UNITAL VERSIONS OF THE HIGHER ORDER PEAK ALGEBRAS

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Abstract. We construct unital extensions of the higher order peak algebras defined by Krob and the third author in [Ann. Comb. 9 (2005), 411–430.], and show that they can be obtained as homomorphic images of certain subalgebras of the Mantaci-Reutenauer algebras of type B. This generalizes a result of Bergeron, Nyman and the first author [Trans. AMS 356 (2004), 2781–2824.].

1. Introduction

A *descent* of a permutation $\sigma \in \mathfrak{S}_n$ is an index i such that $\sigma(i) > \sigma(i+1)$. A descent is a *peak* if moreover $i > 1$ and $\sigma(i) > \sigma(i-1)$. The sums of permutations with a given descent set span a subalgebra of the group algebra, the *descent algebra* Σ_n. The *peak algebra* $\mathring{\mathcal{P}}_n$ of $\mathfrak{S}_n$ is a subalgebra of its descent algebra, spanned by sums of permutations having the same peak set. This algebra has no unit.

Descent algebras can be defined for all finite Coxeter groups [19]. In [2], it is shown that the peak algebra of $\mathfrak{S}_n$ can be naturally extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of the hyperoctahedral group B_n.

As explained in [5], it turns out that a fair amount of results on the peak algebras can be deduced from the case $q = -1$ of a q-identity of [11]. Specializing q to other roots of unity, Krob and the third author introduced and studied *higher order peak algebras* in [12]. Again, these are non-unital, and it is natural to ask whether the construction of [2] can be extended to this case.

We will show that this is indeed possible. We first construct the unital versions of the higher order peak algebras by a simple manipulation of generating series. We then show that they can be obtained as homomorphic images of the *Mantaci-Reutenauer algebras* of type B. Hence no Coxeter groups other than B_n and $\mathfrak{S}_n$ are involved in the process; in fact, the construction is related to the notion of superization, as defined in [16], rather than to root systems or wreath products.

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2. Notations and background

2.1. **Noncommutative symmetric functions.** This article is a continuation of [12]. We will assume familiarity with the notations of [9] and with the main results of [12]. We recall a few definitions for the convenience of the reader.

The Hopf algebra of noncommutative symmetric functions is denoted by **Sym**, or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet A. Linear bases of $\mathbf{Sym}_n$ are labelled by compositions $I = (i_1, \dots, i_r)$ of n (we write $I \vDash n$). The noncommutative complete and elementary functions are denoted by S_n and Λ_n, and $S^I = S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by R_I. The *descent set* of I is $\mathrm{Des}(I) = \{i_1,\ i_1 + i_2, \dots, i_1 + \cdots + i_{r-1}\}$. The *descent composition* of a permutation $\sigma \in \mathfrak{S}_n$ is the composition $I = D(\sigma)$ of n whose descent set is the descent set of σ.

Recall from [8] that for an infinite totally ordered alphabet A, $\mathbf{FQSym}(A)$ is the subalgebra of $\mathbb{C}\langle A\rangle$ spanned by the polynomials

$$\mathbf{G}_\sigma(A) = \sum_{\mathrm{std}(w)=\sigma} w, \tag{1}$$

that is, the sum of all words in A^n whose standardization is the permutation $\sigma \in \mathfrak{S}_n$. The noncommutative ribbon Schur function $R_I \in \mathbf{Sym}$ is then

$$R_I = \sum_{\mathrm{D}(\sigma)=I} \mathbf{G}_\sigma\,. \tag{2}$$

This defines a Hopf embedding $\mathbf{Sym} \to \mathbf{FQSym}$. The Hopf algebra **FQSym** is self-dual under the pairing $(\mathbf{G}_\sigma\,,\, \mathbf{G}_\tau) = \delta_{\sigma,\tau^{-1}}$ (Kronecker symbol). Let $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$, so that $\{\mathbf{F}_\sigma\}$ is the dual basis of $\{\mathbf{G}_\sigma\}$.

The *internal product* $*$ of **FQSym** is induced by composition $\circ$ in $\mathfrak{S}_n$ in the basis **F**, that is,

$$\mathbf{F}_\sigma * \mathbf{F}_\tau = \mathbf{F}_{\sigma\circ\tau} \quad \text{and} \quad \mathbf{G}_\sigma * \mathbf{G}_\tau = \mathbf{G}_{\tau\circ\sigma}\,. \tag{3}$$

Each subspace $\mathbf{Sym}_n$ is stable under this operation, and anti-isomorphic to the descent algebra Σ_n of $\mathfrak{S}_n$.

For $f_i \in \mathbf{FQSym}$ and $g \in \mathbf{Sym}$, we have the splitting formula

$$(f_1 \dots f_r) * g = \mu_r \cdot (f_1 \otimes \cdots \otimes f_r) *_r \Delta^r g\,, \tag{4}$$

where μ_r is r-fold multiplication, and Δ^r the iterated coproduct with values in the r-th tensor power.

