Math 209A, Homework 1

due September 30 is class.

1. Show that

(a) $A \triangle B = \emptyset \Leftrightarrow A = B$.

(b) $A \triangle \emptyset = A$ and $A \triangle X = \tilde{A}$. Here X is the ambient space.

2. Let f map X into Y and $A \subset X$, $B \subset Y$.

(a) Show that

$$f[f^{-1}[B]] \subset B$$

and

$$f^{-1}[f[A]] \supset A.$$

(b) Give examples to show that we need not have equality.

(c) Show that if f maps X *onto* Y and $B \subset Y$, then

$$f[f^{-1}[B]] = B.$$

3. Half-open intervals are intervals of the form $[a, b)$, where each of a, b can be a real number or $\pm\infty$ (so in particular, $\emptyset = [1, 0)$ and $\mathbb{R} = (-\infty, \infty)$ are of this form). Let $\mathcal{A}$ be the collection of all *finite unions* of half-open intervals. Show that $\mathcal{A}$ is an algebra.
Hint: first check the properties for single intervals, then extend to finite unions.

4. Let $S \subset \mathbb{R}$ be a countable set. Show that for any $\varepsilon > 0$, there is a (countable) collection of intervals $\{I_i : i \in \mathbb{N}\}$ such that the sum of their lengths

$$\sum_{i=1}^{\infty} |I_i| < \varepsilon$$

and

$$S \subset \bigcup_{i=1}^{\infty} I_i.$$

(This says that any countable set has measure zero.)

5. For a sequence $\{E_k : k \in \mathbb{N}\}$ of sets define

$$\limsup E_k = \bigcap_{j=1}^{\infty}\Big(\bigcup_{k=j}^{\infty} E_k\Big), \qquad \liminf E_k = \bigcup_{j=1}^{\infty}\Big(\bigcap_{k=j}^{\infty} E_k\Big).$$

Find $\limsup E_k$ and $\liminf E_k$ for

$$E_k = \begin{cases} [-(1/k), 1] & \text{for } k \text{ odd}, \\ [-1, (1/k)] & \text{for } k \text{ even}. \end{cases}$$