

Lecture III: Applications of Voiculescu's random matrix model to operator algebras

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Voiculescu's Random Matrix Model

Theorem [Voiculescu]. For each n in \mathbb{N} , let $X_1^{(n)}, \dots, X_r^{(n)}$ be independent random matrices from $\text{GUE}(n, \frac{1}{n})$, and let i_1, i_2, \dots, i_p be indices from $\{1, 2, \dots, r\}$. Let further $D_1^{(n)}, \dots, D_p^{(n)}$ be non-random *diagonal* $n \times n$ matrices such that

$$\forall j \in \{1, \dots, p\}: \sup_{n \in \mathbb{N}} \|D_j^{(n)}\| < \infty.$$

and

$$\forall p \in \mathbb{C}\langle X_1, \dots, X_p \rangle: \lim_{n \rightarrow \infty} \text{tr}_n(p(D_1^{(n)}, \dots, D_p^{(n)})) \text{ exists in } \mathbb{C}.$$

Voiculescu's Random Matrix Model (continued)

We then have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \text{tr}_n \left(X_{i_1}^{(n)} D_1^{(n)} X_{i_2} D_2^{(n)} \cdots X_{i_p}^{(n)} D_p^{(n)} \right) \right\} = \tau(x_{i_1} d_1 x_{i_2} d_2 \cdots x_{i_p} d_p),$$

where

- $\{d_1, \dots, d_p\}$ and $\{x_1, \dots, x_r\}$ are free sets of operators in a non-commutative probability space (\mathcal{A}, τ) .
- $\{x_1, x_2, \dots, x_r\}$ is a free semi-circular system in (\mathcal{A}, τ) , i.e.
 - (a) x_1, x_2, \dots, x_r are free selfadjoint operators.
 - (b) For each i in $\{1, 2, \dots, r\}$ and any p in \mathbb{N} ,

$$\tau(x_i^p) = \frac{1}{2\pi} \int_{\mathbb{R}} t^p \sqrt{4 - t^2} dt.$$
- $\tau(p(d_1, \dots, d_p)) = \lim_{n \rightarrow \infty} \text{tr}_n(p(D_1^{(n)}, \dots, D_p^{(n)}))$ for all p in $\mathbb{C}\langle X_1, \dots, X_p \rangle$.

When there are no constant diagonals

In the case where

$$D_1^{(n)} = D_2^{(n)} = \dots = D_p^{(n)} = \mathbf{1}_n,$$

one may verify (as in the illustration of Step II) that for any γ in Γ_p ,

$$\Delta_n(\gamma; \mathbf{1}_n, \dots, \mathbf{1}_n) = \frac{1}{n} \sum_{1 \leq i_1, \dots, i_p \leq n} \prod_{j=1}^p (n^{-1} \delta_{i_j, i_{\gamma(j)+1}}) = n^{-p-1+d(\gamma)},$$

where $d(\gamma)$ denotes the number of equivalence classes for the equivalence relation \sim_γ on $\{1, 2, \dots, p\}$ generated by the condition

$$j \sim_\gamma \gamma(j) + 1, \quad (j = 1, 2, \dots, p).$$

One may then prove that

- $p + 1 - d(\gamma) \in 2\mathbb{N}_0$.
- $p + 1 - d(\gamma) = 0 \iff \gamma$ is non-crossing.

When there are no constant diagonals (continued)

Hence the proof of Voiculescu's theorem outlined yesterday leads to the formula

$$\begin{aligned} & \mathbb{E}\{\mathrm{tr}_n(X_{i_1}^{(n)} X_{i_2}^{(n)} \cdots X_{i_p}^{(n)})\} \\ &= \sum_{\gamma \in \Gamma(i_1, \dots, i_p)} \Delta_n(\gamma, \mathbf{1}_n, \dots, \mathbf{1}_n) \\ &= \sum_{\gamma \in \Gamma(i_1, \dots, i_p)} n^{-p-1+d(\gamma)} \\ &= \#\{\gamma \in \Gamma(i_1, \dots, i_p) \mid \gamma \text{ is non-crossing}\} + O(n^{-2}). \end{aligned} \tag{1}$$

Since we know that the limit of the left hand side of (1) is $\tau(x_{i_1} x_{i_2} \cdots x_{i_p})$, it follows that we must have

$$\tau(x_{i_1} x_{i_2} \cdots x_{i_p}) = \#\{\gamma \in \Gamma(i_1, \dots, i_p) \mid \gamma \text{ is non-crossing}\}$$

and hence we get that

$$\mathbb{E}\{\mathrm{tr}_n(X_{i_1}^{(n)} X_{i_2}^{(n)} \cdots X_{i_p}^{(n)})\} = \tau(x_{i_1} x_{i_2} \cdots x_{i_p}) + O(n^{-2}). \quad (2)$$

Strong version of Voiculescu's RMM.