2.2. **The Mantaci-Reutenauer algebra.** We denote by **MR** the free product $\mathbf{Sym} \star \mathbf{Sym}$ of two copies of the Hopf algebra of noncommutative symmetric functions [14]. That is, **MR** is the free associative algebra on two sequences (S_n) and $(S_{\bar n})$ $(n \geq 1)$. We regard the two copies of **Sym** as noncommutative symmetric functions on two auxiliary alphabets: $S_n = S_n(A)$ and $S_{\bar n} = S_n(\bar A)$. We denote by $F \mapsto \bar F$ the involutive automorphism which exchanges S_n and $S_{\bar n}$. The bialgebra structure is defined by the requirement that the series

$$\sigma_1 = \sum_{n\geq 0} S_n \text{ and } \bar\sigma_1 = \sum_{n\geq 0} S_{\bar n} \tag{5}$$

are grouplike.

The internal product of **MR** can be computed from the splitting formula and the conditions that σ_1 is neutral, $\bar\sigma_1$ is central, and $\bar\sigma_1 * \bar\sigma_1 = \sigma_1$.

In [15], an embedding of **MR** in the Hopf algebra **BFQSym** of free quasi-symmetric functions of type B (spanned by colored permutations) is described. Under this embedding, left $*$-multiplication by $\Lambda_n = \mathbf{G}_{n\,n-1\dots 2,1}$ corresponds to right multiplication by $n\,n-1\dots 2,1$ in the group algebra of B_n. This implies that left $*$-multiplication by λ_1 is an involutive anti-automorphism of **BFQSym**, hence of **MR**.

2.3. **Noncommutative symmetric functions of type** B**.** The hyperoctahedral analogue **BSym** of **Sym**, defined in [6], is the right **Sym**-module freely generated by another sequence $(\tilde S_n)$ $(n \geq 0,\ \tilde S_0 = 1)$ of homogeneous elements, with $\tilde\sigma_1$ grouplike. This is a coalgebra, but not an algebra. It is endowed with an internal product, for which each homogeneous component $\mathbf{BSym}_n$ is anti-isomorphic to the descent algebra of B_n.

2.4. **Descents in** B_n**.** The hyperoctahedral group B_n is the group of signed permutations. A signed permutation can be denoted by $w = (\sigma, \epsilon)$ where σ is an ordinary permutation and $\epsilon \in \{\pm 1\}^n$, such that $w(i) = \epsilon_i \sigma(i)$. If we set $w(0) = 0$, then, $i \in [0, n-1]$ is a descent of w if $w(i) > w(i+1)$. Hence, the descent set of w is a subset $D = \{i_0, i_0 + i_1, \dots, i_0 + i_1 + \cdots i_{r-1}\}$ of $[0, n-1]$. We then associate to D a so-called type-B composition (a composition whose first part can be zero) $(i_0 - 0, i_1, \dots, i_{r-1}, n - i_{r-1})$.

For example, if one encodes ϵ as a boolean vector for readability, the signed permutation $w = (231546, 100100)$ has as type-B composition $I = (0, 2, 1, 3)$. The signed permutation $w = (231546, 000100)$ has as type-B composition $I = (2, 3, 1)$.

The sum of all signed permutations whose descent set is contained in D is mapped to $\tilde S^I := \tilde S_{i_0} S^{I'}$ by Chow's anti-isomorphism [6], where $I' = (i_1, \dots, i_r)$.

3. Embedding noncommutative symmetric functions of type B into the Mantaci-Reutenauer algebra

3.1. An embedding of **BSym** as a sub-coalgebra and sub-**Sym**-module of **MR** can be deduced from [14]. To describe it, let us define, for $F \in \mathbf{Sym}(A)$,

$$F^\sharp = F(A|\bar A) = F(A - q\bar A)|_{q=-1} \qquad (6)$$

(the supersymmetric version of F). The superization of $F \in \mathbf{Sym}(A)$ can also be given by

$$F^\sharp = F * \sigma_1^\sharp\,. \qquad (7)$$

Indeed, $\sigma_1^\sharp$ is grouplike, and for $F = S^I$, the splitting formula gives

$$(S_{i_1} \cdots S_{i_r}) * \sigma_1^\sharp = \mu_r[(S_{i_1} \otimes \cdots \otimes S_{i_r}) * (\sigma_1^\sharp \otimes \cdots \otimes \sigma_1^\sharp)] = S^{I\sharp}\,. \qquad (8)$$

We have

$$\sigma_1^\sharp = \bar\lambda_1 \sigma_1 = \sum \Lambda_{\bar i} S_j\,. \qquad (9)$$

The element $\bar\sigma_1$ is central for the internal product, and

$$\bar\sigma_1 * F = \bar F = F * \bar\sigma_1 \,. \tag{10}$$

Hence,

$$\bar\sigma_1 * \sigma_1^\sharp = \lambda_1\bar\sigma_1 =: \sigma_1^\flat \,. \tag{11}$$

The basis element $\tilde S^I$ of **BSym**, where $I = (i_0, i_1, \ldots, i_r)$ is a type B-composition, can be embedded as

$$\tilde S^I = S_{i_0}(A) S^{i_1 i_2 \cdots i_r}(A|\bar A) \,. \tag{12}$$

We will identify **BSym** with its image under this embedding.

