

Lecture II: Voiculescu's random matrix model for a free semi-circular system

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Voiculescu's Random Matrix Model

Theorem [Voiculescu]. For each n in \mathbb{N} , let $X_1^{(n)}, \dots, X_r^{(n)}$ be independent random matrices from $\text{GUE}(n, \frac{1}{n})$, and let i_1, i_2, \dots, i_p be indices from $\{1, 2, \dots, r\}$. Let further $D_1^{(n)}, \dots, D_p^{(n)}$ be non-random *diagonal* $n \times n$ matrices such that

$$\forall j \in \{1, \dots, p\}: \sup_{n \in \mathbb{N}} \|D_j^{(n)}\| < \infty.$$

and

$$\forall p \in \mathbb{C}\langle X_1, \dots, X_p \rangle: \lim_{n \rightarrow \infty} \text{tr}_n(p(D_1^{(n)}, \dots, D_p^{(n)})) \text{ exists in } \mathbb{C}.$$

Voiculescu's Random Matrix Model (continued)

We then have

$$\lim_{n \rightarrow \infty} \mathbb{E} \{ \text{tr}_n (X_{i_1}^{(n)} D_1^{(n)} X_{i_2}^{(n)} D_2^{(n)} \cdots X_{i_p}^{(n)} D_p^{(n)}) \} = \tau(x_{i_1} d_1 x_{i_2} d_2 \cdots x_{i_p} d_p),$$

where

- $\{d_1, \dots, d_p\}$ and $\{x_1, \dots, x_r\}$ are free sets of operators in a non-commutative probability space (\mathcal{A}, τ) .
- $\{x_1, x_2, \dots, x_r\}$ is a free semi-circular system in (\mathcal{A}, τ) , i.e.
 - (a) x_1, x_2, \dots, x_r are free selfadjoint operators.
 - (b) For each i in $\{1, 2, \dots, r\}$ and any p in \mathbb{N} ,

$$\tau(x_i^p) = \frac{1}{2\pi} \int_{\mathbb{R}} t^p \sqrt{4 - t^2} dt.$$
- $\tau(p(d_1, \dots, d_p)) = \lim_{n \rightarrow \infty} \text{tr}_n(p(D_1^{(n)}, \dots, D_p^{(n)}))$ for all p in $\mathbb{C}\langle X_1, \dots, X_p \rangle$.

Non-vanishing of mixed moments

Lemma. Let $X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)}$ be independent random matrices from $\text{GUE}(n, \frac{1}{n})$, let i_1, i_2, \dots, i_p be indices from $\{1, 2, \dots, r\}$ and for each i in $\{1, 2, \dots, r\}$ put

$$K(i) = \{j \in \{1, 2, \dots, p\} \mid i_j = i\}.$$

Let further $D_1^{(n)}, D_2^{(n)}, \dots, D_p^{(n)}$ be arbitrary non-random $n \times n$ matrices. Then

$$\mathbb{E}\{X_{i_1}^{(n)} D_1^{(n)} X_{i_2}^{(n)} D_2^{(n)} \dots X_{i_p}^{(n)} D_p^{(n)}\} = 0,$$

unless $\#K(i)$ is an even number for all i , i.e., unless there exists a permutation γ from S_p such that

- (i) $\gamma \circ \gamma = \text{id}$,
- (ii) $\gamma(j) \neq j$, for all j in $\{1, 2, \dots, p\}$.
- (iii) $\gamma(K(i)) = K(i)$, for all i in $\{1, 2, \dots, r\}$.

Proof.