Proposition. For each n in \mathbb{N} , let $X_1^{(n)}, \dots, X_r^{(n)}$ be independent random matrices from $\text{GUE}(n, \frac{1}{n})$, and let i_1, i_2, \dots, i_p be indices from $\{1, 2, \dots, r\}$. Then for almost all ω in (Ω, \mathcal{F}, P) we have

$$\text{tr}_n(X_{i_1}^{(n)}(\omega)X_{i_2}^{(n)}(\omega)\cdots X_{i_p}^{(n)}(\omega)) \xrightarrow{n \rightarrow \infty} \tau(x_{i_1}x_{i_2}\cdots x_{i_p}),$$

where $\{x_1, x_2, \dots, x_r\}$ is a free semi-circular system in a non-commutative probability space (\mathcal{A}, τ) .

Proof.

Put

$$Q_n = X_{i_1}^{(n)} X_{i_2}^{(n)} \cdots X_{i_p}^{(n)} \quad \text{and} \quad q = x_{i_1} x_{i_2} \cdots x_{i_p}$$

We know from Voiculescu's RMM that

$$\mathbb{E}\{\text{tr}_n(Q_n)\} \longrightarrow \tau(q), \quad \text{as } n \rightarrow \infty,$$

so it suffices to show that

$$\text{tr}_n(Q_n) - \mathbb{E}\{\text{tr}_n(Q_n)\} \xrightarrow{\text{a.s.}} 0, \quad \text{as } n \rightarrow \infty. \quad (3)$$

By standard probability techniques, (3) will follow if we establish that

$$\sum_{n=1}^{\infty} \mathbb{V}\{\text{tr}_n(Q_n)\} < \infty. \quad (4)$$

Note here that

$$\begin{aligned}
 \mathbb{V}\{\mathrm{tr}_n(Q_n)\} &= \mathbb{E}\{|\mathrm{tr}_n(Q_n)|^2\} - |\mathbb{E}\{\mathrm{tr}_n(Q_n)\}|^2 \\
 &= \mathbb{E}\{\mathrm{tr}_n(Q_n)\mathrm{tr}_n(Q_n^*)\} - |\tau(q)|^2 + O(n^{-2}) \\
 &= \mathbb{E}\{\mathrm{tr}_n(YQ_n YQ_n^*)\} - |\tau(q)|^2 + O(n^{-2}) \\
 &= \tau(yqyq^*) - |\tau(q)|^2 + O(n^{-2}) \\
 &= \tau(q)\tau(q^*) - |\tau(q)|^2 + O(n^{-2}) \\
 &= O(n^{-2}),
 \end{aligned} \tag{5}$$

where

- $Y \in \mathrm{GUE}(n, \frac{1}{n})$ which is independent of $\{X_1^{(n)}, \dots, X_r^{(n)}\}$
- y is a semi-circular element which is free from $\{x_1, \dots, x_r\}$.

Now (4) follows immediately from (5). ■

Main Open Problem on the free group factors

Problem. Let q, r be numbers in $\{2, 3, 4, \dots\} \cup \{\infty\}$, and consider the free group factors $\mathcal{L}(\mathbb{F}_q)$ and $\mathcal{L}(\mathbb{F}_r)$. Do we have

$$\mathcal{L}(\mathbb{F}_q) \simeq \mathcal{L}(\mathbb{F}_r)?$$

Key observation [Voiculescu]. Let r be an element of $\mathbb{N} \cup \{\infty\}$ and consider the free group factor $\mathcal{L}(\mathbb{F}_r)$. Then $\mathcal{L}(\mathbb{F}_r)$ is generated, as a von Neumann algebra, by a semi-circular system

$$\{x_1, x_2, \dots, x_r\} \subseteq \mathcal{L}(\mathbb{F}_r).$$

The fundamental group of a II_1 -factor

Let \mathcal{A} be a II_1 -factor acting on a Hilbert space \mathcal{H} , i.e., \mathcal{A} is an infinite dimensional von Neumann factor with a faithful tracial state $\tau: \mathcal{A} \rightarrow \mathbb{C}$. Then for any t in $[0, 1]$ there is a projection e in \mathcal{A} , such that $\tau(e) = t$, and we then put

$$\mathcal{A}_t = e\mathcal{A}e,$$

which is again a II_1 -factor acting on the Hilbert space $e(\mathcal{H})$. It is not hard to see that the $*$ -isomorphism class \mathcal{A}_t does not depend on the choice of e in \mathcal{A} with $\tau(e) = t$.