3.2. A proof that BSym is $*$-stable. We are now in a position to understand why **BSym** is a $*$-subalgebra of **MR**. The argument will be extended below to the case of unital peak algebras.

Let $F, G \in$ **Sym**. We want to understand why $\sigma_1 F^\sharp * \sigma_1 G^\sharp$ is in **BSym**. Using the splitting formula, we rewrite this as

$$\mu[(\sigma_1 \otimes F^\sharp) * \Delta\sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G^\sharp_{(1)})(F^\sharp * \sigma_1 G^\sharp_{(2)}). \tag{13}$$

We now only have to show that each term $F^\sharp * \sigma_1 G^\sharp_{(2)}$ is in $\mathbf{Sym}^\sharp$. We may assume that $F = S^I$, and for any $G \in$ **Sym**,

$$S^{I\sharp} * \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S^\sharp_{i_1} \otimes \cdots \otimes S^\sharp_{i_r}) * (\sigma_1 G^\sharp_{(1)} \otimes \cdots \otimes \sigma_1 G^\sharp_{(r)})] \tag{14}$$

so that it is sufficient to prove the property for $F = S_n$. Now,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= (\bar\lambda_1 \sigma_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\bar\lambda_1 * \sigma_1 G^\sharp_{(1)})(\sigma_1 G^\sharp_{(2)}) \\ &= \sum_{(G)} (\bar\sigma_1 * \lambda_1 * \sigma_1 G^\sharp_{(1)}) \cdot \sigma_1 \cdot G^\sharp_{(2)} \end{aligned} \tag{15}$$

Now,

$$\lambda_1 * \sigma_1 G^\sharp_{(1)} = (\lambda_1 * G^\sharp_{(1)})(\lambda_1 * \sigma_1) = (\lambda_1 * G^\sharp_{(1)})\lambda_1, \tag{16}$$

since λ_1 is an anti-automorphism. We then get

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= \sum_{(G)} (\bar\sigma_1 * ((\lambda_1 * G^\sharp_{(1)})\lambda_1) \cdot \sigma_1 \cdot G^\sharp_{(2)} \\ &= \sum_{(G)} (\bar\sigma_1 * \lambda_1 * G^\sharp_{(1)}) \cdot (\bar\sigma_1 * \lambda_1)\sigma_1 \cdot G^\sharp_{(2)} \\ &= \sum_{(G)} (\bar\lambda_1 * G^\sharp_{(1)}) \cdot \sigma_1^\sharp \cdot G^\sharp_{(2)} \end{aligned} \tag{17}$$

Now, the result will follow if we can prove that $\bar\lambda_1 * G^\sharp$ is in $\mathbf{Sym}^\sharp$ for any $G \in$ **Sym**.

For $G = S^I$,

$$\bar\lambda_1 * S^{I\sharp} = \lambda_1 * \bar\sigma_1 * S^I * \sigma_1^\sharp = \lambda_1 * S^I * \bar\sigma_1 * \sigma_1^\sharp = \lambda_1 * S^I * \sigma_1^\flat\,. \tag{18}$$

Since left $*$-multiplication by λ_1 in an anti-automorphism, we only need to prove that $\bar\lambda_1 * S_n^\flat$ is of the form $G^\sharp$. And indeed,

$$\begin{aligned}\bar\lambda_1 * S_n^\flat &= \sum_{i+j=n} \lambda_1 * (\Lambda_i S_{\bar j}) \\ &= \sum_{i+j=n} (\lambda_1 * S_{\bar j})(\lambda_1 * \Lambda_i) \\ &= \sum_{i+j=n} \Lambda_{\bar j} S_i = S_n^\sharp\,.\end{aligned} \tag{19}$$

This concludes the proof that **BSym** is a $*$-subalgebra of **BFQSym**.

4. Unital versions of the higher order peak algebras

4.1. As shown in [5], much of the theory of the peak algebra can be deduced from a formula of [11] for $R_I((1-q)A)$, in the special case $q=-1$. In [12], this formula was studied in the case where q is an arbitrary root of unity, and higher order analogs of the peak algebra were obtained. In [2], it was shown that the classical peak algebra can be extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of type B.

