x})\, t^j\Big).$$

This is Waring's formula. $\square$

There is another proof by Cartier via an explicit description of the free commutative Baxter algebra. This one works for all θ.

From Baxter algebras to associative algebras

Let (A, β) be a Baxter algebra of weight θ.

Define a new operation on A:

$$a * b := \beta(a)b + a\beta(b) - \theta\, ab.$$

Proposition: the operation $*$ is associative.

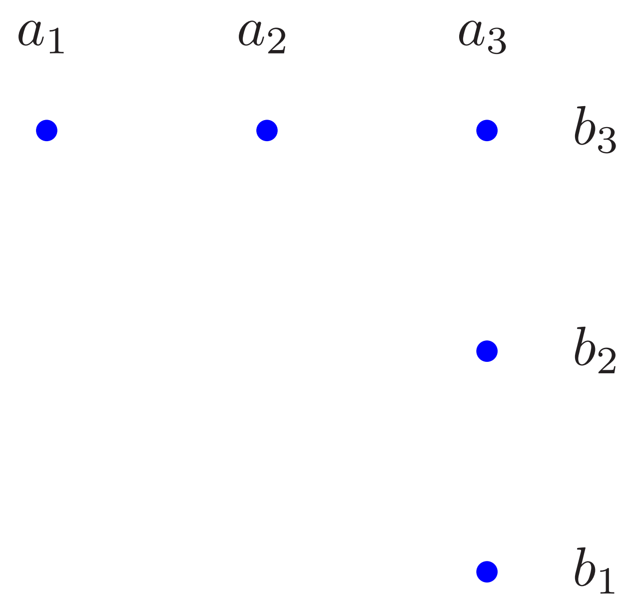
Example: $A = \mathbb{K}^P$ with $P = \{1 < 2 < 3 < \cdots\}$.

Let $\mathbf{a} := (a_1, a_2, a_3, \ldots)$, $\mathbf{b} := (b_1, b_2, b_3, \ldots)$. Then

$$\mathbf{a} * \mathbf{b} = (a_1 b_1,\ a_1 b_2 + a_2 b_2 + a_2 b_1,\\ a_1 b_3 + a_2 b_3 + a_3 b_3 + a_3 b_2 + a_3 b_1, \ldots)$$

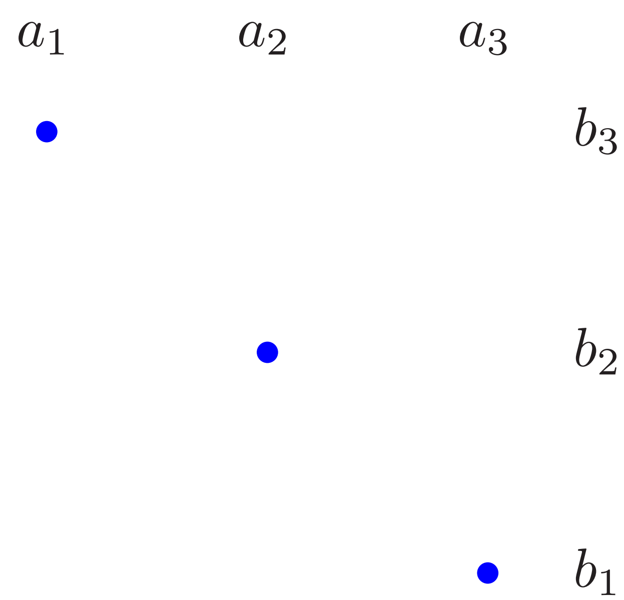
A term in $\mathbf{a} * \mathbf{b}$:

$$a_1 b_3 + a_2 b_3 + a_3 b_3 + a_3 b_2 + a_3 b_1$$



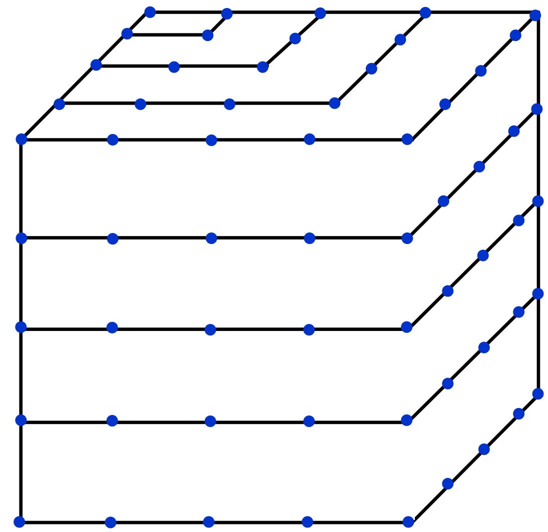
A term in the Cauchy product:

$$a_1b_3 + a_2b_2 + a_3b_1$$

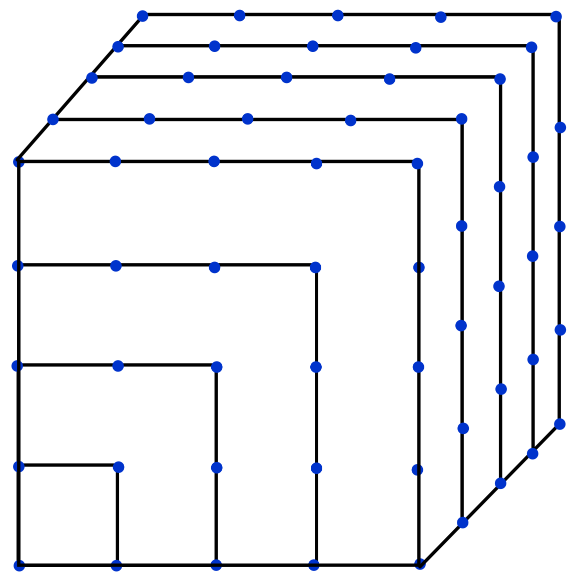


Why is $\mathbf{a} * \mathbf{b}$ associative?

$(\mathbf{a} * \mathbf{b}) * \mathbf{c}$



$\mathbf{a} * (\mathbf{b} * \mathbf{c})$



Baxter vs associative

There are two functors from Baxter algebras to associative algebras

$$\left\{ \begin{array}{c} \text{Baxter} \\ \text{algebras} \end{array} \right\} \underset{F_*}{\overset{F_\bullet}{\rightrightarrows}} \left\{ \begin{array}{c} \text{Associative} \\ \text{algebras} \end{array} \right\}$$

and a natural transformation

$$F_* \xrightarrow{\beta} F_\bullet.$$

We are interested in their left adjoints

$$\left\{ \begin{array}{c} \text{Baxter} \\ \text{algebras} \end{array} \right\} \underset{F_\bullet^\#}{\overset{F_*^\#}{\leftleftarrows}} \left\{ \begin{array}{c} \text{Associative} \\ \text{algebras} \end{array} \right\}$$

and

$$F_*^\# \xleftarrow{\beta^\#} F_\bullet^\#.$$

Other algebraic structures related to Baxter (Loday)

A dendriform algebra of weight θ is a space D with 3 operations $(\succ, \prec, \cdot)$
such that

$$(x \prec y) \prec z = x \prec (y * z) \qquad (x \succ y) \cdot z = x \succ (y \cdot z)$$
$$(x \succ y) \prec z = x \succ (y \prec z) \qquad (x \prec y) \cdot z = x \cdot (y \succ z)$$
$$(x * y) \succ z = x \succ (y \succ z) \qquad (x \cdot y) \prec z = x \cdot (y \prec z)$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

where

$$x * y := x \succ y + x \prec y - \theta\, x \cdot y\,.$$

Proposition: the operation $*$ is associative.

From Baxter to dendriform

Let (A, β) be a Baxter algebra of weight θ.
Define new operations on A:

$$a \succ b := \beta(a)b, \qquad a \prec b := a\beta(b), \qquad a \cdot b := ab.$$

Proposition: $(A, \succ, \prec, \cdot)$ is dendriform of weight θ.

Note: the associative products $*$ corresponding to (A, β) and $(A, \succ, \prec, \cdot)$ are the same.

Baxter, dendriform, associative

There are functors

We are interested in their left adjoints:

These and related constructions appear in work of Cartier, Loday and Ronco, Hazewinkel, Hoffman, Guo and Keigher, Ebrahimi-Fard and Guo, and others.

The quasishuffle product (Cartier, Ehrenborg, Fares, Hazewinkel, Hoffman,...)

Let A be an associative algebra.

Given two tensors

$$\mathbf{a} = a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}, \qquad \mathbf{b} = b_1 \otimes \cdots \otimes b_m \in A^{\otimes m},$$

