

Lecture I: Asymptotics for large GUE random matrices

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Random Matrices

Definition. Let (Ω, \mathcal{F}, P) be a probability space and let n be a positive integer. Then a random $n \times n$ matrix A on (Ω, \mathcal{F}, P) is an $n \times n$ -matrix

$$A = (a_{ij})_{1 \leq i, j \leq n},$$

where all the entries are complex valued random variables on (Ω, \mathcal{F}, P) . In other words, A is a measurable mapping

$$A: (\Omega, \mathcal{F}, P) \rightarrow (M_n(\mathbb{C}), \mathcal{B}(M_n(\mathbb{C}))),$$

when $M_n(\mathbb{C})$ is equipped with its Borel σ -algebra $\mathcal{B}(M_n(\mathbb{C}))$.

The spectral distribution of a selfadjoint random matrix

Let $A: \Omega \rightarrow M_n(\mathbb{C})$ be a selfadjoint random matrix, i.e., $A^*(\omega) = A(\omega)$ for all ω . Then for each ω we consider the ordered eigenvalues

$$\lambda_1(\omega) \leq \lambda_2(\omega) \leq \dots \leq \lambda_n(\omega)$$

of $A(\omega)$. For each fixed ω the *empirical eigenvalue distribution* of $A(\omega)$ is the probability measure

$$\mu_{A(\omega)} = \frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j(\omega)},$$

where δ_c denotes the Dirac measure at the constant c .

Then the *spectral distribution* of A is the *mixture* of the family $(\mu_{A(\omega)})_{\omega \in \Omega}$ with respect to P , i.e.....

The spectral distribution (continued)

$$\mu_A(B) = \int_{\Omega} \mu_{A(\omega)}(B) dP(\omega),$$

for any Borel subset B of \mathbb{R} . Thus, by a standard extension argument, for any bounded Borel function $f: \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\begin{aligned} \int_{\mathbb{R}} f(t) \mu_A(dt) &= \int_{\Omega} \left(\int_{\mathbb{R}} f(t) \mu_{A(\omega)}(dt) \right) dP(\omega) \\ &= \int_{\Omega} \frac{1}{n} \sum_{j=1}^n f(\lambda_j(\omega)) dP(\omega) \\ &= \int_{\Omega} \text{tr}_n(f(A(\omega))) dP(\omega) \\ &= \mathbb{E}\{\text{tr}_n(f(A))\}. \end{aligned}$$

The Gaussian Unitary Ensemble

Definition. By $\text{GUE}(n, \sigma^2)$ we denote the set of random $n \times n$ matrices $X = (x_{ij})_{1 \leq i, j \leq n}$, defined on (Ω, \mathcal{F}, P) , which satisfy the following conditions:

- $\forall i \geq j: x_{ij} = \overline{x_{ji}}$.
- the random variables x_{ij} , $1 \leq i \leq j \leq n$, are independent.
- $\forall i < j: \text{Re}(x_{ij}), \text{Im}(x_{ij}) \sim \text{i.i.d. } N(0, \frac{1}{2}\sigma^2)$.
- $\forall i: x_{ii} \sim N(0, \sigma^2)$.

The spectral distribution of a GUE random matrix.

Let X_n be a random matrix in $\text{GUE}(n, \frac{1}{n})$. For any bounded Borel function $f: \mathbb{R} \rightarrow \mathbb{R}$ we then have

$$\mathbb{E}\{\text{tr}_n(f(X_n))\} = \int_{\mathbb{R}} f(x)h_n(x) dx,$$

where the function $h_n: \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$h_n(x) = \frac{1}{\sqrt{2n}} \sum_{j=0}^{n-1} \varphi_j(\sqrt{\frac{n}{2}}x)^2,$$

and where

- $\varphi_0, \varphi_1, \varphi_2, \dots$, is the sequence of Hermite functions:

$$\varphi_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} H_k(x) \exp(-\frac{x^2}{2}), \quad (k \in \mathbb{N}_0),$$

- H_0, H_1, H_2, \dots , are the Hermite polynomials:

$$H_k(x) = (-1)^k \exp(x^2) \cdot \left(\frac{d^k}{dx^k} \exp(-x^2) \right).$$

The moment generating function of a GUE random matrix

Theorem. Let X_n be a random matrix in $\text{GUE}(n, \frac{1}{n})$. Then for any complex number z we have

$$\mathbb{E}\{\text{tr}_n(\exp(zX_n))\} = \exp\left(\frac{z^2}{2n}\right) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \left(\frac{z^2}{n}\right)^j.$$

Sketch of Proof.