In this section, we construct unital extensions of the higher order peak algebras.

4.2. Let q be a primitive r-th root of unity. All objects introduced below will depend on q (and r), although this dependence will not be made explicit in the notation.

`We should address the following question: do different primitive` r`-th roots of unity lead to the same (or at least, isomorphic) algebras? Presumably the answer is yes, so that` $\mathring{\mathcal{P}}$ `and` $\mathcal{P}$ `depend only on` r`. In this case, perhaps we could use the notation` $\mathring{\mathcal{P}}^{(r)}$ `instead of` $\mathring{\mathcal{P}}$`, etc.`

We denote by θ_q the endomorphism of **Sym** defined by

$$\tilde f = \theta_q(f) = f((1-q)A) = f(A) * \sigma_1((1-q)A)\,. \tag{20}$$

We denote by $\mathring{\mathcal{P}}$ the image of θ_q and by $\mathcal{P}$ the right $\mathring{\mathcal{P}}$-module generated by the S_n for $n \ge 0$. Note that $\mathring{\mathcal{P}}$ is by definition a left $*$-ideal of **Sym**.

Theorem 4.1. *$\mathcal{P}$ is a unital $*$-subalgebra of* **Sym**. *Its Hilbert series is*

$$\sum_{n\ge 0} \dim \mathcal{P}_n t^n = \frac{1}{1-t-t^2-\cdots-t^r}\,. \tag{21}$$

Proof – Since the internal product of homogeneous elements of different degrees is zero, it is enough to show that, for any $f, g \in$ **Sym**, $\sigma_1 \tilde f * \sigma_1 \tilde g$ is in $\mathcal{P}$. Thanks to the splitting formula,

$$\begin{aligned}\sigma_1\tilde{f} * \sigma_1\tilde{g} &= \mu[(\sigma_1 \otimes \tilde{f}) * \sum_{(g)} \sigma_1\tilde{g}_{(1)} \otimes \sigma_1\tilde{g}_{(2)}] \\ &= \sum_{(g)} (\sigma_1\tilde{g}_{(1)})(\tilde{f} * \sigma_1\tilde{g}_{(2)})\,.\end{aligned} \tag{22}$$

Thus, it is enough to check that $\tilde{f} * \sigma_1\tilde{h}$ is in $\mathring{\mathcal{P}}$ for any $f, h \in \mathbf{Sym}$. Now,

$$\tilde{f} * \sigma_1\tilde{h} = f * \sigma_1((1-q)A) * \sigma_1\tilde{h}\,, \tag{23}$$

and since $\mathring{\mathcal{P}}$ is a **Sym** left $*$-ideal, we only have to show that $\sigma_1((1-q)A) * \sigma_1\tilde{h}$ is in $\mathring{\mathcal{P}}$. One more splitting yields

$$\begin{aligned}\sigma_1((1-q)A) * \sigma_1\tilde{h} &= (\lambda_{-q}\sigma_1) * \sigma_1\tilde{h} \\ &= \mu[(\lambda_{-q} \otimes \sigma_1) * \sum_{(h)} \sigma_1\tilde{h}_{(1)} \otimes \sigma_1\tilde{h}_{(2)}] \\ &= \sum_{(h)} (\lambda_{-q} * \sigma_1\tilde{h}_{(1)})(\sigma_1\tilde{h}_{(2)}) \\ &= \sum_{(h)} (\lambda_{-q} * \tilde{h}_{(1)})\lambda_{-q}\sigma_1\tilde{h}_{(2)}\end{aligned} \tag{24}$$

(since left $*$-multiplication by λ_{-q} is an anti-automorphism, namely the composition of the antipode and q^{degree}). The first parentheses $(\lambda_{-q} * \tilde{h}_{(1)})$ are in $\mathring{\mathcal{P}}$ since it is a left $*$-ideal. The middle term is $\sigma_1((1-q)A)$, and the last one is in $\mathring{\mathcal{P}}$ by definition.

Recall from [12, Prop. 3.5] that the Hilbert series of $\mathring{\mathcal{P}}$ is

$$\sum_{n\geq 0} \dim \mathring{\mathcal{P}}_n t^n = \frac{1-t^r}{1-t-t^2-\ldots-t^r}\,. \tag{25}$$

From [12, Lemma 3.13 and Eq. (3.9)], it follows that $S_n \in \mathring{\mathcal{P}}$ if and only if $n \equiv 0$ mod r, so that the Hilbert series of $\mathcal{P}$ is

$$\sum_{n\geq 0} \dim \mathcal{P}_n t^n = \frac{1}{1-t-t^2-\ldots-t^r}\,. \tag{26}$$

■

5. Subalgebras of the Mantaci-Reutenauer algebra

The above proofs are in fact special cases of a master calculation in the Mantaci-Reutenauer algebra, which we carry out in this section.

