Free Entropy Dimension in finite von Neumann Algebras

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von Neumann algebras

Let H be a separable complex Hilbert space.

Let $\mathcal{B}(H)$ be the set of all bounded linear operators from H to H.

The adjoint of a bounded linear operator T is the operator T^* characterized by the identity

$$< T^* v, w >=< v, Tw > \qquad \forall v, w \in H.$$

Weak operator topology (WOT) on $B(H)$ is the topology such that a sequence (or a net) $\{T_\alpha\}$ converges to T in the weak operator topology if and only if

$$\langle T_\alpha v_1, v_2 \rangle \to \langle T v_1, v_2 \rangle$$

for all $v_1, v_2 \in H$.

A von Neumann algebra $\mathcal{M}$ is defined to be a selfadjoint subalgebra of $\mathcal{B}(H)$ which is closed in weak operator topology, i.e.

(i) $\forall\ T \in \mathcal{M}$, then $T^* \in \mathcal{M}$; and

(ii) $\overline{\mathcal{M}}^{WOT} = \mathcal{M}$.

Example: $L^\infty(\mathbb{T}, \mu)$, where μ is the Lebesgue measure on the unit circle $\mathbb{T}$.

Suppose $\mathcal{M}$ is a von Neumann algebra. Define

$$\mathcal{M}' = \{T \in B(H) \mid ST = TS,\ \forall S \in \mathcal{M}\}$$

to be the commutant of $\mathcal{M}$ and

$$\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$$

to be the center of $\mathcal{M}$.

Factors are the von Neumann algebras whose centers are scalar multiples of the identity, i.e. $\mathcal{Z}(\mathcal{M}) = \mathbb{C}I$. The factors are classified by means of a relative dimension function into type I_n, II_1, II_∞, III factors.

1. A factor $\mathcal{M}$ is a type I_n factor if it is * isomorphic to $\mathcal{B}(H)$ for some $n-$dimensional Hilbert space H.

2. A factor $\mathcal{M}$ is a type II_1 factor if and only if that it is infinite dimensional algebra and there is a positive linear mapping τ from $\mathcal{M}$ to $\mathbb{C}$ such that $\tau(AB) = \tau(BA)$ for each A and B in $\mathcal{M}$ and $\tau(I) = 1$. Such positive linear mapping τ is also called a "trace" on $\mathcal{M}$.

3. A factor $\mathcal{M}$ is a type II_∞ factor if it is * isomorphic to $\mathcal{R} \otimes \mathcal{B}(H)$ for some II_1 factor $\mathcal{R}$ and a infinite dimensional Hilbert space H.

4. All other factors are type III factors.

Examples of factors of type $\mathbf{II}_1$

We assume that G is discrete countable group and the Hilbert space H is $l^2(G)$. For each g in G, let L_g denote the left translation of functions in $l^2(G)$ by g^{-1}. Then $g \to L_g$ is a faithful unitary representation of G on H. Let

$$L(G) = \overline{* - alg\{L_g : \ g \in G\}}^{WOT}$$

be the von Neumann algebra generated by $\{L_g : \ g \in G\}$.

Proposition 1 *(Murray and von Neumann) If G is an* infinity conjugacy class *group ($\forall e \neq g \in G$, the cardinality of the set $\{h^{-1}gh \mid h \in G\}$ is infinite), then $L(G)$ is a II_1 factor.*

1. $L(F(n))$ ($n \geq 2$), where $F(n)$ is the free group with n generators.

2. $L(\Pi)$, where Π is the permutation group of $\mathbb{Z}$ (consisting of those permutations that leave fixed all but a finite set of $\mathbb{Z}$).

3. $L(SL(3, \mathbb{Z}))$, where $SL(3, \mathbb{Z})$ is the special linear group with integer entries.