By analytic continuation it suffices to consider the case $z \in \mathbb{R}$.

Step I. We know that

$$\mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} = \frac{1}{\sqrt{2n}} \int_{\mathbb{R}} \exp(zt) \left(\sum_{k=0}^{n-1} \varphi_k(\sqrt{\frac{n}{2}}t)^2 \right) dt,$$

so by a substitution, it follows that we have to show that

$$\begin{aligned} F(s) &:= \int_{\mathbb{R}} \exp(st) \left(\sum_{k=0}^{n-1} \varphi_k(t)^2 \right) dt \\ &= n \exp\left(\frac{s^2}{4}\right) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \left(\frac{s^2}{2}\right)^j, \end{aligned}$$

for any real number s .

Sketch of Proof (continued).

Step II. By partial integration we find that

$$F(s) := \int_{\mathbb{R}} \exp(st) \left(\sum_{k=0}^{n-1} \varphi_k(t)^2 \right) dt = -\frac{1}{s} \int_{\mathbb{R}} \exp(st) \frac{d}{dt} \left(\sum_{k=0}^{n-1} \varphi_k(t)^2 \right) dt$$

and using properties of the Hermite polynomials, one may verify that

$$\frac{d}{dt} \left(\sum_{k=0}^{n-1} \varphi_k(t)^2 \right) = -\sqrt{2n} \varphi_n(t) \varphi_{n-1}(t),$$

so that

$$\begin{aligned} F(s) &= \frac{\sqrt{2n}}{s} \int_{\mathbb{R}} \exp(st) \varphi_n(t) \varphi_{n-1}(t) dt \\ &= \frac{\sqrt{2n}}{s} (2^{2n-1} n! (n-1)! \pi)^{-1/2} \int_{\mathbb{R}} \exp(st - t^2) H_n(t) H_{n-1}(t) dt. \end{aligned}$$

Sketch of Proof (continued).

Step III Using the substitution $y = t - \frac{1}{2}s$ we have that $\exp(st - t^2) = \exp(-y^2) \exp(\frac{1}{4}s^2)$, and hence

$$\begin{aligned} F(s) &= \frac{\sqrt{2n}}{s} (2^{2n-1} n! (n-1)! \pi)^{-1/2} \int_{\mathbb{R}} \exp(st - t^2) H_n(t) H_{n-1}(t) dt \\ &= \frac{1}{s} (2^{(n-1)} ((n-1)!)^2 \pi)^{-1/2} \exp(\frac{1}{4}s^2) \\ &\quad \times \int_{\mathbb{R}} \exp(-y^2) H_n(y + \frac{1}{2}s) H_{n-1}(y + \frac{1}{2}s) dy. \end{aligned}$$

Using then the following properties of the Hermite polynomials:

$$H_k(x+a) = \sum_{j=0}^k \binom{k}{j} (2a)^{k-j} H_j(x), \quad (x, a \in \mathbb{R}, k \in \mathbb{N})$$

and

$$\int_{\mathbb{R}} H_k(y) H_l(y) \exp(-y^2) dy = \delta_{k,l} (2^k k! \sqrt{\pi}), \quad (k, l \in \mathbb{N}_0),$$

Sketch of Proof (continued).

we find that

$$\begin{aligned} F(s) &= \frac{1}{s} (2^{(n-1)} ((n-1)!)^2 \pi)^{-1/2} \exp\left(\frac{1}{4}s^2\right) \\ &\quad \times \sum_{j=0}^{n-1} \binom{n}{j} \binom{n-1}{j} s^{n-j} s^{n-1-j} (2^j j! \sqrt{\pi}) \\ &= \dots \\ &= n \exp\left(\frac{s^2}{4}\right) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \left(\frac{s^2}{2}\right)^j, \end{aligned}$$

as desired. ■

Wigner's semi-circle law

Theorem. For each n in \mathbb{N} , let X_n be a random matrix in $\text{GUE}(n, \frac{1}{n})$ and consider its spectral distribution μ_{X_n} . Then

$$\mu_{X_n} \xrightarrow{w} \frac{1}{2\pi} \sqrt{4 - t^2} 1_{[-2,2]}(t) dt, \quad \text{as } n \rightarrow \infty,$$

i.e., for any continuous bounded function $f: \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\mathbb{E}\{\text{tr}_n(f(X_n))\} = \int_{\mathbb{R}} f(x) \mu_{X_n}(dx) \longrightarrow \frac{1}{2\pi} \int_{-2}^2 f(x) \sqrt{4 - x^2} dx,$$

as $n \rightarrow \infty$.