5.1. The $\sharp$ transform. Let q be an arbitrary complex number or an indeterminate, and define, for any $F \in \mathbf{MR}$,

$$F^\sharp = F * \sigma_1(A - q\bar{A}) = F * \sigma_1^\sharp\,. \tag{27}$$

Since $\sigma_1^\sharp$ is grouplike, it follows from the splitting formula that

$$F \mapsto F^\sharp$$

is an endomorphism of $\mathbf{MR}$ for the Hopf structure. In addition, it is clear from the definition that it is also a endomorphism of left $*$-modules. We refer to it as the $\sharp$ *transform.*

5.2. Definition of the subalgebras. We define

$$\mathring{\mathcal{Q}} = \mathbf{MR}^\sharp,$$

the image of the $\sharp$ transform. Since the latter is an endomorphism of Hopf algebras and of left $*$-modules, $\mathring{\mathcal{Q}}$ is both a Hopf subalgebra of $\mathbf{MR}$ and a left $*$-ideal.

When q is a root of unity, its image under the specialization $\bar{A} = A$ is the non-unital peak algebra $\mathring{\mathcal{P}}$ of Section 4.2 (and for generic q, it is $\mathbf{Sym}$).

Let $\mathcal{Q}$ be the right $\mathring{\mathcal{Q}}$-module generated by the S_n, for all $n \geq 0$. Clearly, the identification $\bar{A} = A$ maps $\mathcal{Q}$ to $\mathcal{P}$, the unital peak algebra of Section 4.2.

`I suppose this maps` $\mathcal{Q}$ *`onto`* $\mathcal{P}$`.  If so, we should say it.`

Theorem 5.1. *$\mathcal{Q}$ is a $*$-subalgebra of $\mathbf{MR}$, containing $\mathring{\mathcal{Q}}$ as a left ideal.*

Proof – Let $F, G \in \mathbf{MR}$. As above, we want to show that $\sigma_1 F^\sharp * \sigma_1 G^\sharp$ is in $\mathcal{Q}$. Using the splitting formula, we rewrite this as

$$\mu[(\sigma_1 \otimes F^\sharp) * \Delta\sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G^\sharp_{(1)})(F^\sharp * \sigma_1 G^\sharp_{(2)}) \tag{28}$$

and we only have to show that each term $F^\sharp * \sigma_1 G^\sharp_{(2)}$ is in $\mathring{\mathcal{Q}}$. We may assume that $F = S^I$, where I is now a bicolored composition, and for any $G \in \mathbf{MR}$,

$$S^{I\sharp} * \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S^\sharp_{i_1} \otimes \cdots \otimes S^\sharp_{i_r}) * (\sigma_1 G^\sharp_{(1)} \otimes \cdots \otimes \sigma_1 G^\sharp_{(r)})] \tag{29}$$

so that it is sufficient to prove the property for $F = S_n$ or $S_{\bar{n}}$. Now,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= (\bar{\lambda}_{-q}\sigma_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_{-q} 1 * \sigma_1 G^\sharp_{(1)})(\sigma_1 G^\sharp_{(2)}) \\ &= \sum_{(G)} (\bar{\lambda}_{-q} * G^\sharp_{(1)}) \cdot \sigma_1^\sharp \cdot G^\sharp_{(2)} \end{aligned} \tag{30}$$

which is in $\mathring{\mathcal{Q}}$, since it is a subalgebra and a left $*$-ideal, and similarly,

$$
\begin{aligned}
\bar\sigma_1^\sharp * \sigma_1 G^\sharp &= (\lambda_{-q}\bar\sigma_1) * \sigma_1 G^\sharp \\
&= \sum_{(G)} (\lambda_{-q} * \sigma_1 G^\sharp_{(1)})(\bar\sigma_1 \bar G^\sharp_{(2)}) \\
&= \sum_{(G)} (\lambda_{-q} * G^\sharp_{(1)}) \cdot \bar\sigma_1^\sharp \cdot \bar G^\sharp_{(2)}
\end{aligned}
\tag{31}
$$

is also in $\mathring{\mathcal{Q}}$. ■

The various algebras introduced in this paper and their interrelationships are summarized in the following diagram.

$$
\begin{array}{ccccccc}
\mathring{\mathcal{Q}} & \subseteq & \mathcal{Q} & \subseteq & \mathbf{MR} & \subseteq & \mathbf{BFQSym} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathring{\mathcal{P}} & \subseteq & \mathcal{P} & \subseteq & \mathbf{Sym} & \subseteq & \mathbf{FQSym}
\end{array}
$$

Note that in the special case $q = -1$, $\mathcal{Q}_n$ is the descent algebra of B_n, $\mathcal{Q}$ is **BSym**, and $\mathcal{P}$ is the unital peak algebra of [2].

