Free Processes via Matrix Theory
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Definitions

- Non commutative probability space :
  Unital Algebra $\mathscr{A}$ + linear functional $\Phi : \mathscr{A} \mapsto \mathbb{C}$ (**state**).
- Additional requirements :
  1. Involutive Banach algebras $(\mathscr{A}, \star, ||\cdot||)$.
  2. $C^\star$-algebras : $||aa^\star|| = ||a||^2$,
     $\Rightarrow B(H)$ (H : Hilbert space), $\Phi(a) =< a\xi, \xi >$, $||\xi|| = 1$ (pure state) $\Leftrightarrow$ GNS representation.
  3. $W^\star$-algebras: weakly closed subalgebra of $B(H) \Leftrightarrow \mathscr{A}'' = \mathscr{A}$.
- $\cap_{p\geq 1} L^p(\Omega, \mathscr{F}, \mathbb{P}) \otimes \mathbb{M}_m(\mathbb{C})$: the space of random matrices of all order moments with the **normalized trace expectation**:

$$\Phi(a) = \frac{1}{m}\mathbb{E}(\mathrm{tr}(a)) := \mathbb{E}(\mathrm{tr}_m)$$

Asymptotic + Free convergence

- $(X_i(m))_{i\in I}$: a collection of square random matrices.
- $\forall\, i_1, \ldots i_r \in I$ (Asymptotic Convergence)

$$\lim_{m\to\infty} \mathbb{E}\{\mathrm{tr}_m(X_{i_1}(m)\ldots X_{i_r}(m))\} = \Phi(X_{i_1}\ldots X_{i_r})$$

in some NCPS $(\mathscr{A}, \Phi)$.

- $\Leftrightarrow \forall P \in \mathbb{C}\langle X_i, i \in I\rangle$

$$\lim_{m\to\infty} \mathbb{E}\{\mathrm{tr}_m(P(X_{i_1}(m), \ldots, X_{i_r}(m))\} = \Phi(P(X_{i_1}, \ldots, X_{i_r})$$

- $\Rightarrow$

$$\lim_{m\to\infty} \mathbb{E}\{\mathrm{tr}_m(X_i^r(m)\} = \Phi(X_i^r), \quad i \in I.$$

- Free asymptotic convergence : $(X_i)_{i\in I}$ free family.

Voiculescu result

- **Gaussian unitary ensemble** (GUE$(m,1)$):
- $X(m) = (X_{ij})_{1\leq i,j\leq m}$, $X(m) = X(m)^\star$,

$$X_{ij} = \begin{cases} \mathcal{N}_{ii} & \text{if} \quad i = j \\ \dfrac{\mathcal{N}_{ij}^1 + \sqrt{-1}\mathcal{N}_{ij}^2}{\sqrt{2}} & \text{if} \quad i < j \end{cases}$$

$(\mathcal{N}_{ii}), (\mathcal{N}_{ij}^1), (\mathcal{N}_{ij}^2)$: independent standard Normal variables.

- Independent normalized matrices $\in$ GUE$(m, 1/m)$ $m \overset{CA}{\to} \infty$ Free semicircular system.
- Wigner Theorem :

$$\mathbb{E}\left(\frac{1}{m}\sum_{i=1}^{m}\delta_{\lambda_i/\sqrt{m}}\right) \overset{\text{weakly}}{\longrightarrow} \quad \text{semicircle (or Wigner) law}$$

Free stochastic Processes

- **Definition**: a collection of random variables indexed by time $t \geq 0$.
- **Examples**
  1. Free additive Brownian motion.
  2. Free complex Brownian motion.
  3. Free multiplicative Brownian motion.
  4. Free Wishart process.
  5. Free Jacobi process.
  6. Haar unitary random variable.

Definitions

- $X = X^\star$, $X_0 = 0$.
- Free increments : $\{X_t - X_s, \, 0 \leq s < t\}$.
- Stationary increments : $X_t - X_s \overset{\mathscr{L}}{=} X_{t-s}$.
- $$\mathscr{L}(X_t) : \sigma_t(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}\mathbf{1}_{[-2\sqrt{t}, 2\sqrt{t}]}(x)dx$$
- Complex free BM: X^1, X^2 **free** free additive BMs,
$$W := \frac{X^1 + \sqrt{-1}X^2}{\sqrt{2}}$$
- Multiplicative free BM: $Y_0 = \mathbf{1}$ (identity element) $+ Y_t$ unitary $+$ free and stationary (left or right) multiplicative increments.
- $\mathscr{L}(Y_t) : \Sigma_t(z) = e^{\frac{t}{2}\frac{1+z}{1-z}}$ (Bercovicci-Voiculescu).

Voiculescu's result Consequence

- From GUE to **Hermitian Brownian motion**:

  Independent standard Normal variables

  $\Downarrow$

  Independent standard Brownian motions

  $\Downarrow$

  Hermitian Brownian motion

  $\Downarrow$

  independent normalized increments $\in$ GUE

  $\Downarrow$

  Free additive BM (Voiculescu result).

- Similar results:

Voiculescu's result Consequence

- From GUE to **Hermitian Brownian motion**:

  Independent standard Normal variables

  $\Downarrow$

  Independent standard Brownian motions

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  independent normalized increments $\in$ GUE

  $\Downarrow$

  Free additive BM (Voiculescu result).