If $t \geq 1$ we choose a positive integer n greater than t , and note that $M_n(\mathcal{A})$ is again a II_1 -factor with trace $\text{tr}_n \otimes \tau$. We then define

$$\mathcal{A}_t = M_n(\mathcal{A})_{\frac{t}{n}},$$

and one may verify that the $*$ -isomorphism class of \mathcal{A}_t does not depend on the choice of n .

The fundamental group of a II_1 -factor (continued)

Now the *fundamental group* of \mathcal{A} is defined as

$$\text{FG}(\mathcal{A}) = \{t \in (0, \infty) \mid \mathcal{A}_t \simeq \mathcal{A}\}.$$

One may verify that $\text{FG}(\mathcal{A})$ is a subgroup of the multiplicative group $(0, \infty)$.

Main application: Isomorphism results for free group factors

Theorem [Voiculescu]. For any m in $\{2, 3, 4, \dots\} \cup \{\infty\}$ and any positive integer q , we have

$$\mathcal{L}(\mathbb{F}_m)_{\frac{1}{q}} \simeq \mathcal{L}(\mathbb{F}_{q^2(m-1)+1}).$$

Outline of Proof of $\mathcal{L}(\mathbb{F}_2)_{\frac{1}{2}} \simeq \mathcal{L}(\mathbb{F}_5)$.

First idea. We know that $\mathcal{L}(\mathbb{F}_2) \simeq W^*\{X_1, X_2\}$, where $\{X_1, X_2\}$ is a free semi-circular system. We want to realize $\{X_1, X_2\}$ in $(M_2(\mathcal{A}), \text{tr}_2 \otimes \tau)$ for a suitable C^* -probability space (\mathcal{A}, τ) . In $M_2(\mathcal{A})$ we can cut down by the projection

$$e = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix},$$

which satisfies that $\text{tr}_n \otimes \tau(e) = \frac{1}{2}$.

Outline of Proof (continued)

Lemma. Let (\mathcal{A}, τ) be a W^* -probability space containing a free semi-circular system $\{x_{11}, x'_{12}, x''_{12}, x_{22}\}$ and a Haar unitary u , such that

$$W^*\{x_{11}, x'_{12}, x''_{12}, x_{22}\} \quad \text{and} \quad W^*\{u\} \quad \text{are free subalgebras.}$$

Consider further the 2×2 matrices over \mathcal{A} :

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} x_{11} & \frac{1}{\sqrt{2}}(x'_{12} + ix''_{12}) \\ \frac{1}{\sqrt{2}}(x'_{12} - ix''_{12}) & x_{22} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u & 0 \\ 0 & 2u \end{bmatrix}.$$

We then have

- (i) X and U are $*$ -free operators in $(M_2(\mathcal{A}), \text{tr}_2 \otimes \tau)$.
- (ii) X is a semi-circular element with respect to $\text{tr}_2 \otimes \tau$.
- (iii) $W^*\{X, U\} \simeq \mathcal{L}(\mathbb{F}_2)$.

Outline of Proof (continued)

Proof of Lemma. For each n in \mathbb{N} consider independent $n \times n$ random matrices

$$X(1, 1, n), X(1, 2, n)', X(1, 2, n)'', X(2, 2, n) \in \text{GUE}(n, \frac{1}{n}),$$

and consider further the constant diagonal $n \times n$ matrix

$$V(n) = \begin{pmatrix} \exp(\frac{2\pi i}{n}) & & & 0 \\ & \exp(\frac{4\pi i}{n}) & & \\ & & \ddots & \\ 0 & & & \exp(\frac{2n\pi i}{n}). \end{pmatrix}$$

It is not hard to see that

$$V(n) \xrightarrow{*D} u \quad \text{i.e.,} \quad \{V(n), V(n)^*\} \xrightarrow{D} \{u, u^*\},$$

as $n \rightarrow \infty$.