Assume that $\#K(i)$ is odd for some i from $\{1, 2, \dots, r\}$. Then the random matrix $X_i^{(n)}$ appears an odd number of times in the product

$$X_{i_1}^{(n)} D_1^{(n)} X_{i_2}^{(n)} D_2^{(n)} \dots X_{i_p}^{(n)} D_p^{(n)}.$$

But since $-X_i^{(n)} \in \text{GUE}(n, \frac{1}{n})$ too, this implies that

$$\mathbb{E}\{X_{i_1}^{(n)} D_1^{(n)} X_{i_2}^{(p)} D_2^{(n)} \dots X_{i_p}^{(n)} D_p^{(n)}\} = -\mathbb{E}\{X_{i_1}^{(n)} D_1^{(n)} X_{i_2} D_2^{(n)} \dots X_{i_p}^{(n)} D_p^{(n)}\},$$

from which the assertion follows. \blacksquare

Sets of permutations

Definition. In the setting of the previous lemma, we put

$$\Gamma(i_1, i_2, \dots, i_p) = \{\gamma \in S_p \mid \text{(i),(ii) and (iii) are satisfied}\}.$$

We note in particular that

$$\Gamma(i_1, i_2, \dots, i_p) \neq \emptyset \iff \#K(i) \text{ is even for all } i \text{ in } \{1, 2, \dots, r\}.$$

Definition. Let p be an even positive integer. Then we put

$$\Gamma_p = \{\gamma \in S_p \mid \gamma \circ \gamma = \text{id} \text{ and } \gamma(j) \neq j \text{ for all } j\}.$$

Remark. Note that for any even positive integer p and any indices i_1, i_2, \dots, i_p from $\{1, 2, \dots, r\}$ we have the inclusion

$$\Gamma(i_1, i_2, \dots, i_p) \subseteq \Gamma_p.$$

Pairs and crossings

Definition. Let p be a positive integer and let γ be a permutation in Γ_p . Then if $j \in \{1, 2, \dots, p-1\}$ such that

$$\gamma(j) = j + 1,$$

we say that $(j, j+1)$ is a pair for γ .

If $p \geq 4$ and $(j, j+1)$ is a pair for γ , then we denote by $\mathcal{C}_j\gamma$ the permutation in Γ_{p-2} obtained by *cancelling the pair* $(j, j+1)$. More precisely $\mathcal{C}_j\gamma$ is given by:

$$\mathcal{C}_j\gamma(k) = \begin{cases} \gamma(k), & \text{if } 1 \leq k, \gamma(k) \leq j-1, \\ \gamma(k) - 2, & \text{if } 1 \leq k \leq j-1 \text{ and } \gamma(k) \geq j+2, \\ \gamma(k+2), & \text{if } j \leq k \leq p-2 \text{ and } \gamma(k+2) \leq j-1, \\ \gamma(k+2) - 2, & \text{if } j \leq k \leq p-2 \text{ and } \gamma(k+2) \geq j+2. \end{cases}$$

Pairs and crossings (continued)

Definition. Let p be a positive integer and let γ be a permutation in Γ_p . Then if $j, k, l, m \in \{1, 2, \dots, p\}$ such that

$$1 \leq j < k < l < m \quad \text{and} \quad \gamma(j) = l, \gamma(k) = m,$$

we say that (j, k, l, m) constitutes a crossing for γ .

If γ has no crossings, we say that γ is a *non-crossing* permutation.

For any subset F of S_p we put

$$F^{\text{nc}} = \{\gamma \in F \mid \gamma \text{ is non-crossing}\}.$$

Remarks.

- (a) If γ is non-crossing, then γ must have a pair.
- (b) If $(j, j+1)$ is a pair for γ , then

$$\gamma \text{ is non-crossing} \iff \mathbb{C}_j \gamma \text{ is non-crossing.}$$

The canonical GUE mixed moment associated to a permutation

Definition. Let i_1, i_2, \dots, i_p be indices from $\{1, 2, \dots, r\}$, and let γ be a permutation from $\Gamma(i_1, i_2, \dots, i_p)$. Let further $D_1^{(n)}, \dots, D_p^{(n)}$ be non-random $n \times n$ matrices. Then we put

$$\Delta_n(\gamma; D_1^{(n)}, D_2^{(n)}, \dots, D_p^{(n)}) = \mathbb{E}\{\mathrm{tr}_n(Z_1^{(n)} D_1^{(n)} Z_2^{(n)} D_2^{(n)} \dots Z_p^{(n)} D_p^{(n)})\},$$

where

- $Z_1^{(n)}, Z_2^{(n)}, \dots, Z_p^{(n)}$ are $\mathrm{GUE}(n, \frac{1}{n})$ random matrices.
- The set $\{Z_1^{(n)}, Z_2^{(n)}, \dots, Z_p^{(n)}\}$ consists of $\frac{p}{2}$ distinct random matrices, each of which appears twice. These $\frac{p}{2}$ distinct matrices are independent.
- $\forall j, j' \in \{1, 2, \dots, p\}: Z_j = Z_{j'} \iff j' = j \text{ or } j' = \gamma(j)$.

