

Lecture IV: Convergence of norms of polynomials of GUE's and applications

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Convergence of norms of polynomials of GUE's – the hard part

Theorem [Haagerup+T]. For each $n \in \mathbb{N}$, let $X_1^{(n)}, \dots, X_r^{(n)}$ be independent random matrices from the class $\text{GUE}(n, \frac{1}{n})$, and let $\{x_1, \dots, x_r\}$ be a semi-circular system in a C^* -probability space (\mathcal{A}, τ) with a faithful state τ .

Then for any polynomial p in r non-commuting variables, we have with probability one that

$$\limsup_{n \rightarrow \infty} \|p(X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)})\| \leq \|p(x_1, \dots, x_r)\|,$$

and hence combined with a proposition from yesterday's talk,

$$\lim_{n \rightarrow \infty} \|p(X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)})\| = \|p(x_1, \dots, x_r)\|.$$

Outline of proof Theorem.

Theorem A. For each n in \mathbb{N} , let $X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)}$ be independent random matrices from $\text{GUE}(n, \frac{1}{n})$ and let $\{x_1, x_2, \dots, x_r\}$ be a semicircular system in a C^* -probability space (\mathcal{A}, τ) . Let further p be a selfadjoint polynomial of degree d in r non-commuting variables and put

$$Q_n = p(X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)}) \in M_n(\mathbb{C}), \quad (n \in \mathbb{N}),$$

$$q = p(x_1, x_2, \dots, x_r).$$

Then for any function φ in $C_c^\infty(\mathbb{R}, \mathbb{R}) + \mathbb{R}$ we have

$$\mathbb{E}\{\text{tr}_n[\varphi(Q_n)]\} = \tau[\varphi(q)] + O(n^{-2})$$

$$\mathbb{V}\{\text{tr}_n[\varphi(Q_n)]\} \leq \frac{C(p)}{n^2} \mathbb{E}\left\{\text{tr}_n[|\varphi'|^2(Q_n)] \left(1 + \sum_{j=1}^r \|X_j^{(n)}\|^{d-1}\right)^2\right\},$$

for some constant $C(p)$ depending only on p .

Outline of proof Theorem (continued)

Corollary B. Let $\{X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)}\}$, $\{x_1, x_2, \dots, x_r\}$, p , Q_n and q be as in Theorem A.

(i) If $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R}) + \mathbb{R}$ such that $\varphi = 0$ on $\text{sp}(q)$, then

$$\mathbb{E}\{\text{tr}_n[\varphi(Q_n)]\} = O(n^{-2}).$$

(ii) If $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R}) + \mathbb{R}$ such that $\varphi' = 0$ on $\text{sp}(q)$, then

$$\mathbb{V}\{\text{tr}_n[\varphi(Q_n)]\} = O(n^{-4}).$$

Outline of proof Theorem (continued)

Theorem C. Let $\{X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)}\}$, $\{x_1, x_2, \dots, x_r\}$, p , Q_n and q be as in Theorem A. Then with probability one we have

$$\forall \epsilon > 0: \text{sp}(Q_n) \subseteq \text{sp}(q) +] - \epsilon, \epsilon[\quad \text{for all sufficiently large } n.$$

Proof of Theorem C.

It suffices to prove that for any fixed $\epsilon > 0$ we have with probability one that

$$\text{sp}(Q_n) \subseteq \text{sp}(q) +] - \epsilon, \epsilon[, \quad \text{for all sufficiently large } n.$$

So let $\epsilon > 0$ be given, and choose a function φ in $C_c^\infty(\mathbb{R}) + \mathbb{R}$, such that

- $\varphi(t) \in [0, 1]$ for all t in \mathbb{R} ,
- $\varphi(t) = 0$ for all t in $\text{sp}(q) +] - \epsilon/2, \epsilon/2[$,
- $\varphi(t) = 1$ for all t outside $\text{sp}(q) +] - \epsilon, \epsilon[$.

Proof of Theorem C (continued).