Outline of Proof (continued)

Hence, it follows from Voiculescu's random matrix model that

$$\{X(1, 1, n), X(1, 2, n)', X(1, 2, n)'', X(2, 2, n), V(n)\}$$

$$\xrightarrow{*D} \{x, x'_{12}, x''_{12}, x_{22}, u\}.$$

This implies that with

$$X(n) :=$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} X(1, 1, n) & \frac{1}{\sqrt{2}}(X(1, 2, n)' + iX(1, 2, n)'') \\ \frac{1}{\sqrt{2}}(X(1, 2, n)' - iX(1, 2, n)'') & X(2, 2, n) \end{bmatrix}$$

and

$$U(n) := \begin{bmatrix} V(n) & 0 \\ 0 & 2V(n) \end{bmatrix}$$

we have that

$$\{X(n), U(n)\} \xrightarrow{*D} \{X, U\} \quad \text{as } n \rightarrow \infty. \quad (6)$$

Outline of Proof (continued)

On the other hand it is straightforward to check that

$$X(n) =$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} X(1, 1, n) & \frac{1}{\sqrt{2}}(X(1, 2, n)' + iX(1, 2, n)'') \\ \frac{1}{\sqrt{2}}(X(1, 2, n)' - iX(1, 2, n)'') & X(2, 2, n) \end{bmatrix}$$


$$\in \text{GUE}(2n, \frac{1}{2n}),$$

and we know that $U(n)$ is a constant diagonal matrix such that

$$U(n) := \begin{bmatrix} V(n) & 0 \\ 0 & 2V(n) \end{bmatrix} \xrightarrow{*D} U,$$

so by another application of Voiculescu's random matrix model it follows that

$$\{X(n), U(n)\} \xrightarrow{*D} \{X', U'\}, \quad (7)$$

where X' and U' are $*$ -free operators in a non-commutative probability space (\mathcal{A}', τ') , X' is a semi-circular element and $U' \stackrel{D}{=} U$. 

Outline of Proof (continued)

Combining now (6) and (7) it follows that

$$\{X, U\} \stackrel{*D}{=} \{X', U'\},$$

which proves (i) and (ii) of the lemma.

To prove (iii), note that by spectral calculus, $W^*\{U\} = W^*\{X_2\}$ for some semi-circular element $X_2 \in W^*\{U\}$. Therefore

$$W^*\{X, U\} = W^*\{X, X_2\} \simeq \mathcal{L}(\mathbb{F}_2),$$

since X and X_2 are free. ■

Immediate consequences

Corollary [Voiculescu].

- (i) If m is a positive integer of the form $m = n^2 + 1$ for some positive integer n , then

$$\mathcal{L}(\mathbb{F}_m) \simeq \mathcal{L}(\mathbb{F}_2)_{1/\sqrt{m-1}}.$$

- (ii) For the free group factor $\mathcal{L}(\mathbb{F}_\infty)$ we have

$$\text{FG}(\mathcal{L}(\mathbb{F}_\infty)) \supseteq \mathbb{Q}_+.$$

Further results on the free group factors

Theorem [Voiculescu]. For any m in $\{2, 3, 4, \dots\}$ and any positive integer n we have

$$\mathcal{L}((\mathbb{Z}/n\mathbb{Z}) * \mathbb{F}_m)_{\frac{1}{n}} \simeq \mathcal{L}(\mathbb{F}_{mn^2-n+1}).$$

Theorem [Dykema]. For any m in $\{2, 3, 4, \dots\}$ and any positive integer n , we have

$$(M_n(\mathbb{C}) * \mathcal{L}(\mathbb{F}_m))_{\frac{1}{n}} \simeq \mathcal{L}(\mathbb{F}_{kn^2}).$$

Theorem [Radulescu]. For any positive integer m in $\{2, 3, \dots\}$ we have

$$\mathcal{L}(\mathbb{F}_m) \simeq \mathcal{L}(\mathbb{F}_2)_{1/\sqrt{m-1}}.$$

Further results on the free group factors (continued)

Definition [Dykema/Radulescu]. For any t in $(1, \infty)$ we define a *-isomorphism class $\mathcal{L}(\mathbb{F}_t)$ of II_1 -factors by

$$\mathcal{L}(\mathbb{F}_t) \simeq \mathcal{L}(\mathbb{F}_2)_{1/\sqrt{t-1}}.$$

These are called the *interpolated free group factors*.

Theorem [Dykema]. The interpolated free group factors defined above satisfy

- (i) $\mathcal{L}(\mathbb{Z}/n\mathbb{Z}) * \mathcal{L}(\mathbb{F}_t) \simeq \mathcal{L}(\mathbb{F}_{t+1-\frac{1}{n}})$, whenever $t > 1$.
- (ii) $M_m(\mathbb{C}) * \mathcal{L}(\mathbb{F}_t) \simeq \mathcal{L}(\mathbb{F}_{t+1-\frac{1}{n^2}})$, whenever $t > 1$.
- (iii) $\mathcal{R} * \mathcal{L}(\mathbb{F}_t) \simeq \mathcal{L}(\mathbb{F}_{t+1})$, whenever $t > 1$, and where \mathcal{R} denotes the hyperfinite II_1 -factor.