The canonical semi-circular mixed moment associated to a permutation

Definition. Let i_1, i_2, \dots, i_p be indices from $\{1, 2, \dots, r\}$, and let γ be a permutation from $\Gamma(i_1, i_2, \dots, i_p)$. Let further $d_1^{(n)}, \dots, d_p^{(n)}$ be operators in a non-commutative probability space (\mathcal{A}, τ) . Then we put

$$\Delta(\gamma; d_1, d_2, \dots, d_p) = \tau(z_1 d_1 z_2 d_2 \cdots z_p d_p),$$

where

- $\{z_1, z_2, \dots, z_p\}$ and $\{d_1, d_2, \dots, d_r\}$ are free sets of operators in (\mathcal{A}, τ) .
- The set $\{z_1, z_2, \dots, z_p\}$ consists of $\frac{p}{2}$ distinct selfadjoint operators, each of which appears twice. These $\frac{p}{2}$ distinct operators form a free semi-circular system.
- $\forall j, j' \in \{1, 2, \dots, p\}: z_j = z_{j'} \iff j' = j \text{ or } j' = \gamma(j)$.

Sketch of Proof of Voiculescu's Random matrix Model

Step I (The \sqrt{s} -trick).

For each fixed n in \mathbb{N} we have that

$$\begin{aligned}\mathbb{E}\{\mathrm{tr}_n(X_{i_1}^{(n)} D_1^{(n)} X_{i_2}^{(n)} D_2^{(n)} \cdots X_{i_p}^{(n)} D_p^{(n)})\} \\ = \sum_{\gamma \in \Gamma(i_1, i_2, \dots, i_p)} \Delta_n(\gamma; D_1^{(n)}, \dots, D_p^{(n)})\end{aligned}$$

and that

$$\tau(x_{i_1} d_1 x_{i_2} d_2 \cdots x_{i_p} d_p) = \sum_{\gamma \in \Gamma(i_1, i_2, \dots, i_p)} \Delta(\gamma; d_1, \dots, d_p).$$

Sketch of Proof of Voiculescu's Random matrix Model (continued)

Step II (Vanishing of terms from permutations with crossings).

If $\gamma \in \Gamma_p$ has a crossing, then

$$\lim_{n \rightarrow \infty} \Delta_n(\gamma; D_1^{(n)}, \dots, D_p^{(n)}) = 0,$$

and

$$\Delta(\gamma; d_1, \dots, d_r) = 0.$$

Sketch of Proof of Voiculescu's Random matrix Model (continued)

Step III (Cancellation properties).

Suppose $p \geq 4$ and that $\gamma \in \Gamma_p$ has a pair $(j, j+1)$ for some j in $\{1, 2, \dots, p\}$, and let $\mathfrak{C}_j \gamma$ be the permutation in Γ_{p-2} obtained by cancelling $(j, j+1)$. Then for all n in \mathbb{N}

$$\begin{aligned} \Delta_n(\gamma, D_1^{(n)}, \dots, D_r^{(n)}) \\ = \operatorname{tr}_n(D_j^{(n)}) \cdot \Delta_n(\mathfrak{C}_j \gamma; D_1^{(n)}, \dots, D_{j-2}^{(n)}, \tilde{D}_j^{(n)}, D_{j+2}^{(n)}, \dots, D_p^{(n)}), \end{aligned}$$

where $\tilde{D}_j^{(n)} := D_{j-1}^{(n)} D_{j+1}^{(n)}$. Similarly we have that

$$\Delta(\gamma, d_1, \dots, d_r) = \tau(d_j) \cdot \Delta(\mathfrak{C}_j \gamma; d_1, \dots, d_{j-2}, \tilde{d}_j, d_{j+2}, \dots, d_p),$$

where $\tilde{d}_j = d_{j-1} d_{j+1}$.