Note then that

$$\begin{aligned}
 & \#(\text{eigenvalues for } Q_n(\omega) \text{ outside } \text{sp}(q)_+] - \epsilon, \epsilon[) \\
 &= n \text{tr}_n [\mathbf{1}_{(\text{sp}(q)_+] - \epsilon, \epsilon[}^c(Q_n(\omega)))] \quad (1) \\
 &\leq n \text{tr}_n [\varphi(Q_n(\omega))].
 \end{aligned}$$

Since the left hand side of (1) is an integer, it then suffices to show that

$$n \text{tr}_n [\varphi(Q_n)] \xrightarrow{\text{a.s.}} 0, \quad \text{as } n \rightarrow \infty. \quad (2)$$

Proof of Theorem C (continued).

To prove (2) it suffices to show that

$$n\mathbb{E}\{\text{tr}_n[\varphi(Q_n)]\} \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

and

$$\sum_{n=1}^{\infty} n^2 \mathbb{V}\{\text{tr}_n[\varphi(Q_n)]\} < \infty.$$

But since $\varphi = 0$ and $\varphi' = 0$ on $\text{sp}(q)$, Corollary B asserts that

$$\mathbb{E}\{\text{tr}_n[\varphi(Q_n)]\} = O(n^{-2})$$

and

$$\mathbb{V}\{\text{tr}_n[\varphi(Q_n)]\} = O(n^{-4}).$$

Theorem.

For each $n \in \mathbb{N}$, let $X_1^{(n)}, \dots, X_r^{(n)}$ be independent random matrices from the class $\text{GUE}(n, \frac{1}{n})$, and let $\{x_1, \dots, x_r\}$ be a semi-circular system in a C^* -probability space (\mathcal{A}, τ) with a faithful state τ . Then for any polynomial p in r non-commuting variables, we have with probability one that

$$\limsup_{n \rightarrow \infty} \|p(X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)})\| \leq \|p(x_1, \dots, x_r)\|.$$

Sketch of Proof of Theorem

- Put $Q_n = p(X_1^{(n)}, \dots, X_r^{(n)})$ and $q = p(x_1, \dots, x_r)$.
- Note that

$$\|Q_n\|^2 = \lambda_{\max}(Q_n^* Q_n) = \max\{t \mid t \in \text{sp}(Q_n^* Q_n)\}$$

and similarly

$$\|q\|^2 = \max\{t \mid t \in \text{sp}(q^* q)\}.$$

- From Theorem C we have with probability one that

$$\forall \epsilon > 0: \text{sp}(Q_n^* Q_n) \subseteq \text{sp}(q^* q) +]-\epsilon, \epsilon[, \quad \text{for all sufficiently large } n.$$

- Consequently, with probability one

$$\forall \epsilon > 0: \|Q_n\|^2 \leq \|q\|^2 + \epsilon, \quad \text{for all sufficiently large } n.$$

The operator algebras associated to free groups

Consider the free group \mathbb{F}_r on r generators, and consider the Hilbert space

$$\ell^2(\mathbb{F}_r) = \{f: \mathbb{F}_r \rightarrow \mathbb{C} \mid \sum_{g \in \mathbb{F}_r} |f(g)|^2 < \infty\}.$$

For each g in \mathbb{F}_r , let $\lambda(g): \ell^2(\mathbb{F}_r) \rightarrow \ell^2(\mathbb{F}_r)$ be the linear operator given by:

$$(\lambda(g)f)(h) = f(g^{-1}h), \quad (f \in \ell^2(\mathbb{F}_r), h \in \mathbb{F}_r).$$

Then $\lambda(g)$ is a unitary on $\ell^2(\mathbb{F}_r)$, and we put

$$C_{\text{red}}^*(\mathbb{F}_r) = C^*(\{\lambda(g) \mid g \in \mathbb{F}_r\}) \subseteq \mathcal{B}(\ell^2(\mathbb{F}_r))$$

$$\mathcal{L}(\mathbb{F}_r) = W^*(\{\lambda(g) \mid g \in \mathbb{F}_r\}) \subseteq \mathcal{B}(\ell^2(\mathbb{F}_r)).$$

The mapping

$$\lambda: g \mapsto \lambda(g): \mathbb{F}_r \rightarrow \mathcal{B}(\ell^2(\mathbb{F}_r)),$$

is called the *left regular representation* of \mathbb{F}_r .