and a lattice path

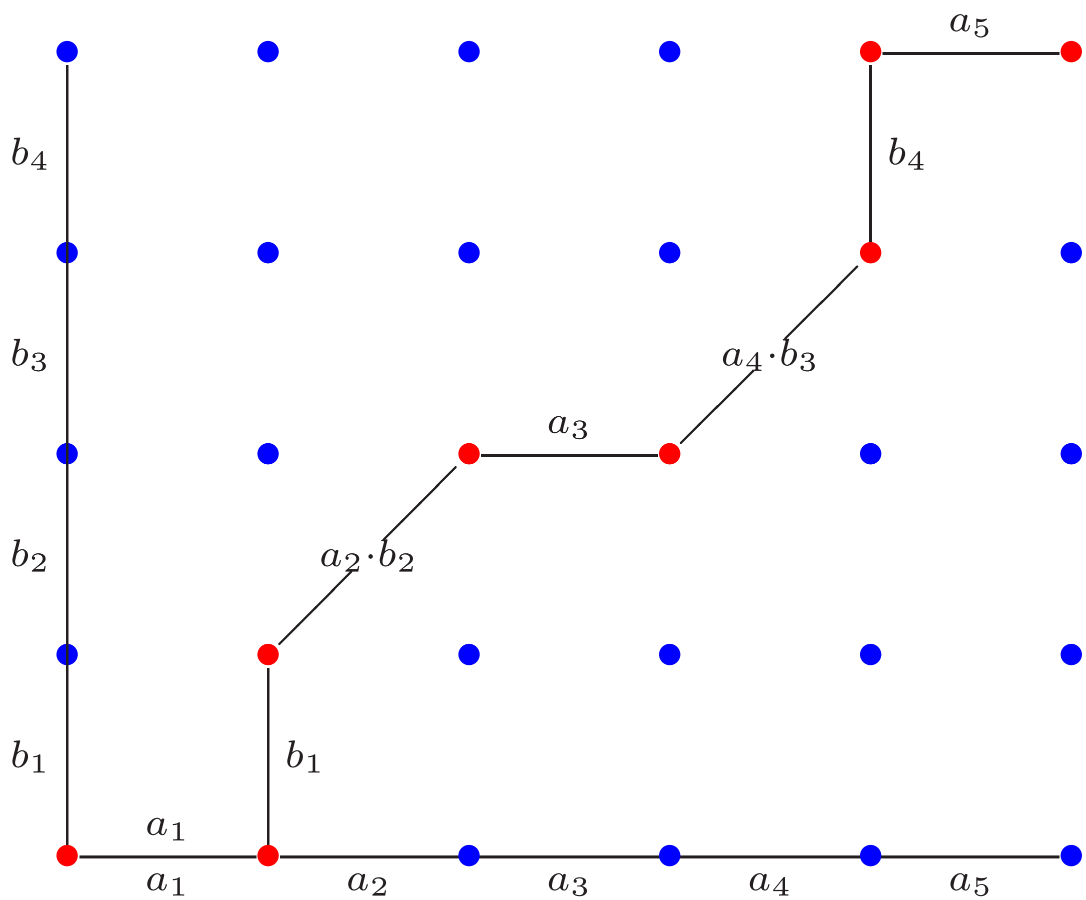
$$(0,0) \xrightarrow{L} (n,m)$$

with horizontal, vertical, or diagonal steps, define a partial product

$$\mathbf{a} *_L \mathbf{b}$$

as in the following example.

Here $n = 5$, $m = 4$, and L is shown in red:



The corresponding partial product is

$$\mathbf{a} *_L \mathbf{b} = a_1 \otimes b_1 \otimes (a_2 \cdot b_2) \otimes a_3 \otimes (a_4 \cdot b_3) \otimes b_4 \otimes a_5 \in A^{\otimes 7}.$$

On the space

$$T^{\vee}(A) := \bigoplus_{n\geq 1} A^{\otimes n},$$

define 3 operations

$$\mathbf{a} \succ \mathbf{b} := \sum_{\text{first}(L)=V} \mathbf{a} *_L \mathbf{b}, \qquad \mathbf{a} \prec \mathbf{b} := \sum_{\text{first}(L)=H} \mathbf{a} *_L \mathbf{b}, \qquad \mathbf{a} \cdot \mathbf{b} := \sum_{\text{first}(L)=D} \mathbf{a} *_L \mathbf{b},$$

so that

$$\mathbf{a} * \mathbf{b} = \mathbf{a} \succ \mathbf{b} + \mathbf{a} \prec \mathbf{b} - \theta\, \mathbf{a} \cdot \mathbf{b}$$

is the sum over all lattice paths L (the quasishuffle product).

Theorem. Let A be a commutative associative algebra. Then $(T^{\vee}(A), \succ, \prec, \cdot)$ is a commutative dendriform algebra. Moreover, the functor $T^{\vee}$ is the left adjoint of $F_{\bullet}$

$$\left\{ \begin{matrix} \text{Commutative} \\ \text{dendriform} \\ \text{algebras} \end{matrix} \right\} \underset{T^{\vee}}{\overset{F_{\bullet}}{\rightleftarrows}} \left\{ \begin{matrix} \text{Commutative} \\ \text{associative} \\ \text{algebras} \end{matrix} \right\}$$

There is a similar construction involving planar trees that yields the left adjoint

$$\left\{ \begin{array}{c} \text{Dendriform} \\ \text{algebras} \end{array} \right\} \underset{\dashleftarrow}{\xrightarrow{F_\bullet}} \left\{ \begin{array}{c} \text{Associative} \\ \text{algebras} \end{array} \right\}$$

(Loday and Ronco, Ebrahimi-Fard and Guo).