2. Free entropy theory

A result proved by Voiculescu using random matrices

Proposition 2 *Let $L(F_n)$ be the free group factor on n generators with the tracial state τ, and $u_1, \ldots, u_n$ be the standard generators of $L(F_n)$. For each $m, k \geq 1$ and $\epsilon > 0$, let*

$$\begin{aligned}\Omega_{m,\epsilon}(k) = &\{(U_1, \ldots, U_n) \in \mathcal{U}(k)^n \mid \\ &\quad |\tau_k(U_{i_1}^{\eta_1} \cdots U_{i_p}^{\eta_p}) - \tau(u_{i_1}^{\eta_1} \cdots u_{i_p}^{\eta_p})| < \epsilon \\ &\quad \text{for all} \quad 1 \leq p \leq m, 1 \leq i_1, \ldots, i_p \leq n, \\ &\quad \{\eta_1, \ldots, \eta_p\} \subset \{1, *\}\}.\end{aligned}$$

Then

$$\lim_{k \to \infty} \mu_k(\Omega_{m,\epsilon}(k)) = 1,$$

where μ_k is normalized Haar measure on $\mathcal{U}(k)^n$.

Free entropy

Let $M_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$ and τ_k be the normalized trace on $M_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}\text{Tr}$, where Tr is the usual trace on $M_k(\mathbb{C})$. Let M_k^{sa} denote the self-adjoint complex matrices. The euclidean norm $\| \ \|_e$ on $(M_k^{sa})^n$ is given by

$$\|(A_1, \ldots, A_n)\|_e^2 = \text{Tr } (A_1^2 + \cdots + A_n^2),$$

for each $(A_1, \ldots, A_n)$ in $(M_k^{sa})^n$. Let Λ denote the Lebesgue measure on $(M_k^{sa})^n$ induced by the euclidean norm $\| \ \|_e$.

Voiculescu's microstate space

Let $(\mathcal{M}, \tau)$ be a free probability space. (In our case, $\mathcal{M}$ is a von Neumann algebra with a faithful normal tracial state τ).

Let $X_1, \ldots, X_n$ be self-adjoint elements in $\mathcal{M}$. For $\epsilon, R > 0$, $m, k \in \mathbb{N}$, let

$$\Gamma_R(X_1, \ldots, X_n; m, k, \epsilon)$$

be a subset of $(M_k^{sa})^n$ consisting of all

$$(A_1, \ldots, A_n)$$

in $(M_k^{sa})^n$ such that

$$|\tau(X_{i_1} \ldots X_{i_p}) - \tau_k(A_{i_1} \ldots A_{i_p})| \leq \epsilon,$$

for all $1 \leq p \leq m$, $1 \leq i_1, \ldots, i_p \leq n$ and $\|A_j\| < R, 1 \leq j \leq m$.

Then, we define successively,

$$\chi_R(X_1, \ldots, X_n; m, k, \epsilon) \\ = \log \Lambda(\Gamma_R(X_1, \ldots, X_n; m, k, \epsilon)),$$

$$\chi_R(X_1, \ldots, X_n; m, \epsilon) \\ = \limsup_{k\to\infty} \left(k^{-2}\chi_R(X_1, \ldots, X_n; m, k, \epsilon) + \frac{n}{2}\log k\right),$$

$$\chi_R(X_1, \ldots, X_n) \\ = \inf\left\{\chi_r(X_1, \ldots, X_n; m, \epsilon) : m \in \mathbb{N}, \epsilon > 0\right\},$$

$$\chi(X_1, \ldots, X_n) \\ = \sup_{R>0} \chi_R(X_1, \ldots, X_n).$$

Proposition 3 *Let* $R > \max\{\|X_1\|, \ldots, \|X_n\|\}$ *be a positive number. Then*

$$\chi(x_1, \ldots, x_n) = \limsup_{\omega\to 0} \chi_R(x_1, , \ldots, x_n).$$

D. Voiculescu proved the following

Theorem 1 Basic Properties of $\chi(X_1, \ldots, X_n)$,

1. Upper Bound

   $$\chi(X_1, \ldots, X_n) \leq 2^{-1} n \log(2\pi e n^{-1} C^2)$$

   where $C^2 = \tau(X_1^2 + \cdots + X_n^2)$. In particular $\chi(X_1, \ldots, X_n)$ is either finite or $-\infty$.