Sketch of Proof of Voiculescu's Random matrix Model (continued)

Step 4 (Conclusion).

Using Steps 1-2 it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \{ \text{tr}_n (X_{i_1}^{(n)} D_1^{(n)} X_{i_2}^{(n)} D_2^{(n)} \cdots X_{i_p}^{(n)} D_p^{(n)}) \} \\ = \lim_{n \rightarrow \infty} \sum_{\gamma \in \Gamma^{\text{nc}}(i_1, i_2, \dots, i_p)} \Delta_n(\gamma; D_1^{(n)}, \dots, D_p^{(n)}) \end{aligned}$$

and that

$$\tau(x_{i_1} d_1 x_{i_2} d_2 \cdots x_{i_p} d_p) = \sum_{\gamma \in \Gamma^{\text{nc}}(i_1, i_2, \dots, i_p)} \Delta(\gamma; d_1, \dots, d_p).$$

Hence it suffices to show that

$$\Delta_n(\gamma; D_1^{(n)}, \dots, D_p^{(n)}) \longrightarrow \Delta(\gamma, d_1, \dots, d_p), \quad \text{as } n \rightarrow \infty, \quad (1)$$

for any γ in Γ_p^{nc} .

Sketch of Proof of Voiculescu's Random matrix Model (continued)

Formula (1) is proved by induction on p :

Assume that the assertion has been established for $p - 2$ and all choices of constant diagonal matrices $\tilde{D}_1^{(n)}, \dots, \tilde{D}_{p-2}^{(n)}$ and corresponding operators $\tilde{d}_1, \dots, \tilde{d}_{p-2}$ as in the theorem. Then let γ from Γ_p^{nc} be given, and recall that γ must have a pair $(j, j + 1)$ for some j in $\{1, 2, \dots, p\}$. Then by Step 3 and the induction hypothesis,

$$\begin{aligned} \Delta_n(\gamma; D_1^{(n)}, \dots, D_p^{(n)}) &= \text{tr}_N(D_j^{(n)}) \cdot \Delta_n(\mathbb{C}_j \gamma; D_1^{(n)}, \dots, D_{j-1}^{(n)}, D_{j+2}^{(n)}, \dots, D_p^{(n)} D_{j+1}^{(n)}) \\ &\xrightarrow{n \rightarrow \infty} \tau(d_j) \cdot \Delta(\mathbb{C}_j \gamma; d_1, \dots, d_{j-1}, d_{j+2}, \dots, d_p d_{j+1}) \\ &= \Delta(\gamma; d_1, \dots, d_p), \end{aligned}$$

as desired.

Illustration of Step 2.

Consider the permutation γ in Γ_4 given by

$$\gamma(1) = 3 \quad \text{and} \quad \gamma(2) = 4.$$

Then if Z_1 and Z_2 are independent $\text{GUE}(n, \frac{1}{n})$ matrices we have

$$\Delta_n(\gamma; D_1, D_2, D_3, D_4) = \mathbb{E}\{\text{tr}_n(Z_1 D_1 Z_2 D_2 Z_3 D_3 Z_4 D_4)\}$$

where $Z_3 = Z_1$ and $Z_4 = Z_2$. Writing

$$Z_j = \sum_{k,l=1}^n Z_j(k,l)e(k,l) \quad \text{and} \quad D_j = \sum_{k,l=1}^n D_j(k,l)e(k,l),$$

with $e(k,l)$, $1 \leq k, l \leq n$ the $n \times n$ matrix units, we find that

Illustration of Step 2 (continued).