6. Further developments

6.1. Inversion of the generic $\sharp$ transform. For generic q, the endomorphism (27) of **MR** is invertible; therefore

$$\mathring{\mathcal{Q}} = \mathbf{MR}.$$

The inverse endomorphism of **MR** arises from the transformation of alphabets

$$A \mapsto (q\bar A + A)/(1-q^2),$$

which is to be understood in the following sense:

$$\sigma_1\left(\frac{q\bar A + A}{1-q^2}\right) := \prod_{k\geq 0} \sigma_{q^{2k+1}}(\bar A)\sigma_{q^{2k}}(A)\,. \tag{32}$$

Indeed,

$$
\begin{aligned}
\sigma_1\left(\frac{q\bar A + A}{1-q^2}\right) * \sigma_1(A - q\bar A) &= \prod_{k\geq 0} \sigma_{q^{2k+1}}(\bar A - qA)\sigma_{q^{2k}}(A - q\bar A) \\
&= \prod_{k\geq 0} \lambda_{-q^{2k+2}}(A)\sigma_{q^{2k+1}}(\bar A)\lambda_{-q^{2k+1}}(\bar A)\sigma_{q^{2k}}(A) \\
&= \sigma_1(A)\,.
\end{aligned}
\tag{33}
$$

By normalizing the term of degree n in (32), we obtain B_n-analogs of the q-Klyachko elements defined in [9]:

$$K_n(q; A, \bar A) := \prod_{i=1}^{n} (1 - q^{2\,i}) S_n\left(\frac{q\bar A + A}{1-q^2}\right) = \sum_{I \vDash n} q^{2\,\mathrm{maj}(I)} R_I(q\bar A + A)\,. \tag{34}$$

This expression can be completely expanded on signed ribbons. From the expression of R_I in **FQSym**, we have

$$R_I(\bar{A}+A) = \sum_{C(\sigma)=I} \mathbf{G}_\sigma(\bar{A}+A) \tag{35}$$

where $\bar{A}+A$ is the ordinal sum. If we order $\bar{A}$ by

$$\bar{a}_1 < \bar{a}_2 < \ldots < \bar{a}_k < \ldots \tag{36}$$

then, arguing as in [16], we have

$$\mathbf{G}_\sigma(\bar{A}+A) = \sum_{\mathrm{std}(\tau,\epsilon)=\sigma} \mathbf{G}_{\tau,\epsilon} \tag{37}$$

so that

$$R_I(\bar{A}+A) = \sum_{\rho(\mathrm{J})=I} R_{\mathrm{J}} \tag{38}$$

where for a signed composition $\mathrm{J} = (J,\epsilon)$, the unsigned composition $\rho(\mathrm{J})$ is defined as the shape of $\mathrm{std}(\sigma,\epsilon)$, where σ is any permutation of shape J.

6.2. Replacing $\bar{A}$ by $q\bar{A}$, one obtains the expansion of the q-Klyachko elements of type B:

$$K_n(q;A,\bar{A}) = \sum_{\mathrm{J}} q^{\mathrm{bmaj}(\mathrm{J})} R_{\mathrm{J}} \tag{39}$$

where

$$\mathrm{bmaj}(\mathrm{J}) = 2\,\mathrm{maj}(\rho(\mathrm{J})) + |\epsilon|\,, \tag{40}$$

where $|\epsilon|$ is the number of minus signs in ϵ.

For example,

$$K_2(q) = R_2 + q^2\,R_{\bar{2}} + q^2\,R_{11} + q^3\,R_{1\bar{1}} + q\,R_{\bar{1}1} + q^4\,R_{\overline{11}}. \tag{41}$$

$$\begin{aligned} K_3(q) &= R_3 + q^3\,R_{\bar{3}} + q^4\,R_{21} + q^5\,R_{2\bar{1}} + q^2\,R_{\bar{2}1} + q^7\,R_{\overline{21}} + q^2\,R_{12} + q^4\,R_{1\bar{2}} \\ &\quad + q\,R_{\bar{1}2} + q^5\,R_{\overline{12}} + q^6\,R_{111} + q^7\,R_{11\bar{1}} + q^3\,R_{1\bar{1}1} + q^8\,R_{1\overline{11}} \\ &\quad + q^5\,R_{\bar{1}11} + q^6\,R_{\bar{1}1\bar{1}} + q^4\,R_{\overline{11}1} + q^9\,R_{\overline{111}}. \end{aligned} \tag{42}$$