   Proof: Let $C_1 > \sqrt{\tau(X_1^2 + \cdots + X_n^2)}$ and

   $$\Gamma = \{(A_1, \ldots, A_n) \in (M_k^{sa})^n \mid \|(A_1, \ldots, A_n)\|_e \leq \sqrt{k} C_1\}.$$

   Then when m is large enough and ϵ is small enough, we have

   $$\Gamma_R(X_1, \ldots, X_n; k, m, \epsilon) \subset \Gamma.$$

But

$$\lambda(\Gamma) = \frac{\pi^{nk^2/2}(\sqrt{k}C_1)^{nk^2}}{\Gamma(1 + nk^2/2)}.$$

Using the sterling's formula, we have

$$\Gamma(1 + nk^2/2) \approx \sqrt{nk^2\pi}\frac{(nk^2/2)^{nk^2/2}}{e^{nk^2/2}}.$$

Thus

$$\limsup_{k\to\infty}(\frac{\log(\lambda(\Gamma))}{k^2} + \frac{n}{2}\log k) \leq 2^{-1}n\log(2\pi en^{-1}C_1^2)$$

2. One Variable Case

$$\chi(X) = \iint \log|s-t|d\mu(s)d\mu(t)+\frac{3}{4}+\frac{1}{2}\log 2\pi,$$

where μ is the distribution of X.

3. If $X_1, \ldots, X_n$ are freely independent, then,

$$\chi(X_1, \ldots, X_n) = \chi(X_1) + \cdots + \chi(X_n)$$

Free entropy dimension

Covering number in complex matrix algebras

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k} Tr$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$.

Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$.

Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \ldots, A_n)\|_2^2 = \tau_k(A_1^* A_1) + \ldots + \tau_k(A_n^* A_n)$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|_2$-ball $Ball(B_1, \ldots, B_n; \omega, \|\cdot\|_2)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\|_2 < \omega.$$

Definition 1 *Suppose that Σ is a set in $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_2(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|_2$-balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.*

Voiculescu's microstate spaces

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra, $X_1, \ldots, X_n$ be elements in $\mathcal{M}$. For $\epsilon, R > 0$, $m, k \in \mathbb{N}$, let

$$\Gamma_R(X_1, \ldots, X_n; m, k, \epsilon)$$

be a subset of $(M_k^{sa})^n$ consisting of all

$$(A_1, \ldots, A_n)$$

in $(M_k^{sa})^n$ such that

$$|\tau(X_{i_1} \ldots X_{i_n}) - \tau_k(A_{i_1}^{\eta_1} \ldots A_{i_n}^{\eta_n})| \leq \epsilon,$$

for all $1 \leq p \leq m$, $(i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$, $\eta_1, \ldots, \eta_n \in \{1, *\}$ and $\|A_j\| < R, 1 \leq j \leq m$.

Now we define, successively,

$$\delta_0(x_1, \ldots, x_n; \omega, R, m, \epsilon) = \limsup_{k\to\infty} \frac{\log(\nu_2(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon), \omega))}{-k^2 \log \omega}$$

$$\delta_0(x_1, \ldots, x_n; \omega, R) = \inf_{m\in\mathbb{N}, \epsilon>0} \delta_0(x_1, \ldots, x_n; \omega, R, m, \epsilon)$$

$$\delta_0(x_1, \ldots, x_n; \omega) = \sup_{R>0} \delta_0(x_1, , \ldots, x_n; \omega, R)$$

$$\delta_0(x_1, \ldots, x_n) = \limsup_{\omega\to 0} \delta_0(x_1, , \ldots, x_n; \omega),$$

where $\delta_0(x_1, , \ldots, x_n)$ is called the *free entropy dimension* of $x_1, \ldots, x_n$. (Here we used an equivalent definition by Jung [12])

Proposition 4 *Let* $R > \max\{\|X_1\|, \ldots, \|X_n\|\}$ *be a positive number. Then*

$$\delta_0(x_1, \ldots, x_n) = \limsup_{\omega \to 0} \delta_0(x_1, , \ldots, x_n; \omega, R).$$

Theorem 2 *Let $L(F_n)$ be the free group factor on n generators with the tracial state τ, and $u_1, \ldots, u_n$ be the standard generators of $L(F_n)$. Then*

$$\delta_0(u_1, \ldots, u_n) = n.$$

Proof: It follows from Proposition 2 that, for every $m \geq 1$ and $\epsilon > 0$, there are some positive integer $k_{m,\epsilon}$ and a sequence of subsets $\{\Omega_{m,\epsilon}(k)\}_{k=k_{m,\epsilon}}^{\infty}$ such that

$$\mu_k(\Omega_{m,\epsilon}(k)) \geq \frac{1}{2}, \qquad \text{for } k \geq k_{m,\epsilon},$$

where μ_k is normalized Haar measure on $\mathcal{U}(k)^n$.