Direct product and sum of the matrix algebras

Consider the C^* -algebra

$$\prod_{n \in \mathbb{N}} M_n(\mathbb{C}) = \left\{ (T_n)_{n \in \mathbb{N}} \mid T_n \in M_n(\mathbb{C}), \sup_{n \in \mathbb{N}} \|T_n\| < \infty \right\}$$

and the closed ideal

$$\bigoplus_{n \in \mathbb{N}} M_n(\mathbb{C}) = \left\{ (T_n)_{n \in \mathbb{N}} \mid T_n \in M_n(\mathbb{C}), \lim_{n \rightarrow \infty} \|T_n\| = 0 \right\}.$$

Then the quotient

$$\prod_{n \in \mathbb{N}} M_n(\mathbb{C}) / \bigoplus_{n \in \mathbb{N}} M_n(\mathbb{C})$$

is again a C^* -algebra with norm given by

$$\|[(T_n)_{n \in \mathbb{N}}]\| = \limsup_{n \rightarrow \infty} \|T_n\|.$$

Corollary [Hagerup+T].

For any r in $\mathbb{N} \cup \{\infty\}$, the C^* -algebra $C_{\text{red}}^*(\mathbb{F}_r)$ has a unital embedding into the quotient C^* -algebra

$$\prod_n M_n(\mathbb{C}) / \sum_n M_n(\mathbb{C}),$$

In particular, $C_{\text{red}}^*(\mathbb{F}_r)$ is an MF-algebra in the sense of Blackadar and Kirchberg.

Semi-circular elements in $\mathcal{L}(\mathbb{F}_2)$

Lemma. Let $\{x_1, x_2\}$ be a free semi-circular system in a W^* -probability space (\mathcal{A}, τ) , and consider the C^1 -function $\varphi: \mathbb{R} \rightarrow [0, 1]$ given by

$$\varphi(t) = \begin{cases} 0, & \text{if } t \leq -2, \\ \frac{1}{2\pi} \int_{-2}^t \sqrt{4-t^2} dt, & \text{if } t \in]-2, 2[, \\ 1, & \text{if } t \geq 2. \end{cases}$$

Then

$$\{e^{2\pi i\varphi(x_1)}, e^{2\pi i\varphi(x_2)}\} \stackrel{*D}{=} \{\lambda(g_1), \lambda(g_2)\}.$$

where g_1, g_2 are the generators of \mathbb{F}_2 . As a consequence

$$C_{\text{red}}^*(\mathbb{F}_2) = C^*\{\lambda(g_1), \lambda(g_2)\} \simeq C^*\{e^{2\pi i\varphi(x_1)}, e^{2\pi i\varphi(x_2)}\} \subseteq C^*\{x_1, x_2\},$$

and

$$\mathcal{L}(\mathbb{F}_2) = W^*\{\lambda(g_1), \lambda(g_2)\} \simeq W^*\{e^{2\pi i\varphi(x_1)}, e^{2\pi i\varphi(x_2)}\} = W^*\{x_1, x_2\}.$$

Proof of Lemma.

We have to show that $e^{2\pi i\varphi(x_1)}, e^{2\pi i\varphi(x_2)}$ are freely independent Haar unitaries.

We know that $e^{2\pi i\varphi(x_1)}, e^{2\pi i\varphi(x_2)}$ are freely independent, since x_1 and x_2 are.