Further results on the free group factors (continued)

Theorem [Dykema/Radulescu]. The interpolated free group factors defined above satisfy

- (iv) $\mathcal{L}(\mathbb{F}_s)_t \simeq \mathcal{L}(\mathbb{F}_{t-2(s-1)+1})$ whenever $s > 1$, $t > 0$.
- (v) $\mathcal{L}(\mathbb{F}_s) * \mathcal{L}(\mathbb{F}_t) \simeq \mathcal{L}(\mathbb{F}_{s+t})$, whenever $s, t > 1$.
- (vi) $\mathcal{L}(\mathbb{Z}) * \mathcal{L}(\mathbb{F}_t) \simeq \mathcal{L}(\mathbb{F}_{t+1})$, whenever $t > 1$.

Theorem [Radulescu]. For the free group factor $\mathcal{L}(\mathbb{F}_\infty)$ we have

$$\text{FG}(\mathcal{L}(\mathbb{F}_\infty)) = (0, \infty).$$

Present status of the main open problem

[Dykema/Radulescu]: One of the following two extremes hold:

- $\mathcal{L}(\mathbb{F}_q) \simeq \mathcal{L}(\mathbb{F}_r)$ for all positive integers q, r .
- $\mathcal{L}(\mathbb{F}_q) \not\simeq \mathcal{L}(\mathbb{F}_r)$ for any distinct positive integers q, r .

Second application of VRMM: Convergence of norms of polynomials of GUE's – the easy part

Theorem. For each $n \in \mathbb{N}$, let $X_1^{(n)}, \dots, X_r^{(n)}$ be independent random matrices from the class $\text{GUE}(n, \frac{1}{n})$, and let $\{x_1, \dots, x_r\}$ be a semi-circular system in a C^* -probability space (\mathcal{A}, τ) with a faithful state τ .

Then for any polynomial p in r non-commuting variables, we have with probability one that

$$\liminf_{n \rightarrow \infty} \|p(X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)})\| \geq \|p(x_1, \dots, x_r)\|.$$

Proof.

For each n in \mathbb{N} put

$$Q_n = p(X_1^{(n)}, \dots, X_r^{(n)})^* p(X_1^{(n)}, \dots, X_r^{(n)})$$

and similarly put

$$q = p(x_1, \dots, x_r)^* p(x_1, \dots, x_r).$$

By the C^* -identity, it suffices to show for any positive ϵ that

$$\liminf_{n \rightarrow \infty} \|Q_n(\omega)\| \geq \|q\| - \epsilon, \quad \text{for almost all } \omega.$$

Note for this, that for any positive integer m we have by the strong version of Voiculescu's theorem

$$\int_{\mathbb{R}} t^m \mu_{Q_n(\omega)}(dt) = \text{tr}_n(Q_n^m(\omega)) \longrightarrow \tau(q^m) = \int_{\mathbb{R}} t^m \mu_q(dt), \quad (8)$$

for almost all ω , where μ_q is the measure on \mathbb{R} defined by the last equality.

Since $\text{supp}(\mu_q) = \text{sp}(q)$ is compact, (8) implies that

$$\mu_{Q_n(\omega)} \xrightarrow{w} \mu_q \quad \text{for almost all } \omega,$$

i.e.,

$$\forall f \in C_b(\mathbb{R}): \int_{\mathbb{R}} f(t) \mu_{Q(\omega)}(dt) \longrightarrow \int_{\mathbb{R}} f(t) \mu_q(dt). \quad (9)$$

Now choose a function f from $C_b(\mathbb{R})$ satisfying that

- $0 \leq f(t) \leq 1$ for all t in \mathbb{R} .
- $f(t) = 0$ for all t in $(-\infty, \|q\| - \epsilon]$.
- $f(t) = 1$ for all t in $[\|q\|, \infty)$.

Then since $\|q\| \in \text{sp}(q) = \text{supp}(\mu_q)$, it follows that $\int_{\mathbb{R}} f(t) \mu_q(dt) > 0$, and hence (9) implies that for almost all ω

$$\mu_{Q_n(\omega)}((\|q\| - \epsilon, \infty)) > 0, \quad \text{for } n \text{ sufficiently large}$$

and therefore

$$\|Q_n(\omega)\| = \lambda_{\max}(Q_n(\omega)) \in (\|q\| - \epsilon, \infty), \quad \text{for } n \text{ sufficiently large.}$$

This implies that

$$\liminf_{n \rightarrow \infty} \|Q_n(\omega)\| \geq \|q\| - \epsilon,$$

for almost all ω . ■