For each $R > 1$ and $m' \geq 1$, $\epsilon' > 0$, it is not hard to verify that, when m is large enough and ϵ is small enough, for any $k \geq k_{m,\epsilon}$,

$$\Omega_{m,\epsilon}(k) \subset \Gamma_R(u_1, \ldots, u_n; k, m', \epsilon')$$

Note there exists constant C (not depending on k) such that for any ball centered at $(U_1, \ldots, U_n)$ with radius ω (with respect to 2-norm) we have

$$\mu_k(Ball((U_1, \ldots, U_n), \omega)) \leq (C\omega)^{nk^2}, \forall\ 0 < \omega < 1.$$

Thus,

$$\nu_2(\Gamma_R(u_1, \ldots, u_n; k, m', \epsilon'), \omega) \geq (C\omega)^{-nk^2}.$$

Therefore

$$\delta_0(u_1, \ldots, u_n) = n.$$

Lemma 1 *Let x be a normal element in a von Neumann algebra $\mathcal{M}$ with a tracial state τ. Let $R > \|x\|$. For every $\omega > 0$, there is some positive integer m such that, for all $k \geq 1$, if A, B are two matrices in $\mathcal{M}_k(\mathbb{C})$ satisfying*

$$A, B \in \Gamma_R(x; k, m, \frac{1}{m}),$$

then there is some unitary matrix U in $\mathcal{U}(k)$ such that

$$\|UAU^* - B\|_2 \leq \omega.$$

Proof: Suppose on the contrary that the following holds: there is some $\omega_0 > 0$ such that for every $m \geq 1$, there is some $k_m \geq 1$ and some self-adjoint matrices A_m, B_m in $\mathcal{M}_{k_m}(\mathbb{C})$ satisfying

$$A_m, B_m \in \Gamma_R(x; k, m, \frac{1}{m}),$$

and $\|UA_mU^* - B_m\|_2 > \omega_0$ for all unitary matrix U in $\mathcal{U}(k_m)$.

Let α be a free filter on $\mathbb{N}$ then denote by $\mathcal{M}_{k_m}(\mathbb{C})^\alpha$ the ultrapower of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ along the free filter α, $(\mathcal{M}_{k_m})^\gamma$ is the quotient of the C* algebra $\prod_m \mathcal{M}_{k_m}(\mathbb{C})$ by the 0-ideal of the norm $\|\cdot\|_\alpha$, where

$$\|(Y_m)_{m=1}^{\infty}\|_{2,\alpha} = \lim_{m\to\alpha} \|Y_m\|_2$$

for each $(Y_m)_{m=1}^{\infty}$ in $\prod_m \mathcal{M}_{k_m}(\mathbb{C})$.

Let ψ or ϕ be the mapping from $W^*(x)$ to $W^*([(A_m)_{m=1}^{\infty}])$, or $W^*([(B_m)_{m=1}^{\infty}])$, induced by

$$x \to [(A_m)_{m=1}^{\infty}]$$
$$x \to [(B_m)_{m=1}^{\infty}]$$

Then ψ and ϕ are two trace preserving $*$-isomorphisms.

For any $\omega > 0$, there are mutually orthogonal projections $p_1, \ldots, p_n$ in $W^*(x)$ and complex numbers $a_1, \ldots, a_n$ such that

$$\|x - \sum_{i=1}^{n} a_i p_i\|_2 \leq \omega/3.$$

Thus

$$\|[(A_m)_{m=1}^{\infty}] - \sum_{i=1}^{n} a_i \psi(p_i)\|_{2,\alpha} \leq \omega/3$$
$$\|[(B_m)_{m=1}^{\infty}] - \sum_{i=1}^{n} a_i \phi(p_i)\|_{2,\alpha} \leq \omega/3$$

Note that $\mathcal{M}_{k_m}(\mathbb{C})^{\alpha}$ is a factor of type II_1. There is a sequence of unitary matrices $\{U_m\}_{m=1}^{\infty}$ with U_m in $\mathcal{M}_{k_m}(\mathbb{C})$ such that

$$[(U_m)_{m=1}^{\infty}](\sum_{i=1}^{n} a_i \psi(p_i))[(U_m)_{m=1}^{\infty}]^* = \sum_{i=1}^{n} a_i \phi(p_i).$$

Hence $\lim_{m\to\alpha} \|U_m A_m U_m^* - B_m\|_2 \le \omega$, which contradicts with the assumption that

$$\|U A_m U^* - B_m\|_2 > \omega_0$$

for all unitary matrix U in $\mathcal{U}(k_m)$. Therefore, the statement of the lemma is true.