Proof of Wigner's semi-circle law.

By the continuity theorem for characteristic functions of probability measures, it suffices to show that

$$\mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} \longrightarrow \frac{1}{2\pi} \int_{-2}^2 \exp(zt) \sqrt{4-t^2} dt, \quad \text{as } n \rightarrow \infty$$

for any complex number z . Given such a z it follows by the previous theorem and dominated convergence (for series) that

$$\begin{aligned} \mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} &= \exp\left(\frac{z^2}{2n}\right) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \left(\frac{z^2}{n}\right)^j \\ &\xrightarrow{n \rightarrow \infty} \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} z^{2j}. \end{aligned}$$

Proof of Wigner's semi-circle law (continued).

On the other hand we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-2}^2 \exp(zt) \sqrt{4-t^2} dt &= \frac{1}{2\pi} \int_{-2}^2 \left(\sum_{j=0}^{\infty} \frac{z^j t^j}{j!} \right) \sqrt{4-t^2} dt \\ &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \left(\frac{1}{2\pi} \int_{-2}^2 t^j \sqrt{4-t^2} dt \right) \\ &= \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!} \left(\frac{1}{j+1} \binom{2j}{j} \right) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} z^{2j}, \end{aligned}$$

as desired. ■

The strong version of Wigner's semi-circle law

Theorem. For each n in \mathbb{N} , let X_n be a random matrix in $\text{GUE}(n, \frac{1}{n})$. Then there is a measurable set $S \subseteq \Omega$ with probability one, such that

$$\mu_{X_n(\omega)} \xrightarrow{w} \frac{1}{2\pi} \sqrt{4 - t^2} 1_{[-2,2]}(t) dt, \quad \text{as } n \rightarrow \infty,$$

for all ω in S .

In other words, for any interval I in \mathbb{R} , we have

$$\frac{1}{n} \#\{j \in \{1, 2, \dots, n\} \mid \lambda_j(X_n(\omega)) \in I\} \xrightarrow{n \rightarrow \infty} \frac{1}{2\pi} \int_{I \cap [-2,2]} \sqrt{4 - t^2} dt,$$

for all ω in S .

Sketch of Proof.

Step 1 (Concentration inequality). Let $G_{N,\sigma}$ denote the Gaussian distribution on \mathbb{R}^N with Lebesgue density

$$\frac{dG_{N,\sigma}(x)}{dx} = (2\pi\sigma^2)^{-N/2} \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right),$$

where $\|x\|$ is the Euclidean norm of x . Furthermore, let $F: \mathbb{R}^N \rightarrow \mathbb{R}$ be a function that satisfies the Lipschitz condition

$$|F(x) - F(y)| \leq c\|x - y\|, \quad (x, y \in \mathbb{R}^N), \quad (1)$$

for some positive constant c . Then for any positive number ϵ , we have that

$$G_{N,\sigma}(\{x \in \mathbb{R}^N \mid |F(x) - \mathbb{E}(F)| > \epsilon\}) \leq 2 \exp\left(-\frac{K\epsilon^2}{c^2\sigma^2}\right),$$

where $\mathbb{E}(F) = \int_{\mathbb{R}^N} F(x) dG_{N,\sigma}(x)$, and $K = \frac{2}{\pi^2}$.

Sketch of Proof (continued).

Step 2. Use (1) on the function

$$F(A) = \operatorname{tr}_n(f(A)), \quad (A \in M_n(\mathbb{C})_{\text{sa}}),$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies a Lipschitz condition:

$$|f(y) - f(x)| \leq c|x - y|, \quad (x, y \in \mathbb{R}).$$

This yields the estimate

$$P(|\operatorname{tr}_n(f(X_n)) - \mathbb{E}\{\operatorname{tr}_n(f(X_n))\}| > \epsilon) \leq \exp(-\frac{n^2 K \epsilon^2}{c^2}).$$

Step 3. Use the Borel-Cantelli lemma!

A differential equation for the spectral density h_n

Proposition. For any positive integer n , the spectral density h_n of a GUE random matrix X_n satisfies the differential equation:

$$\frac{1}{n^2} h_n'''(x) + (4 - x^2) h_n'(x) + x h_n(x) = 0.$$

Sketch of proof.

