Lecture IV: Convergence of norms of polynomials of GUE’s and applications

Steen Thorbjørnsen, University of Aarhus
Convergence of norms of polynomials of GUE’s – the hard part

**Theorem [Haagerup+T].** For each \( n \in \mathbb{N} \), let \( X_1^{(n)}, \ldots, X_r^{(n)} \) be independent random matrices from the class \( \text{GUE}(n, \frac{1}{n}) \), and let \( \{x_1, \ldots, x_r\} \) be a semi-circular system in a \( C^* \)-probability space \((\mathcal{A}, \tau)\) with a faithful state \( \tau \).

Then for any polynomial \( p \) in \( r \) non-commuting variables, we have

\[
\limsup_{n \to \infty} \|p(X_1^{(n)}, X_2^{(n)}, \ldots, X_r^{(n)})\| \leq \|p(x_1, \ldots, x_r)\|,
\]

and hence combined with a proposition from yesterday's talk,

\[
\lim_{n \to \infty} \|p(X_1^{(n)}, X_2^{(n)}, \ldots, X_r^{(n)})\| = \|p(x_1, \ldots, x_r)\|.
\]
Outline of proof Theorem.

**Theorem A.** For each \( n \) in \( \mathbb{N} \), let \( X^{(n)}_1, X^{(n)}_2, \ldots, X^{(n)}_r \) be independent random matrices from \( \text{GUE}(n, \frac{1}{n}) \) and let \( \{x_1, x_2, \ldots, x_r\} \) be a semicircular system in a \( C^* \)-probability space \((\mathcal{A}, \tau)\). Let further \( p \) be a selfadjoint polynomial of degree \( d \) in \( r \) non-commuting variables and put

\[
Q_n = p(X^{(n)}_1, X^{(n)}_2, \ldots, X^{(n)}_r) \in M_n(\mathbb{C}), \quad (n \in \mathbb{N}),
\]

\[
q = p(x_1, x_2, \ldots, x_r).
\]

Then for any function \( \varphi \) in \( C^\infty_c(\mathbb{R}, \mathbb{R}) + \mathbb{R} \) we have

\[
\mathbb{E}\{\text{tr}_n[\varphi(Q_n)]\} = \tau[\varphi(q)] + O(n^{-2})
\]

\[
\mathbb{V}\{\text{tr}_n[\varphi(Q_n)]\} \leq \frac{C(p)}{n^2} \mathbb{E}\{\text{tr}_n[|\varphi'|^2(Q_n)]\left(1 + \sum_{j=1}^r \|X^{(n)}_j\|^{d-1}\right)^2\},
\]

for some constant \( C(p) \) depending only on \( p \).
Outline of proof Theorem (continued)

**Corollary B.** Let \( \{X_1^{(n)}, X_2^{(n)}, \ldots, X_r^{(n)}\} \), \( \{x_1, x_2, \ldots, x_r\} \), \( p, Q_n \) and \( q \) be as in Theorem A.

(i) If \( \varphi \in C_c^\infty(\mathbb{R}, \mathbb{R}) + \mathbb{R} \) such that \( \varphi = 0 \) on \( \text{sp}(q) \), then

\[
\mathbb{E}\{\text{tr}_n[\varphi(Q_n)]\} = O(n^{-2}).
\]

(ii) If \( \varphi \in C_c^\infty(\mathbb{R}, \mathbb{R}) + \mathbb{R} \) such that \( \varphi' = 0 \) on \( \text{sp}(q) \), then

\[
\nabla\{\text{tr}_n[\varphi(Q_n)]\} = O(n^{-4}).
\]
Theorem C. Let \( \{X_1^{(n)}, X_2^{(n)}, \ldots, X_r^{(n)}\}, \{x_1, x_2, \ldots, x_r\}, p, Q_n \) and \( q \) be as in Theorem A. Then with probability one we have

\[
\forall \epsilon > 0: \ sp(Q_n) \subseteq sp(q) + ] - \epsilon, \epsilon[ \ 	ext{for all sufficiently large } n.
\]
Proof of Theorem C.