$$\begin{aligned}
 & \Delta_n(\gamma; D_1, D_2, D_3, D_4) \\
 &= \sum_{1 \leq i_1, j_1, l_1, k_1, \dots, i_4, j_4, l_4, k_4 \leq n} \mathbb{E} \{ Z_1(i_1, j_1) D_1(k_1, l_1) \cdots Z_4(i_4, j_4) D_4(k_4, l_4) \} \\
 & \quad \cdot \operatorname{tr}_n(e(i_1, j_1) e(k_1, l_1) \cdots e(i_4, j_4) e(k_4, l_4)) \\
 &= \frac{1}{n} \sum_{1 \leq i_1, j_1, \dots, i_4, j_4 \leq n} \mathbb{E} \{ Z_1(i_1, j_1) D_1(j_1, i_2) \cdots Z_4(i_4, j_4) D_4(j_4, i_1) \} \\
 &= \frac{1}{n} \sum_{1 \leq i_1, \dots, i_4 \leq n} \mathbb{E} \{ Z_1(i_1, i_2) D_1(i_2, i_2) \cdots Z_4(i_4, i_1) D_4(i_1, i_1) \} \\
 &= \frac{1}{n} \sum_{1 \leq i_1, \dots, i_4 \leq n} D_2(i_2, i_2) \cdots D_4(i_1, i_1) \mathbb{E} \{ Z_1(i_1, i_2) Z_2(i_2, i_3) \cdots Z_4(i_4, i_1) \}
 \end{aligned}$$

Illustration of Step 2 (continued).

Continuing the calculation we have

$$\begin{aligned}
 & \Delta_n(\gamma; D_1, D_2, D_3, D_4) \\
 &= \frac{1}{n} \sum_{1 \leq i_1, \dots, i_4 \leq n} D_2(i_2, i_2) \cdots D_4(i_1, i_1) \mathbb{E}\{Z_1(i_1, i_2) Z_2(i_2, i_3) \cdots Z_4(i_4, i_1)\} \\
 &= \frac{1}{n} \sum_{1 \leq i_1, \dots, i_4 \leq n} D_2(i_2, i_2) \cdots D_4(i_1, i_1) \\
 & \quad \times \mathbb{E}\{Z_1(i_1, i_2) Z_1(i_3, i_4)\} \mathbb{E}\{Z_2(i_2, i_3) Z_2(i_4, i_1)\} \\
 &= \frac{1}{n} \sum_{1 \leq i_1, \dots, i_4 \leq n} D_2(i_2, i_2) \cdots D_4(i_1, i_1) (n^{-1} \delta_{i_1, i_4} \delta_{i_2, i_3}) (n^{-1} \delta_{i_2, i_1} \delta_{i_3, i_4}) \\
 &= \frac{1}{n^3} \sum_{1 \leq i_1, \dots, i_4 \leq n} D_2(i_2, i_2) \cdots D_4(i_1, i_1) \delta_{i_1, i_4} \delta_{i_2, i_3} \delta_{i_2, i_1} \delta_{i_3, i_4}.
 \end{aligned}$$

Illustration of Step 2 (continued).

Since we have assumed that

$$K := \max_{j=1,2,3,4} \left(\sup_{n \in \mathbb{N}} \|D_j\| \right) < \infty,$$

it follows that

$$\begin{aligned} |\Delta_n(\gamma; D_1, D_2, D_3, D_4)| &\leq \frac{K}{n^3} \sum_{1 \leq i_1, \dots, i_4 \leq n} \delta_{i_1, i_4} \delta_{i_2, i_3} \delta_{i_2, i_1} \delta_{i_3, i_4} \\ &= \frac{K}{n^3} n \\ &\xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

as desired. ■

Step II in general

For general p in \mathbb{N} and γ in Γ_p one is lead, by the argument illustrated above, to consider the sum

$$\frac{1}{n} \sum_{1 \leq i_1, \dots, i_p \leq n} \prod_{j=1}^p (n^{-1} \delta_{i_j, i_{\gamma(j)+1}}) = n^{-p-1+d(\gamma)},$$

where $d(\gamma)$ denotes the number of equivalence classes for the equivalence relation \sim_γ on $\{1, 2, \dots, p\}$ generated by the condition

$$j \sim_\gamma \gamma(j) + 1, \quad (j = 1, 2, \dots, p).$$

One may then prove that

- $p + 1 - d(\gamma) \in 2\mathbb{N}_0$.
- $p + 1 - d(\gamma) = 0 \iff \gamma$ is non-crossing.