Theorem 3 *Suppose that x is a normal element in a von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ. Then*

$$\delta_0(x) \le 1.$$

2.4 Some results

1. Voiculescu showed that if x is an self-adjoint element of a von Neumann algebra $\mathcal{M}$ with the tracial state τ, then

$$\delta_0(x) = 1 - \sum_t (\tau(E(\{t\}))^2,$$

where E is the spectral prejection of x in $(\mathcal{M}, \tau)$.

2. Jung showed that if $\mathcal{M}$ is a finite hyperfinite von Neumann algebra, i.e.

$$\mathcal{M} \simeq \mathcal{M}_0 \oplus (\oplus_{i=1}^{N} \mathcal{M}_{n_i}(\mathbb{C}))$$

with a faithful tracial state

$$\tau = t_0 \tau_0 \oplus (\oplus_{i=1}^{N} t_i \tau_{n_i}),$$

where $\mathcal{M}_0$ is a diffuse von Neumann algebra. Then

$$\delta_0(x_1, \ldots, x_n) = 1 - \sum_{i=1}^{N} \frac{t_i^2}{n_i^2}.$$

3. Voiculescu showed that if $u_1, \ldots, u_n$ are the standard generators of $L(F_n)$, then

$$\delta_0(u_1, \ldots, u_n) = n.$$

4. Voiculesu showed that if $X_1, \ldots, X_n$ is a family of generators of a finite von Neumann algebra M with a Cartan subalgebra, then

$$\delta_0(X_1, \ldots, X_n) \leq 1.$$

5. Ge showed that if $X_1, \ldots, X_n$ is a family of generators of nonprime II_1 factor $\mathcal{M}$,

i.e., $\mathcal{M}$ is a tensor product of two type II_1 factors, then

$$\delta_0(X_1, \ldots, X_n) \leq 1.$$

In particular, $L(\mathbf{F}(n))(n > 1)$ is prime.

6. K. Dykema computed the free entropy dimension for the von Neumann algebras with finite multiplicity and the ones with property C.

7. M. Stefan showed that the free group factors $L(\mathbf{F}(n))$ don't have nonprime subfactors with finite index.

8. Let $\mathcal{M}$ be a type II_1 von Neumann algebra. If there is a sequence of "Haar" unitaries (a unitary u is called the Haar unitary if $\tau(u^n) = 0$ for $n \neq 0$)$\{u_j\}_{j=1}^{\infty}$ in $\mathcal{M}$ such that

   (a) $\{u_j\}_{j=1}^{\infty}$ generate $\mathcal{M}$, and

   (b) $u_{j+1}u_ju_{j+1}^*$ is in the von Neumann subalgebra generated by $\{u_1, \ldots, u_j\}$ for all $j \geq 1$,

   then, we showed

$$\delta_0(X_1, \ldots, X_n) \leq 1,$$

   when $X_1, \ldots, X_n$ is a family of generators of $\mathcal{M}$.

9. Jung and Shlyakhtenko showed that all generating set of all property T von

Neumann algebra have free entropy dimension lass than or equal to 1.

Topological free entropy dimension

Covering number in complex matrix algebras $\mathcal{M}_k(\mathbb{C})$

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}Tr$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$.

Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{s.a}(\mathbb{C})$ be the subalgebra of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ be the direct sum of n copies of $\mathcal{M}_k^{s.a}(\mathbb{C})$.