Step 1. We have seen that

$$\psi_n(z) := \int_{\mathbb{R}} \exp(zt) h_n(t) dt = \mathbb{E}\{\mathrm{tr}_n(\exp(zX_n))\} = e^{z^2/2n} \eta_n(z^2/n),$$

with $\eta_n: \mathbb{R} \rightarrow \mathbb{R}$ the function given by

$$\eta_n(s) = \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} s^j.$$

It is a classical result that η_n satisfies the differential equation:

$$s\eta_n''(s) + (2+s)\eta_n'(s) + (n-1)\eta_n(s) = 0. \quad (2)$$

Sketch of proof (continued).

Step II. Using the substitution $s = z^2/n$ it follows from (2) that ψ satisfies the differential equation

$$n^2 z \psi_n''(z) + 3n^2 \psi_n'(z) - (4n^2 z + z^3) \psi_n(z) = 0.$$

Setting $z = -iy$ for y in \mathbb{R} , it follows that the Fourier transform $\widehat{h}_n(y) = \psi_n(-iy)$ satisfies the differential equation:

$$n^2 iy \widehat{h}_n''(y) + 3n^2 i \widehat{h}_n'(y) + (4n^2 iy - iy^3) \widehat{h}_n(y) = 0. \quad (3)$$

Sketch of proof (continued).

Step III. Consider the co-Fourier transform $\overline{\mathcal{F}}: L_1(\mathbb{R}) \rightarrow L_1(\mathbb{R})$ defined by

$$[\overline{\mathcal{F}}f](y) = \int_{\mathbb{R}} \exp(iyt)f(t) dt, \quad (y \in \mathbb{R}),$$

and recall that by Fourier inversion $\widehat{\overline{\mathcal{F}}h_n} = (2\pi)h_n$. Applying now $\overline{\mathcal{F}}$ to (3), we obtain that h_n satisfies

$$\frac{1}{n^2} h_n'''(x) + (4 - x^2)h_n'(x) + xh_n(x) = 0,$$

as desired. ■

The Harer-Zagier Recursion Formulae

Theorem. For each n in \mathbb{N} and p in \mathbb{N}_0 , put

$$\gamma(p, n) = \int_{\mathbb{R}} t^{2p} h_n(t) dt = \mathbb{E}\{\mathrm{tr}_n(X_n^{2p})\},$$

where $X_n \in \mathrm{GUE}(n, \frac{1}{n})$. These moments satisfy the recursion formula

$$(p+2)\gamma(p+1, n) = \frac{(4p^2-1)p}{n^2}\gamma(p-1, n) + (4p+2)\gamma(p, n), \quad (4)$$

for any positive integer p .

Proof of the Harer-Zagier recursion formulae

Since h_n satisfies the differential equation

$$\frac{1}{n^2} h_n'''(x) + (4 - x^2) h_n'(x) + x h_n(x) = 0,$$

it follows by multiplication by x^{2p+1} and partial integration that

$$\begin{aligned} 0 &= \int_{\mathbb{R}} x^{2p+1} (n^{-2} h_n'''(x) + (4 - x^2) h_n'(x) + x h_n(x)) dx \\ &= \int_{\mathbb{R}} (-n^{-2} (2p+1)(2p)(2p-1) x^{2p-2} - 4(2p+1) x^{2p} \\ &\quad + (2p+3) x^{2p+2} + x^{2p+2}) h_n(x) dx \\ &= \frac{-2(4p^2 - 1)p}{n^2} \gamma(p-1, n) - 4(2p+1) \gamma(p, n) + 2(p+2) \gamma(p+1, n), \end{aligned}$$

from which (4) follows readily. ■

Convergence of largest and smallest eigenvalue for a GUE matrix

Theorem. For each positive integer n , let X_n be a random matrix from $\text{GUE}(n, \frac{1}{n})$, and let $\lambda_{\max}(X_n)$ and $\lambda_{\min}(X_n)$ denote, respectively, the largest and smallest eigenvalues of X_n . Then

$$\lim_{n \rightarrow \infty} \lambda_{\max}(X_n) = 2, \quad \text{almost surely,}$$

and

$$\lim_{n \rightarrow \infty} \lambda_{\min}(X_n) = -2, \quad \text{almost surely.}$$

Proof of $\limsup_{n \rightarrow \infty} \lambda_{\max}(X_n) \leq 2$ almost surely

