

Some further topics following Math 10B

1. In Green's theorem, we talked about integration over a closed curve C and also over its interior D . But how do we know each curve divides the plane into the interior and exterior regions? Maybe one can construct a weird curve which gives only one region (the way the Möbius band only has one side)? Or maybe another curve divides the plane into three regions? If you think these are ridiculous questions, take a look at the picture at the very end of the article in the next link. Does this curve have an interior and an exterior? Which of the points are inside, which outside?

The fact that every simple closed curve does have an inside and an outside is important enough and hard enough to prove that it has a name, the Jordan Curve Theorem. It is taught in the Differential Geometry and Topology courses.

2. A very important, and beautiful theorem in mathematics is the Cauchy Integral Formula. Let f be a *complex-valued* function of a complex argument z , C be a simple closed curve. Then for any point a inside C ,

$$f(a) = \int_C \frac{f(z)}{z - a} dz.$$

One can prove this result using Green's Theorem. This formula is crucial in Complex Analysis (Math 165).

3. I emphasized that the Fundamental Theorem of Calculus, Stokes Theorem, and Divergence Theorem all look very similar. Naturally, mathematicians have tried to come up with a single method of proving all of them at once (that's what mathematicians do). The framework that mathematicians came up with is that of *differential forms*. All of the theorems above can be written as

$$\int_{\partial M} \omega = \int_M d\omega,$$

where M is a curve, or a surface, or a region, ∂M is its boundary, ω is a differential form, and $d\omega$ is its differential. Here 0-form is just a function, a 1-form is something like

$$F_1 dx + F_2 dy + F_3 dz,$$

a 2-form is something like

$$F_1 dy dz + F_2 dz dx + F_3 dx dy,$$

and a 3-form is something like

$$f dx dy dz.$$

Moreover, in this language both the identities

$$\operatorname{div}(\operatorname{curl} F) = 0$$

and

$$\operatorname{curl}(\operatorname{grad} f) = 0$$

can be written simply as

$$d(d\omega) = 0.$$

For more about this, see Section 8.6 of the textbook, or Advanced Calculus and Analysis courses (Math 151).