Let $\| \cdot \|_\infty$ denote the operator norm on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \ldots, A_n)\|_\infty = \max\{\|A_1\|, \ldots, \|A_n\|\}$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\| \cdot \|$-ball $Ball(B_1, \ldots, B_n; \omega, \|\cdot\|)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\| < \omega.$$

Definition 2 *Suppose that Σ is a set in $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_\infty(\Sigma, \omega)$ to be the minimal number of ω-$\| \cdot \|$-balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.*

Noncommutative polynomials In this article, we always assume that $\mathcal{A}$ is a C*-algebra. Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ are self-adjoint elements in $\mathcal{A}$. Let

$$1 \in \mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m \rangle$$

be the noncommutative polynomials in the undeteminates $X_1, \ldots, X_n, Y_1, \ldots, Y_m$. Let $\{P_r\}_{r=1}^{\infty}$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m \rangle$ with rational coefficients. (Here "rational coefficients" means that the real and imaginary parts of all coefficients of P_r are rational numbers).

Voiculescu's Norm-microstates For all integers $r, k \geq 1$, real numbers $R, \epsilon > 0$ and noncommutative polynomials $P_1, \ldots, P_r$, we define

$$\Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

to be the subset of $(\mathcal{M}_k^{s.a}(\mathbb{C}))^{n+m}$ consisting of all these

$$(A_1, \ldots, A_n, B_1, \ldots, B_m) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^{n+m}$$

satisfying

$$\max\{\|A_1\|, \ldots, \|A_n\|, \|B_1\|, \ldots, \|B_m\|\} \leq R$$

and

$$\begin{aligned} \|P_j(A_1, \ldots, A_n, B_1, \ldots, B_m) \\ - P_j(x_1, \ldots, x_n, y_1, \ldots, y_m)\| \\ \leq \epsilon, \qquad \forall\ 1 \leq j \leq r. \end{aligned}$$

In the definition of norm-microstates space, we use the following assumption. If

$$P_j(x_1,\ldots,x_n,y_1,\ldots,y_m)$$
$$= \alpha_0 \cdot I_{\mathcal{A}} + \sum_{s=1}^{N} \sum_{1\le i_1,\ldots,i_s\le n+m} \alpha_{i_1\cdots i_s} z_{i_1}\cdots z_{i_s}$$

where $z_1,\ldots,z_{n+m}$ denotes $x_1,\ldots,x_n,y_1,\ldots,y_m$ and $\alpha_0, \alpha_{i_1\cdots i_s}$ are in $\mathbb{C}$, then

$$P_j(A_1,\ldots,A_n,B_1,\ldots,B_m)$$
$$= \alpha_0 \cdot I_k + \sum_{s=1}^{N} \sum_{1\le i_1,\ldots,i_s\le n+m} \alpha_{i_1\cdots i_s} Z_{i_1}\cdots Z_{i_s}$$

where $Z_1,\ldots,Z_{n+m}$ denotes $A_1,\ldots,A_n,B_1,\ldots,B_m$ and I_k is the identity matrix in $\mathcal{M}_k(\mathbb{C})$.

We define the norm-microstates of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_m$, denoted by

$$\Gamma_R(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

as the projection of

$$\Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

onto the space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ via the mapping

$$(A_1, \ldots, A_n, B_1, \ldots, B_m) \rightarrow (A_1, \ldots, A_n).$$

Voiculescu's topological entropy dimension

We define

$$\nu_\infty(\Gamma_R(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega)$$

to be the covering number of the set

$$\Gamma_R(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

by ω-$\|\cdot\|$-balls in the metric space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ equipped with operator norm.

Define

$$\delta_{top}(x_1, \ldots, x_n, y_1, \ldots, y_m; r, \epsilon, R, \omega) = \limsup_{k\to\infty} \frac{\nu_\infty(\Gamma_R(\vec{x}, \vec{y}; k, \epsilon, P_1, \ldots, P_r), \omega)}{-k^2 \log \omega}$$

$$\delta_{top}(x_1, \ldots, x_n, y_1, \ldots, y_m; \omega) = \sup_R \inf_{r,\epsilon} \delta_{top}(x_1, \ldots, x_n, y_1, \ldots, y_m; r, \epsilon, R, \omega)$$

The topological entropy dimension of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_m$ is defined by

$$\delta_{top}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \limsup_{\omega\to\infty} \delta_{top}(x_1, \ldots, x_n, y_1, \ldots, y_m; \omega)$$

Results of topological free entropy dimension

1. Suppose that $x_1, \ldots, x_n$ is a free family of semicircular elements. Voiculescu showed, using the result by Haagerup and Thorbjornsen, that

$$\delta_{top}(x_1, \ldots, x_n) = n.$$

2. Suppose that x is a self-adjoint element in a unital C* algebra $\mathcal{A}$. Then

$$\delta_{top}(x) = 1 - \frac{1}{n},$$

   where n is the cardinality of the spectrum of x in $\mathcal{A}$.