It suffices to show that

$$\forall \epsilon > 0: P\left(\limsup_{n \rightarrow \infty} \lambda_{\max}(X_n) \leq 2 + \epsilon\right) = 1,$$

which will follow if we show that

$$\forall \epsilon > 0: P(\lambda_{\max}(X_n) \leq 2 + \epsilon, \text{ for all sufficiently large } n) = 1,$$

which in turn follows from the Borel-Cantelli Lemma if we show that

$$\forall \epsilon > 0: \sum_{n=1}^{\infty} P(\lambda_{\max}(X_n) \geq 2 + \epsilon) < \infty. \quad (5)$$

Proof of $\limsup_{n \rightarrow \infty} \lambda_{\max}(X_n) \leq 2$ (continued)

So let $\epsilon > 0$ be given, and then note that for any $t > 0$ we have by Chebychev's inequality that for any $t > 0$,

$$\begin{aligned} P(\lambda_{\max}(X_n) \geq 2 + \epsilon) &= P(\exp(t\lambda_{\max}(X_n)) \geq \exp(t(2 + \epsilon))) \\ &= \exp(-(2 + \epsilon)t) \mathbb{E}\{\exp(t\lambda_{\max}(X_n))\}. \end{aligned} \quad (6)$$

Note here that

$$\begin{aligned} \exp(t\lambda_{\max}(X_n)) &= \lambda_{\max}(\exp(tX_n)) \\ &\leq \sum_{j=1}^n \lambda_j(\exp(tX_n)) = n \operatorname{tr}_n(\exp(tX_n)), \end{aligned}$$

since all the eigenvalues of $\exp(tX_n)$ are positive. Thus.....

$$\begin{aligned}\mathbb{E}\left\{\exp(t\lambda_{\max}(X_n))\right\} &\leq n\mathbb{E}\left\{\operatorname{tr}_n(\exp(tX_n))\right\} \\ &= n\exp\left(\frac{t^2}{2n}\right)\sum_{j=0}^{n-1}\frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\left(\frac{t^2}{n}\right)^j. \\ &\leq n\exp\left(\frac{t^2}{2n}\right)\sum_{j=0}^{\infty}\frac{n^j}{j!(j+1)!}\left(\frac{t^2}{n}\right)^j. \\ &\leq n\exp\left(\frac{t^2}{2n}\right)\sum_{j=0}^{\infty}\left(\frac{t^j}{j!}\right)^2. \\ &\leq n\exp\left(\frac{t^2}{2n}\right)\left(\sum_{j=0}^{\infty}\frac{t^j}{j!}\right)^2. \\ &= n\exp\left(\frac{t^2}{2n} + 2t\right).\end{aligned}$$

Proof of $\limsup_{n \rightarrow \infty} \lambda_{\max}(X_n) \leq 2$ (continued)

Comparing with (6) we conclude that

$$\begin{aligned} P(\lambda_{\max}(X_n) \geq 2 + \epsilon) &\leq \exp(-(2 + \epsilon)t) \mathbb{E}\{\exp(t\lambda_{\max}(X_n))\} \\ &\leq n \exp(-(2 + \epsilon)t) \exp\left(\frac{t^2}{2n} + 2t\right) \\ &= n \exp\left(-\epsilon t + \frac{t^2}{2n}\right), \end{aligned}$$

which holds for all $t > 0$. Putting $t = n\epsilon$ we obtain

$$P(\lambda_{\max}(X_n) \geq 2 + \epsilon) \leq n \exp\left(\frac{-n\epsilon^2}{2}\right),$$

from which (5) follows immediately. \blacksquare

Proof of $\liminf_{n \rightarrow \infty} \lambda_{\max}(X_n) \geq 2$ almost surely

Let ϵ be a positive number. Then for almost all ω we have

$$\begin{aligned} \#\{j \in \{1, \dots, n\} \mid \lambda_j(X_n(\omega)) \geq 2 - \epsilon\} &= \sum_{j=1}^n \delta_{\lambda_j(X_n(\omega))}([2 - \epsilon, \infty)) \\ &= n\mu_{X_n(\omega)}([2 - \epsilon, \infty)) \\ &\xrightarrow{n \rightarrow \infty} \infty, \end{aligned}$$

since, according to the strong version of Wigner's semi-circle law,

$$\mu_{X_n(\omega)}([2 - \epsilon, \infty)) \xrightarrow{n \rightarrow \infty} \frac{1}{2\pi} \int_{2-\epsilon}^2 \sqrt{4-t^2} dt > 0,$$

for almost all ω . It follows in particular that

$$\liminf_{n \rightarrow \infty} \lambda_{\max}(X_n(\omega)) \geq 2 - \epsilon, \quad \text{for almost all } \omega.$$