It suffices to prove that for any fixed $\epsilon > 0$ we have with probability one that

$$\text{sp}(Q_n) \subseteq \text{sp}(q) + ] - \epsilon, \epsilon[,$$

for all sufficiently large $n$.

So let $\epsilon > 0$ be given, and choose a function $\varphi$ in $C_c^\infty(\mathbb{R}) + \mathbb{R}$, such that

- $\varphi(t) \in [0, 1]$ for all $t$ in $\mathbb{R}$,
- $\varphi(t) = 0$ for all $t$ in $\text{sp}(q) + ] - \epsilon/2, \epsilon/2[$,
- $\varphi(t) = 1$ for all $t$ outside $\text{sp}(q) + ] - \epsilon, \epsilon[$.
Note then that
\[
\#(\text{eigenvalues for } Q_n(\omega) \text{ outside } \text{sp}(q)+] - \epsilon, \epsilon[)
\]
\[
= ntr_n[1_{\text{sp}(q)+] - \epsilon, \epsilon[}c(Q_n(\omega))]
\]
\[
\leq ntr_n[\phi(Q_n(\omega))].
\]

Since the left hand side of (1) is an integer, it then suffices to show that
\[
\text{ntr}_n[\phi(Q_n)] \overset{\text{a.s.}}{\longrightarrow} 0, \quad \text{as } n \to \infty.
\]
Proof of Theorem C (continued).

To prove (2) it suffices to show that

$$n\mathbb{E}\{\text{tr}_n[\varphi(Q_n)]\} \to 0, \quad \text{as } n \to \infty$$

and

$$\sum_{n=1}^{\infty} n^2 \mathbb{V}\{\text{tr}_n[\varphi(Q_n)]\} < \infty.$$  

But since $\varphi = 0$ and $\varphi' = 0$ on $\text{sp}(q)$, Corollary B asserts that

$$\mathbb{E}\{\text{tr}_n[\varphi(Q_n)]\} = O(n^{-2})$$

and

$$\mathbb{V}\{\text{tr}_n[\varphi(Q_n)]\} = O(n^{-4}).$$
For each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_r^{(n)}$ be independent random matrices from the class $\text{GUE}(n, \frac{1}{n})$, and let $\{x_1, \ldots, x_r\}$ be a semi-circular system in a $C^*$-probability space $(\mathcal{A}, \tau)$ with a faithful state $\tau$. Then for any polynomial $p$ in $r$ non-commuting variables, we have with probability one that

$$\limsup_{n \to \infty} \|p(X_1^{(n)}, X_2^{(n)}, \ldots, X_r^{(n)})\| \leq \|p(x_1, \ldots, x_r)\|.$$
Sketch of Proof of Theorem

- Put $Q_n = p(X_1^{(n)}, \ldots, X_r^{(n)})$ and $q = p(x_1, \ldots, x_r)$.

- Note that

$$\|Q_n\|^2 = \lambda_{\text{max}}(Q_n^*Q_n) = \max\{t \mid t \in \text{sp}(Q_n^*Q_n)\}$$

and similarly

$$\|q\|^2 = \max\{t \mid t \in \text{sp}(q^*q)\}.$$ 

- From Theorem C we have with probability one that

$$\forall \epsilon > 0: \text{sp}(Q_n^*Q_n) \subseteq \text{sp}(q^*q) + ]-\epsilon, \epsilon[,$$

for all sufficiently large $n$.

- Consequently, with probability one

$$\forall \epsilon > 0: \|Q_n\|^2 \leq \|q\|^2 + \epsilon,$$

for all sufficiently large $n$. 

The operator algebras associated to free groups

Consider the free group \( \mathbb{F}_r \) on \( r \) generators, and consider the Hilbert space

\[
\ell^2(\mathbb{F}_r) = \{ f : \mathbb{F}_r \to \mathbb{C} \mid \sum_{g \in \mathbb{F}_r} |f(g)|^2 < \infty \}.
\]

For each \( g \) in \( \mathbb{F}_r \), let \( \lambda(g) : \ell^2(\mathbb{F}_r) \to \ell^2(\mathbb{F}_r) \) be the linear operator given by:

\[
(\lambda(g)f)(h) = f(g^{-1}h), \quad (f \in \ell^2(\mathbb{F}_r), \ h \in \mathbb{F}_r).
\]