This major index of type B is the flag major index defined in [1]. Following [1] and considering the signed composition (where ϵ is encoded as boolean vector for readability)

$$\mathrm{J} = (2,1,1,\bar{3},\bar{1},\bar{2},4,\bar{1},2,2) = (2113124122, 0000111111000010000 0) \tag{43}$$

we can take the smallest permutation of shape $(2,1,1,3,1,2,4,1,2,2)$, which is

$$\alpha = 1\,5\,4\,3\,2\,6\,9\,8\,7\,11\,10\,12\,13\,16\,15\,14\,18\,17\,19 \tag{44}$$

sign it according to ϵ, which yields

$$1\,5\,4\,3\,\bar{2}\,\bar{6}\,\bar{9}\,\bar{8}\,\bar{7}\,\overline{11}\,10\,12\,13\,16\,\overline{15}\,14\,18\,17\,19 \tag{45}$$

whose standardized is

$$8\,11\,10\,9\,1\,2\,5\,4\,3\,6\,12\,13\,14\,16\,7\,15\,18\,17\,19 \tag{46}$$

and has shape $\rho(\mathrm{J}) = (2, 1, 1, 3, 1, 6, 3, 2)$. The major index of $\rho(\mathrm{J})$ is 55, the number of minus signs in ϵ is 7, so $\mathrm{bmaj(J)} = 2 \times 55 + 7 = 117$.

6.3. The major index of type B can be read directly on signed compositions without reference to signed permutations as follows: one can get $\rho(\mathrm{J})$ by first adding the absolute values of two consecutive parts if the left one is signed and the second one is not, then remove the signs and proceed as before.

A different solution consists in reading the composition from right to left, then associate weight 0 (resp. 1) to the rightmost part if it is positive (resp. negative) and then proceed left by adding 2 to the weight if the two parts are of the same sign and 1 if not. Finally, add up the product of the absolute values of the parts with their weight.

For example, with the same J as above we have the following weights:

$$\begin{aligned} \mathrm{J} =&(2, 1, 1,\ \bar{3}, \bar{1}, \bar{2}, 4, \bar{1}, 2, 2) \\ \text{weights :}&14\;12\;10\;\;9\;\;7\;\;5\;\;4\;\;3\;\;2\;\;0 \end{aligned} \tag{47}$$

so that we get $2\cdot14+1\cdot12+1\cdot10+3\cdot9+1\cdot7+2\cdot5+4\cdot4+1\cdot3+2\cdot2+2\cdot0 = 117$.

This technique generalizes immediately to colored compositions with a fixed number c of colors $0, 1, \ldots, c-1$: the weight of the rightmost cell is its color and the weight of a part is equal to the sum of the weight of the next part and the unique representative of the difference of the colors of those parts modulo c belonging to the interval $[1, c]$.

6.4. Generators and Hilbert series. For $n \geq 0$, let

$$S_n^{\pm} = S_n(A) \pm S_n(\bar{A}), \tag{48}$$

and denote by $\mathcal{H}_n$ the subalgebra of $\mathbf{MR}$ generated by the $S_k^{\pm}$ for $k \leq n$. For $n \geq 0$, we have

$$(S_n^{\pm})^{\sharp} \equiv (1 \mp q^n) S_n^{\pm} \mod \mathcal{H}_{n-1}, \tag{49}$$

so that the $(S_n^{\pm})^{\sharp}$ such that $1 \mp q^n \neq 0$ form a set of free generators in $\mathbf{MR}^{\sharp}$.

Conjecture 6.1. *If r is odd, a basis of $\mathbf{MR}^{\sharp}$ will be parametrized by colored compositions such that parts of color 0 are not $\equiv 0 \mod r$ and parts of color 1 are arbitrary. The Hilbert series is then*

$$H_r(t) = \frac{1 - t^r}{1 - 2(t + t^2 + \cdots + t^r)}. \tag{50}$$

If r is even, there is the extra condition that parts of color 1 are not $\equiv r/2 \mod r$. The Hilbert series is then

$$H_r(t) = \frac{1 - t^r}{1 - 2(t + t^2 + \cdots + t^r) + t^{r/2}}. \tag{51}$$

For example,

$$H_2(t) = 1 + t + 2\,t^2 + 4\,t^3 + 8\,t^4 + 16\,t^5 + 32\,t^6 + 64\,t^7 + 128\,t^8 + O\left(t^9\right) \tag{52}$$

$$H_3(t) = 1 + 2\,t + 6\,t^2 + 17\,t^3 + 50\,t^4 + 146\,t^5 + 426\,t^6 + 1244\,t^7 + 3632\,t^8 + O\left(t^9\right) \tag{53}$$

$$H_4(t) = 1 + 2\,t + 5\,t^2 + 14\,t^3 + 38\,t^4 + 104\,t^5 + 284\,t^6 + 776\,t^7 + 2120\,t^8 + O\left(t^9\right) \tag{54}$$

If these conjectures are correct, the Hilbert series of the right $\mathbf{MR}^\sharp$-modules generated by the S_n are respectively

$$\frac{1}{1 - 2(t + t^2 + \ldots + t^r)}\,, \tag{55}$$

or

$$\frac{1}{1 - 2(t + t^2 + \ldots + t^r) + t^{r/2}}\,. \tag{56}$$

according to whether r is odd or even.