Furthermore, for any integer p

$$\begin{aligned}\tau((e^{2\pi i\varphi(x_j)})^p) &= \tau(e^{2p\pi i\varphi(x_j)}) = \int_{\mathbb{R}} e^{2p\pi i\varphi(t)} \mu_{x_j}(dt) \\ &= \int_{\mathbb{R}} e^{2p\pi is} \varphi(\mu_{x_j})(ds) = \int_0^1 e^{2p\pi is} ds \\ &= \delta_{0,p},\end{aligned}$$

as desired. ■

Proof of Corollary in the case $r = 2$

According to the previous lemma, it suffices to show that we have an embedding

$$C^*\{x_1, x_2\} \hookrightarrow \prod_n M_n(\mathbb{C}) / \sum_n M_n(\mathbb{C}),$$

where $\{x_1, x_2\}$ is a free semi-circular system in a W^* -probability space (\mathcal{A}, τ) .

For each n in \mathbb{N} , let $X_1^{(n)}$ and $X_2^{(n)}$ be two independent random matrices from $\text{GUE}(n, \frac{1}{n})$, and choose one ω such that

$$\forall p \in \mathbb{C}\langle X_1, X_2 \rangle: \lim_{n \rightarrow \infty} \|p(X_1^{(n)}, X_2^{(n)})\| = \|p(x_1, x_2)\|.$$

Then we may consider the operators

$$x_j(\omega) = [(X_j^{(n)}(\omega))_{n \in \mathbb{N}}] \in \prod_n M_n(\mathbb{C}) / \sum_n M_n(\mathbb{C}), \quad (j = 1, 2).$$

Proof of Corollary (continued)

Now for any p in $\mathbb{C}\langle X_1, X_2 \rangle$ we have

$$\begin{aligned} \|p(x_1(\omega), x_2(\omega))\| &= \left\| \left[(p(X_1^{(n)}(\omega), X_2^{(n)}(\omega)))_{n \in \mathbb{N}} \right] \right\| \\ &= \limsup_{n \rightarrow \infty} \|p(X_1^{(n)}(\omega), X_2^{(n)}(\omega))\| \\ &= \|p(x_1, x_2)\|. \end{aligned}$$

Hence the expression:

$$p(x_1, x_2) \mapsto p(x_1(\omega), x_2(\omega)) : \text{Alg}\{x_1, x_2\} \rightarrow \text{Alg}\{x_1(\omega), x_2(\omega)\},$$

is a well-defined isometric $*$ -homomorphism, so it extends by continuity to an isometric $*$ -isomorphism

$$\iota : C^*\{x_1, x_2\} \simeq C^*\{x_1(\omega), x_2(\omega)\} \subseteq \prod_n M_n(\mathbb{C}) / \sum_n M_n(\mathbb{C}),$$

as desired. \blacksquare

The Ext semi-group.

Let \mathcal{A} be a separable, unital C^* -algebra and consider the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$. Consider, further, the *Calkin algebra*:

$$\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

Then

$$\text{Ext}(\mathcal{A}) = \{ * \text{-monomorphisms } \pi: \mathcal{A} \rightarrow \mathcal{C}(\mathcal{H}) \} / \sim$$

where

$$\pi_1 \sim \pi_2 \iff \exists u \in \mathcal{U}(\mathcal{B}(\mathcal{H})) \forall a \in \mathcal{A}: \pi_1(a) = \rho(u)\pi_2(a)\rho(u)^*,$$

and $\rho: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$ is the quotient mapping.

The Ext semi-group (continued)

$\text{Ext}(\mathcal{A})$ has a natural semi-group structure:

$$\pi_1 \oplus \pi_2(a) = (\pi_1(a), \pi_2(a)) \in \mathcal{C}(\mathcal{H}) \oplus \mathcal{C}(\mathcal{H}) \hookrightarrow \mathcal{C}(\mathcal{H} \oplus \mathcal{H}) \simeq \mathcal{C}(\mathcal{H}),$$

where the embedding $\mathcal{C}(\mathcal{H}) \oplus \mathcal{C}(\mathcal{H}) \hookrightarrow \mathcal{C}(\mathcal{H} \oplus \mathcal{H})$ is given by

$$([S], [T]) \mapsto \left[\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \right], \quad (S, T \in \mathcal{B}(\mathcal{H})).$$

Main Application

Corollary [Haagerup+T]. For any r in $\{2, 3, 4, \dots\}$, $\text{Ext}(C_{\text{red}}^*(\mathbb{F}_r))$ is *not* a group.