Then \( \lambda(g) \) is a unitary on \( \ell^2(\mathbb{F}_r) \), and we put

\[
C^*_\text{red}(\mathbb{F}_r) = C^*\left(\{\lambda(g) \mid g \in \mathbb{F}_r\}\right) \subseteq \mathcal{B}(\ell^2(\mathbb{F}_r))
\]

\[
\mathcal{L}(\mathbb{F}_r) = \mathcal{W}^*\left(\{\lambda(g) \mid g \in \mathbb{F}_r\}\right) \subseteq \mathcal{B}(\ell^2(\mathbb{F}_r)).
\]

The mapping

\[
\lambda : g \mapsto \lambda(g) : \mathbb{F}_r \to \mathcal{B}(\ell^2(\mathbb{F}_r)),
\]

is called the left regular representation of \( \mathbb{F}_r \).
Direct product and sum of the matrix algebras

Consider the $C^*$-algebra

$$\prod_{n \in \mathbb{N}} M_n(\mathbb{C}) = \{(T_n)_{n \in \mathbb{N}} \mid T_n \in M_n(\mathbb{C}), \sup_{n \in \mathbb{N}} \|T_n\| < \infty\}$$

and the closed ideal

$$\bigoplus_{n \in \mathbb{N}} M_n(\mathbb{C}) = \{(T_n)_{n \in \mathbb{N}} \mid T_n \in M_n(\mathbb{C}), \lim_{n \to \infty} \|T_n\| = 0\}.$$

Then the quotient

$$\prod_{n \in \mathbb{N}} M_n(\mathbb{C}) / \bigoplus_{n \in \mathbb{N}} M_n(\mathbb{C})$$

is again a $C^*$-algebra with norm given by

$$\|[(T_n)_{n \in \mathbb{N}}]\| = \limsup_{n \to \infty} \|T_n\|.$$
Corollary [Haagerup+T].

For any $r$ in $\mathbb{N} \cup \{\infty\}$, the $C^*$-algebra $C^*_{\text{red}}(F_r)$ has a unital embedding into the quotient $C^*$-algebra

$$\prod_n M_n(\mathbb{C}) \big/ \sum_n M_n(\mathbb{C}),$$

In particular, $C^*_{\text{red}}(F_r)$ is an MF-algebra in the sense of Blackadar and Kirchberg.
**Lemma.** Let \( \{x_1, x_2\} \) be a free semi-circular system in a \( \mathcal{W}^* \)-probability space \((\mathcal{A}, \tau)\), and consider the \( C^1 \)-function \( \varphi : \mathbb{R} \rightarrow [0, 1] \) given by

\[
\varphi(t) = \begin{cases} 
0, & \text{if } t \leq -2, \\
\frac{1}{2\pi} \int_{-2}^{t} \sqrt{4 - t^2} \, dt, & \text{if } t \in ]-2, 2[, \\
1, & \text{if } t \geq 2.
\end{cases}
\]

Then

\[
\{e^{2\pi i \varphi(x_1)}, e^{2\pi i \varphi(x_2)}\} \overset{\mathcal{D}}{=} \{\lambda(g_1), \lambda(g_2)\}.
\]

where \( g_1, g_2 \) are the generators of \( \mathbb{F}_2 \). As a consequence

\[
C^*_\text{red}(\mathbb{F}_2) = C^*\{\lambda(g_1), \lambda(g_2)\} \simeq C^*\{e^{2\pi i \varphi(x_1)}, e^{2\pi i \varphi(x_2)}\} \subseteq C^*\{x_1, x_2\},
\]
and

\[
\mathcal{L}(\mathbb{F}_2) = \mathcal{W}^*\{\lambda(g_1), \lambda(g_2)\} \simeq \mathcal{W}^*\{e^{2\pi i \varphi(x_1)}, e^{2\pi i \varphi(x_2)}\} = \mathcal{W}^*\{x_1, x_2\}.
\]
Proof of Lemma.

We have to show that \( e^{2\pi i \varphi(x_1)} \), \( e^{2\pi i \varphi(x_2)} \) are freely independent Haar unitaries.

