Math 128a, Homework 9

due November 13.

1. Write a program to explore the sensitivity of roots of polynomials to perturbations in their coefficients. The program should take as inputs a vector r of roots of the initial polynomial, a maximum relative perturbation e of the coefficients, and the total number m of random polynomials to be generated. Starting with r, the program should construct the polynomial using `poly`, perturb its coefficients by a random *relative* amount between $-e$ and e using `rand` (remember to allow negative perturbations), and calculate the perturbed roots using `roots`. Plot the roots of m such polynomials, for m a few hundred, on a single plot, and use this to estimate the relative error in each root specified below. You will probably need to change the scale of each plot a number of times, using `axis`. Compare with the theoretical prediction of equations (5.8.1) and (5.8.3) of the textbook. Here the function `polyder` might be helpful.

Try the following examples with $e = 10^{-4}$.

(a) $r = [1, 2, 3]$, root 1.
(b) $r = [1, 1.1, 3]$, roots $1, 3$.
(c) $r = [1, 2, 2, 3, 3, 3]$, roots $1, 2, 3$.
(d) $r = [1 : 10]$, roots $1, 4, 7, 10$.
(e) $r = [2, 4, 8, \ldots, 1024]$, roots $2, 16, 256, 1024$.
(f) A polynomial of your choice with a pair of complex conjugate roots.

Discuss your observations, both about absolute perturbations and about their relation to the theoretical prediction.

2. A sequence $\{x_n\}$ is *superlinearly convergent* to ξ if

$$\lim_{n\to\infty} \frac{|x_{n+1} - \xi|}{|x_n - \xi|} = 0.$$

(a) Show that if $x_n \to \xi$ of order $p > 1$, then $\{x_n\}$ is superlinearly convergent to ξ.
(b) Show that $x_n = 1/n^n$ is superlinearly convergent to 0 but does not converge to 0 of order p for any $p > 1$.
(c) Suppose that $\{x_n\}$ is superlinearly convergent to ξ. Show that

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|x_n - \xi|} = 1.$$

Hint: use the triangle inequality twice.

3. Let $\Phi(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x - \frac{1}{12}$. Plot Φ. Prove that Φ has three fixed points. For each of the fixed points, explain if Φ is locally convergent at it, and if it is, of what order. Now let Ψ be Steffensen's transformation of Φ. Plot Ψ. Check numerically that Ψ is locally convergent at each of the fixed points of Φ (use the graph to choose good starting values), and that the convergence is quadratic.