3. Suppose that $\mathcal{A}$ is a finite dimensional C* algebra and $dim_{\mathbb{C}}\mathcal{A}$ is the complex

dimension of $\mathcal{A}$. If $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$, then

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{dim_{\mathbb{C}} \mathcal{A}}.$$

4. Suppose that $\mathcal{A}$ is a C* algebra with a unique tracial state τ. Then

$$\delta_{top}(x_1, \ldots, x_n) \leq \delta_0(x_1, \ldots, x_n : \tau),$$

for any family of self-adjoint generators $x_1, \ldots, x_n$ of $\mathcal{A}$.

5. Suppose that $\mathcal{A}$ is a infinite dimensional, unital, simple C* algebra with a unique tracial state and suppose that $\mathcal{A}$ has the approximation property. Then

$$\delta_{top}(x_1, \ldots, x_n) \geq 1,$$

where $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$.

Suppose $\mathcal{A}$ is a C* algebra and $x_1, \ldots, x_n$ is a family of self-adjoint elements of $\mathcal{A}$ that generates $\mathcal{A}$ as a C* algebra. If for any $R > \max\{\|x_1\|, \ldots, \|x_n\|, \|y_1\|, \ldots, \|y_m\|\}$, $r > 0$, $\epsilon > 0$, there is a sequence of positive integers $k_1 < k_2 < \cdots$ such that

$\Gamma_R^{(top)}(\vec{x} : \vec{y}; k_s, \epsilon, P_1, \ldots, P_r) \neq \varnothing, \ \forall\, s \geq 1$

then $\mathcal{A}$ is called having approximation property.

6. We have

$$\delta_{top}(x_1, \ldots, x_n) = 1$$

where $x_1, \ldots, x_n$ is a family of self-adjoint generators of UFH algebra, or irrational rotation algebra, or $C_{red}(F_2) \otimes_{min} C_{red}(F_2)$.

7. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C* algebras and $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is a family of self-adjoint elements that generates $\mathcal{A} \bigoplus \mathcal{B}$. Assume

$$s = \delta_{top}(x_1, \ldots, x_n)$$

and

$$t = \delta_{top}(y_1, \ldots, y_n).$$

(i) If $s \geq 1$ or $t \geq 1$, then

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \max\{s, t\}$$

(ii) If $s < 1$, $t < 1$ and both families $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ are stable, then

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \frac{st - 1}{s + t - 2};$$

and the family of elements $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is also stable.

A family of elements $x_1, \ldots, x_n$ in $\mathcal{A}$ is called stable if for any $\alpha < \delta_{top}(x_1, \ldots, x_n)$ there are positive numbers $C_3 > 0$ and $\omega_0 > 0$, $r_0 \geq 1$, $k_0 \geq 1$ so that

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; q \cdot k_0, \frac{1}{r}, P_1, \ldots, P_r), \omega) \geq C_3^{(q \cdot k_0)^2} \left(\frac{1}{\omega}\right)^{\alpha \cdot (q \cdot k_0)^2},$$

$\forall\ 0 < \omega < \omega_0, r > r_0, q \in \mathbb{N}$.

8. (Work in progress) Suppose that $\mathcal{A}$ a unital C* algebras and $\mathcal{B} = \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$. Suppose

$$x_1 = \sum x_{st}^{(1)} \otimes e_{st}, \ldots, x_n = \sum x_{st}^{(n)} \otimes e_{st}$$

is a family of self-adjoint elements in $\mathcal{B}$ that generates $\mathcal{B}$, where $\{e_{st}\}_{st=1}^n$ is the canonical system of matrix units of

$\mathcal{M}_n(\mathbb{C})$. Then

$$\delta_{top}(x_1, \ldots, x_n) = 1 + \frac{\delta_{top}(\langle x_{st}^{(i)} \rangle) - 1}{n^2}.$$

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