We know that \( e^{2\pi i \varphi(x_1)} \), \( e^{2\pi i \varphi(x_2)} \) are freely independent, since \( x_1 \) and \( x_2 \) are.

Furthermore, for any integer \( p \)

\[
\tau\left(\left(e^{2\pi i \varphi(x_j)}\right)^p\right) = \tau\left(e^{2p\pi i \varphi(x_j)}\right) = \int_{\mathbb{R}} e^{2p\pi i \varphi(t)} \mu_{x_j}(dt)
\]

\[
= \int_{\mathbb{R}} e^{2p\pi is} \varphi(\mu_{x_j})(ds) = \int_0^1 e^{2p\pi is} \, ds
\]

\[
= \delta_{0,p},
\]

as desired.  ■
Proof of Corollary in the case $r = 2$

According to the previous lemma, it suffices to show that we have an embedding

$$C^*\{x_1, x_2\} \hookrightarrow \prod_n M_n(\mathbb{C}) \bigg/ \sum_n M_n(\mathbb{C}),$$

where $\{x_1, x_2\}$ is a free semi-circular system in a $W^*$-probability space $(\mathcal{A}, \tau)$.

For each $n$ in $\mathbb{N}$, let $X_1^{(n)}$ and $X_2^{(n)}$ be two independent random matrices from $\text{GUE}(n, \frac{1}{n})$, and choose one $\omega$ such that

$$\forall p \in \mathbb{C}\langle X_1, X_2 \rangle : \lim_{n \to \infty} \|p(X_1^{(n)}, X_2^{(n)})\| = \|p(x_1, x_2)\|.$$

Then we may consider the operators

$$x_j(\omega) = \left[ (X_j^{(n)}(\omega))_{n \in \mathbb{N}} \right] \in \prod_n M_n(\mathbb{C}) \bigg/ \sum_n M_n(\mathbb{C}), \quad (j = 1, 2).$$
Proof of Corollary (continued)

Now for any $p$ in $\mathbb{C}\langle X_1, X_2 \rangle$ we have

$$\|p(x_1(\omega), x_2(\omega))\| = \| \left[ (p(X_1^{(n)}(\omega), X_2^{(n)}(\omega)))_{n \in \mathbb{N}} \right] \|$$

$$= \limsup_{n \to \infty} \|p(X_1^{(n)}(\omega), X_2^{(n)}(\omega))\|$$

$$= \|p(x_1, x_2)\|.$$

Hence the expression:

$$p(x_1, x_2) \mapsto p(x_1(\omega), x_2(\omega)) : \text{Alg}\{x_1, x_2\} \to \text{Alg}\{x_1(\omega), x_2(\omega)\},$$

is a well-defined isometric $*$-homomorphism, so it extends by continuity to an isometric $*$-isomorphism

$$\nu : C^*\{x_1, x_2\} \simeq C^*\{x_1(\omega), x_2(\omega)\} \subseteq \prod_n M_n(\mathbb{C}) \bigg/ \sum_n M_n(\mathbb{C}),$$

as desired. ■
Let $\mathcal{A}$ be a separable, unital $C^*$-algebra and consider the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$. Consider, further, the Calkin algebra:

$$C(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

Then

$$\text{Ext}(\mathcal{A}) = \{ *\text{-monomorphisms } \pi : \mathcal{A} \to C(\mathcal{H}) \}/\sim$$

where

$$\pi_1 \sim \pi_2 \iff \exists u \in \mathcal{U}(\mathcal{B}(\mathcal{H})) \forall a \in \mathcal{A} : \pi_1(a) = \rho(u)\pi_2(a)\rho(u)^*,$$

and $\rho : \mathcal{B}(\mathcal{H}) \to C(\mathcal{H})$ is the quotient mapping.
The $\text{Ext}$ semi-group (continued)

$\text{Ext}(\mathcal{A})$ has a natural semi-group structure:

$$\pi_1 \oplus \pi_2(a) = (\pi_1(a), \pi_2(a)) \in C(\mathcal{H}) \oplus C(\mathcal{H}) \hookrightarrow C(\mathcal{H} \oplus \mathcal{H}) \cong C(\mathcal{H}),$$

where the embedding $C(\mathcal{H}) \oplus C(\mathcal{H}) \hookrightarrow C(\mathcal{H} \oplus \mathcal{H})$ is given by

$$([S], [T]) \mapsto \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}, \quad (S, T \in B(\mathcal{H})).$$
Main Application

**Corollary [Haagerup+T].** For any \( r \) in \( \{2, 3, 4, \ldots\} \), 
\( \text{Ext}(C_{\text{red}}^*(\mathbb{F}_r)) \) is not a group.