Historical notes.

1973 $\text{Ext}(\mathcal{A})$ introduced [Brown, Douglas and Fillmore].

1976 $\text{Ext}(\mathcal{A})$ is a group for any unital, separable, nuclear C^* -algebra [Choi and Effros].

1976 $\text{Ext}(\mathcal{A})$ always has a unit (\mathcal{A} separable, unital C^* -algebra) [Voiculescu].

1978 There is a projection p in $\mathcal{B}(\ell^2(\mathbb{F}_2))$, such that the semi-group $\text{Ext}(C_{\text{red}}^*(\mathbb{F}_2) \vee \{p\})$ is not a group [Anderson].

1980's KK-theory....

Yet another result of Voiculescu's

Theorem [Voiculescu, 1991]. Suppose there exists a sequence $(\pi_n)_{n \in \mathbb{N}}$ of unitary representations

$$\pi_n: \mathbb{F}_r \rightarrow \mathcal{U}(n),$$

such that

$$\lim_{n \rightarrow \infty} \left\| \sum_{h \in \mathbb{F}_r} f(h) \pi_n(h) \right\| = \left\| \sum_{h \in \mathbb{F}_r} f(h) \lambda(h) \right\|,$$

for any function $f: \mathbb{F}_r \rightarrow \mathbb{C}$ with finite support, and where λ is the left regular representation of \mathbb{F}_r .

Then $\text{Ext}(C_{\text{red}}^*(\mathbb{F}_r))$ cannot be a group.

Proof of: $\text{Ext}(C_{\text{red}}^*(\mathbb{F}_2))$ is not a group.

We prove the existence of a sequence $(\pi_n)_{n \in \mathbb{N}}$ of unitary representations:

$$\pi_n: \mathbb{F}_2 \rightarrow \mathcal{U}(n)$$

as in Voiculescu's 1991 theorem.

Let g_1, g_2 be the canonical generators of \mathbb{F}_2 and let $\{x_1, x_2\}$ be a semicircular system in a C^* -probability space (\mathcal{A}, τ) . Consider again the function $\varphi: \mathbb{R} \rightarrow [0, 1]$ given by

$$\varphi(t) = \begin{cases} 0, & \text{if } t \leq -2, \\ \frac{1}{2\pi} \int_{-2}^t \sqrt{4-t^2} dt, & \text{if } t \in]-2, 2[, \\ 1, & \text{if } t \geq 2. \end{cases}$$

It follows then from the previous lemma that

$$\|p(e^{2\pi i \varphi(x_1)}, e^{2\pi i \varphi(x_2)})\| = \|p(\lambda(g_1), \lambda(g_2))\|,$$

for any polynomial p in 2 non-commuting variables.

Proof of: $\text{Ext}(C_{\text{red}}^*(\mathbb{F}_2))$ is not a group (continued).

Now, for each n in \mathbb{N} , let $X_1^{(n)}, X_2^{(n)}$ be independent random matrices from $\text{GUE}(n, \frac{1}{n})$, and consider the random unitaries:

$$e^{2\pi i \varphi(X_1^{(n)})}, e^{2\pi i \varphi(X_2^{(n)})} \in \mathcal{U}(n).$$

By the universal property of \mathbb{F}_2 we have

$$\forall n \in \mathbb{N} \forall \omega \in \Omega \exists \pi_{n,\omega} : \mathbb{F}_2 \rightarrow \mathcal{U}(n) : \pi_{n,\omega}(g_j) = e^{2\pi i \varphi(X_j^{(n)}(\omega))}.$$

Claim: For almost all ω , the representation $(\pi_{n,\omega})_{n \in \mathbb{N}}$ works!