The cases $r = 1$ and $r = 2$ are easily proved as follows. Assume first that $q = 1$. Set

$$f = 1 + (\sigma_1^+)^\sharp = (\sigma_1 + \lambda_{-1})(A - \bar{A})\,, \tag{57}$$

$$g = (\sigma_1^-)^\sharp - 1 = (\sigma_1 - \lambda_{-1})(A - \bar{A})\,. \tag{58}$$

Then, $f^2 = g^2 + 4$, so that

$$f = 2\left(1 + \frac{1}{4}g^2\right)^{\frac{1}{2}} \tag{59}$$

which proves that the $(S_n^+)^\sharp$ can be expressed in terms of the $(S_m^-)^\sharp$.

Similarly, for $q = -1$, one can express

$$f = \sum_{n\geq 1}(S_{2n}^+)^\sharp + \sum_{n\geq 0}(S_{2n+1}^-)^\sharp \tag{60}$$

in terms of

$$g = \sum_{n\geq 1}(S_{2n}^-)^\sharp + \sum_{n\geq 0}(S_{2n+1}^+)^\sharp \tag{61}$$

since, as is easily verified,

$$(f+2)^2 = g^2 + 4\,,\ \ i.e.,\ f = -2 + 2\left(1 + \frac{1}{4}g^2\right)^{\frac{1}{2}}\,. \tag{62}$$

Apparently, this approach does not work anymore for higher roots of unity.

7. Appendix: monomial expansion of the $(1-q)$-kernel

The results of [16, 7] allow us to write down a new expansion of $S_n((1-q)A)$, in terms of the monomial basis of [4]. The special case $q = 1$ gives back a curious expression of Dynkin's idempotent, first obtained in [3].

Let σ be a permutation. We then define its *left-right minima* set $\mathrm{LR}(\sigma)$ as the values of σ that have no smaller value to their left. We will denote by $\mathrm{lr}(\sigma)$ the cardinality of $\mathrm{LR}(\sigma)$. For example, with $\sigma = 46735182$, we have $\mathrm{LR}(\sigma) = \{4, 3, 1\}$, and $\mathrm{lr}(\sigma) = 3$.

Let us now compute how $S_n((1-q)A)$ decomposes on the monomial basis $\mathbf{M}_\sigma$ (see [4]) of **FQSym**. Thanks to the Cauchy formula of **FQSym** [7], we have

$$S_n((1-q)A) = \sum_{\sigma} \mathbf{S}^\sigma(1-q)\mathbf{M}_\sigma(A), \tag{63}$$

where $\mathbf{S}$ is the dual basis of $\mathbf{M}$. Given the transition matrix between $\mathbf{M}$ and $\mathbf{G}$, we immediately deduce that

$$\mathbf{S}^\sigma = \sum_{\tau<\sigma^{-1}} \mathbf{F}_\tau, \tag{64}$$

where $<$ stands for the right weak order on permutations, so that, for example.

$$\mathbf{S}^{312} = \mathbf{F}_{123} + \mathbf{F}_{213} + \mathbf{F}_{231}. \tag{65}$$

Thanks to [16], we know that $\mathbf{F}_\sigma(1-q)$ is either $(-q)^k$ if $\mathrm{Des}(\sigma) = \{1, \ldots, k\}$ or 0 otherwise. Let us define *hook permutations* of hook k the permutations σ such that $\mathrm{Des}(\sigma) = \{1, \ldots, k\}$. Now, $\mathbf{S}^\sigma(1-q)$ amounts to compute the list of *hook permutations* smaller than σ. Note that hook permutations are completely characterized by their left-right minima. Moreover, if τ is smaller than σ in the right weak order, then $\mathrm{LR}(\tau) \subset \mathrm{LR}(\sigma)$.

Hence all hook permutations smaller than a given permutation σ belong to the set of hook permutations with left-right minima in $\mathrm{LR}(\sigma)$. Since by elementary transpositions decreasing the length, one can get from σ to the hook permutation with the same left-right minima and then from this permutation to all the others, we have:

Theorem 7.1. *Let n be an integer. Then*

$$S_n((1-q)A) = \sum_{\sigma\in\mathfrak{S}_n} (1-q)^{\mathrm{lr}(\sigma)}\mathbf{M}_\sigma. \tag{66}$$

■

In the particular case $q = 1$, we recover a result of [3]:

$$\Psi_n = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(1)=1}} \mathbf{M}_\sigma, \tag{67}$$

where Ψ_n is the *first Eulerian idempotent* [, COMPLETE].

`Refer to NCSF where an expression connecting (67) to (66) is given.`

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