**Historical notes.**

1973 \( \text{Ext}(\mathcal{A}) \) introduced [Brown, Douglas and Fillmore].

1976 \( \text{Ext}(\mathcal{A}) \) is a group for any unital, separable, nuclear \( C^* \)-algebra [Choi and Effros].

1976 \( \text{Ext}(\mathcal{A}) \) always has a unit (\( \mathcal{A} \) separable, unital \( C^* \)-algebra) [Voiculescu].

1978 There is a projection \( p \) in \( B(\ell^2(\mathbb{F}_2)) \), such that the semi-group \( \text{Ext}(C_{\text{red}}^*(\mathbb{F}_2) \vee \{p\}) \) is not a group [Anderson].

1980’s KK-theory....
Yet another result of Voiculescu’s

**Theorem [Voiculescu, 1991].** Suppose there exists a sequence \((\pi_n)_{n \in \mathbb{N}}\) of unitary representations

\[ \pi_n : \mathbb{F}_r \to U(n), \]

such that

\[
\lim_{n \to \infty} \left\| \sum_{h \in \mathbb{F}_r} f(h) \pi_n(h) \right\| = \left\| \sum_{h \in \mathbb{F}_r} f(h) \lambda(h) \right\|, 
\]

for any function \( f : \mathbb{F}_r \to \mathbb{C} \) with finite support, and where \( \lambda \) is the left regular representation of \( \mathbb{F}_r \).

Then \( \operatorname{Ext}(C^*_\text{red}(\mathbb{F}_r)) \) cannot be a group.
Proof of: $\text{Ext}(C^*_{\text{red}}(\mathbb{F}_2))$ is not a group.

We prove the existence of a sequence $(\pi_n)_{n \in \mathbb{N}}$ of unitary representations:

$$\pi_n : \mathbb{F}_2 \to \mathcal{U}(n)$$

as in Voiculescu’s 1991 theorem.

Let $g_1, g_2$ be the canonical generators of $\mathbb{F}_2$ and let $\{x_1, x_2\}$ be a semicircular system in a $C^*$-probability space $(\mathcal{A}, \tau)$. Consider again the function $\varphi : \mathbb{R} \to [0, 1]$ given by

$$\varphi(t) = \begin{cases} 
0, & \text{if } t \leq -2, \\
\frac{1}{2\pi} \int_{-2}^{t} \sqrt{4 - t^2} \, dt, & \text{if } t \in ]-2, 2[, \\
1, & \text{if } t \geq 2.
\end{cases}$$

It follows then from the previous lemma that

$$\|p(e^{2\pi i \varphi(x_1)}, e^{2\pi i \varphi(x_2)})\| = \|p(\lambda(g_1), \lambda(g_2))\|,$$

for any polynomial $p$ in 2 non-commuting variables.
Proof of: $\text{Ext}(C^*_\text{red}(\mathbb{F}_2))$ is not a group (continued).

Now, for each $n$ in $\mathbb{N}$, let $X_1^{(n)}, X_2^{(n)}$ be independent random matrices from $\text{GUE}(n, \frac{1}{n})$, and consider the random unitaries:

$$e^{2\pi i \varphi(X_1^{(n)})}, e^{2\pi i \varphi(X_2^{(n)})} \in \mathcal{U}(n).$$

By the universal property of $\mathbb{F}_2$ we have

$$\forall n \in \mathbb{N} \ \forall \omega \in \Omega \ \exists \pi_{n,\omega} : \mathbb{F}_2 \to \mathcal{U}(n) : \pi_{n,\omega}(g_j) = e^{2\pi i \varphi(X_j^{(n)}(\omega))}.$$