Consider for example the function $f : \mathbb{F}_2 \rightarrow \mathbb{C}$ given by

$$f = 1_{g_1 g_2} + 1_{g_1^2 g_2}.$$

Then for almost all ω , we have.....

Proof of: $\text{Ext}(C_{\text{red}}^*(\mathbb{F}_2))$ is not a group (continued).

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left\| \sum_{g \in \mathbb{F}_2} f(g) \pi_{n,\omega}(g) \right\| &= \lim_{n \rightarrow \infty} \left\| \pi_{n,\omega}(g_1 g_2) + \pi_{n,\omega}(g_1^2 g_2) \right\| \\
 &= \lim_{n \rightarrow \infty} \left\| e^{2\pi i \varphi(X_1^{(n)}(\omega))} e^{2\pi i \varphi(X_2^{(n)}(\omega))} + e^{4\pi i \varphi(X_1^{(n)}(\omega))} e^{2\pi i \varphi(X_2^{(n)}(\omega))} \right\| \\
 &\approx \lim_{n \rightarrow \infty} \left\| \rho(X_1^{(n)}(\omega)) \rho(X_2^{(n)}(\omega)) + \rho(X_1^{(n)}(\omega))^2 \rho(X_2^{(n)}(\omega)) \right\| \\
 &= \left\| \rho(x_1) \rho(x_2) + \rho(x_1)^2 \rho(x_2) \right\| \\
 &\approx \left\| e^{2\pi i \varphi(x_1)} e^{2\pi i \varphi(x_2)} + e^{4\pi i \varphi(x_1)} e^{2\pi i \varphi(x_2)} \right\| \\
 &= \left\| \lambda(g_1) \lambda(g_2) + \lambda(g_1)^2 \lambda(g_2) \right\| \\
 &= \left\| \sum_{g \in \mathbb{F}_2} f(g) \lambda(g) \right\|,
 \end{aligned}$$

Proof of: $\text{Ext}(C_{\text{red}}^*(\mathbb{F}_2))$ is not a group (continued).

where p is a polynomial in one variable, which approximates $e^{2\pi i\varphi}$ well on, say, $[-3, 3]$.

The lack of projections in $C_{\text{red}}^*(\mathbb{F}_2)$.

Theorem D [Voiculescu]. Let $\{x_1, \dots, x_r\}$ be a semi-circular system in a C^* -probability space (\mathcal{A}, τ) . Then for any projection e in $C^*(x_1, \dots, x_r, \mathbf{1})$, we have

$$\tau(e) \in \{0, 1\}.$$

In particular, there are no non-trivial projections in $C^*(x_1, \dots, x_r, \mathbf{1})$ (since τ is faithful!).

Corollary [Pimsner-Voiculescu]. For any $r \in \{2, 3, 4, \dots\}$, there are no non-trivial projections in $C_{\text{red}}^*(\mathbb{F}_r)$.

Sketch of proof of Theorem D.

- (1) Choose a selfadjoint polynomial p in r non-commuting variables, such that

$$\|e - p(x_1, \dots, x_r)\| < \frac{1}{8},$$

and put $q = p(x_1, \dots, x_r)$.

- (2) Choose a function φ in $C_c^\infty(\mathbb{R}, \mathbb{R})$ such that $\varphi(q)$ is a projection in $C^*(x_1, \dots, x_r, \mathbf{1})$ and

$$\tau(e) = \tau(\varphi(q)).$$

Sketch of proof of Theorem D (continued).

(3) Put $Q_n = p(X_1^{(n)}, \dots, X_r^{(n)})$ and use Theorem C to show that

$\varphi(Q_n)$ is a projection in $M_n(\mathbb{C})$ eventually,

with probability one. In particular

$$n \text{tr}_n [\varphi(Q_n)] \in \{0, 1, 2, \dots, n\},$$

eventually with probability one.

(4) Use Theorem A to show that

$$n \text{tr}_n [\varphi(Q_n)] - n\tau [\varphi(q)] \xrightarrow{\text{a.s.}} 0, \quad \text{as } n \rightarrow \infty.$$