**Claim:** For almost all $\omega$, the representation $(\pi_{n,\omega})_{n \in \mathbb{N}}$ works!

Consider for example the function $f : \mathbb{F}_2 \to \mathbb{C}$ given by

$$f = 1_{g_1g_2} + 1_{g_1^2g_2}.$$

Then for almost all $\omega$, we have....
Proof of: $\text{Ext}(C^*_\text{red}(\mathbb{F}_2))$ is not a group (continued).

$$\lim_{n \to \infty} \left\| \sum_{g \in \mathbb{F}_2} f(g) \pi_{n,\omega}(g) \right\| = \lim_{n \to \infty} \left\| \pi_{n,\omega}(g_1 g_2) + \pi_{n,\omega}(g_1^2 g_2) \right\|$$

$$= \lim_{n \to \infty} \left\| e^{2\pi i \varphi(X_1^{(n)}(\omega))} e^{2\pi i \varphi(X_2^{(n)}(\omega))} + e^{4\pi i \varphi(X_1^{(n)}(\omega))} e^{2\pi i \varphi(X_2^{(n)}(\omega))} \right\|$$

$$\approx \lim_{n \to \infty} \left\| p(X_1^{(n)}(\omega)) p(X_2^{(n)}(\omega)) + p(X_1^{(n)}(\omega))^2 p(X_2^{(n)}(\omega)) \right\|$$

$$= \left\| p(x_1) p(x_2) + p(x_1)^2 p(x_2) \right\|$$

$$\approx \left\| e^{2\pi i \varphi(x_1)} e^{2\pi i \varphi(x_2)} + e^{4\pi i \varphi(x_1)} e^{2\pi i \varphi(x_2)} \right\|$$

$$= \left\| \lambda(g_1) \lambda(g_2) + \lambda(g_1)^2 \lambda(g_2) \right\|$$

$$= \left\| \sum_{g \in \mathbb{F}_2} f(g) \lambda(g) \right\|.$$
Proof of: \( \text{Ext}(\mathbb{C}_\text{red}(\mathbb{F}_2)) \) is not a group (continued).

where \( p \) is a polynomial in one variable, which approximates \( e^{2\pi i \varphi} \) well on, say, \([-3, 3]\).
The lack of projections in $C^*_{\text{red}}(\mathbb{F}_2)$.

**Theorem D [Voiculescu].** Let $\{x_1, \ldots, x_r\}$ be a semi-circular system in a $C^*$-probability space $(A, \tau)$. Then for any projection $e$ in $C^*(x_1, \ldots, x_r, 1)$, we have

$$\tau(e) \in \{0, 1\}.$$

In particular, there are no non-trivial projections in $C^*(x_1, \ldots, x_r, 1)$ (since $\tau$ is faithful!).

**Corollary [Pimsner-Voiculescu].** For any $r \in \{2, 3, 4, \ldots\}$, there are no non-trivial projections in $C^*_{\text{red}}(\mathbb{F}_r)$. 
Sketch of proof of Theorem D.

(1) Choose a selfadjoint polynomial \( p \) in \( r \) non-commuting variables, such that

\[
\| e - p(x_1, \ldots, x_r) \| < \frac{1}{8},
\]

and put \( q = p(x_1, \ldots, x_r) \).

(2) Choose a function \( \varphi \) in \( \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}) \) such that \( \varphi(q) \) is a projection in \( C^*(x_1, \ldots, x_r, 1) \) and

\[
\tau(e) = \tau(\varphi(q)).
\]
Sketch of proof of Theorem D (continued).

(3) Put $Q_n = p(X_1^{(n)}, \ldots, X_r^{(n)})$ and use Theorem C to show that

$$\varphi(Q_n)$$ is a projection in $M_n(\mathbb{C})$ eventually, with probability one. In particular

$$n \text{tr}_n [\varphi(Q_n)] \in \{0, 1, 2, \ldots, n\},$$

eventually with probability one.

(4) Use Theorem A to show that

$$n \text{tr}_n [\varphi(Q_n)] - n \tau [\varphi(q)] \xrightarrow{\text{a.s.}} 0, \quad \text{as } n \to \infty.$$