- Similar results:
- Non selfadjoint Brownian matrix $\stackrel{CA}{\rightarrow}$ Free complex BM.

Voiculescu's result Consequence

- From GUE to **Hermitian Brownian motion**:

$$\begin{array}{c}\text{Independent standard Normal variables}\\ \Downarrow \\ \text{Independent standard Brownian motions}\\ \Downarrow \\ \text{Hermitian Brownian motion}\\ \Downarrow \\ \text{independent normalized increments} \in \text{GUE}\\ \Downarrow \\ \text{Free additive BM (Voiculescu result).}\end{array}$$

- Similar results:
- Non selfadjoint Brownian matrix $\stackrel{CA}{\rightarrow}$ Free complex BM.
- Unitary Brownian matrix $\stackrel{CA}{\rightarrow}$ Free multiplicative BM.

Voiculescu's result Consequence

- From GUE to **Hermitian Brownian motion**:

  Independent standard Normal variables

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  independent normalized increments $\in$ GUE

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  Free additive BM (Voiculescu result).

- Similar results:
- Non selfadjoint Brownian matrix $\overset{CA}{\rightarrow}$ Free complex BM.
- Unitary Brownian matrix $\overset{CA}{\rightarrow}$ Free multiplicative BM.
- Haar unitary matrix $\overset{CA}{\rightarrow}$ Haar unitary variable.

Free stochastic integration (Biane-Speicher)

- Bi-process : $U_t := \sum_{i=1}^r A_t^i \otimes B_t^i \in (\mathscr{A} \otimes \mathscr{A}^{op}, \Phi \otimes \Phi)$, $t \geq 0$.
- Simple bi-processes : $0 = t_0 < t_1 \cdots < t_{k-1} < t_k = T$

$$\begin{cases} U_t = \sum_{i=1}^r \alpha_p^i \otimes \beta_p^i, \in \mathscr{A} \otimes \mathscr{A} \; t \in [t_p, t_{p+1}[, \\ U_t = 0 \; t > T, \end{cases}$$

- X : free additive Brownian motion.
- U : simple bi-process

$$\int_0^\infty U_s \sharp dX_s := \sum_{i=1}^r \sum_{p=1}^{k-1} \alpha_p^i (X_{t_{p+1}} - X_{t_p}) \beta_p^i$$

- Extension to bi-processes: Itô's free formula for polynomials and globally Lipschitz functions (Biane-Speicher).
- Example : free multiplicative Brownian motion:

$$dY_t = \sqrt{-1} dX_t Y_t - \frac{1}{2} Y_t dt$$

Complex Wishart and free Wishart processes

- Complex Wishart : $B(p, m)$ complex Brownian matrix $p \times m$

$$Z_t(m) := B_t^\star(m, p)B_t(p, m), \qquad W(p, m, Z_0(m))$$

- Free Wishart process $W(\lambda, Z_0)$ (Capitaine-Donati): $m \vee p(m) \leq d(m)$

$$\begin{aligned} Z_t &:= \lim_{m\to\infty} B_t^\star(m, p(m))B_t(p(m), m) \\ &= \lim_{m\to\infty} P_m B_t^\star(d(m))Q_{p(m)}B_t(d(m))P_m \\ &= PW_t^\star QW_tP \end{aligned}$$

where

$$P_m = \begin{pmatrix} I_m & \\ & 0 \end{pmatrix} \qquad Q_{p(m)} = \begin{pmatrix} I_{p(m)} & \\ & 0 \end{pmatrix}$$

such that

$$\lim_{m\to\infty} \frac{p(m)}{m} = \lambda > 0.$$

Spectral distribution

- Free Wishart variable ($t = 1$) : Marchencko-Pastur or **free Poisson** distribution

$$g(x) = \begin{cases} \dfrac{\lambda\sqrt{(b-x)(x-a)}}{2\pi x} & \text{if} \quad \lambda \geq 1 \\ \dfrac{\lambda\sqrt{(b-x)(x-a)}}{2\pi x} + (1-\lambda)\delta_0 & \text{if} \quad 0 < \lambda < 1 \end{cases}$$

where $b = (1+\sqrt{\lambda})^2$, $a = (1-\sqrt{\lambda})^2$.

- $\lambda \geq 0$: additive convolution semi group of the variable λ (Anshelevich).

Complex Jacobi and Free Jacobi processes

- $\Theta(d)$: $d \times d$ unitary Brownian matrix.
- $X(m,p)$: $m \times p$ upper left corner.
- $J := X(m,p)X^\star(m,p)$: $m \times m$ (Doumerc).
- Free Jacobi process $FJ(\lambda, \theta, J_0)$ (Demni): $m \vee p(m) \leq d(m)$

$$\begin{aligned} J_t &:= \lim_{m\to\infty} X(m,p(m))X^\star(m,p(m)) \\ &= \lim_{m\to\infty} P_m \Theta_t(d(m)) Q_{p(m)} \Theta_t^\star(d(m)) P_m \\ &= P Y_t Q Y_t^\star P \end{aligned}$$

where

$$P_m = \begin{pmatrix} I_m & \\ & 0 \end{pmatrix} \qquad Q_{p(m)} = \begin{pmatrix} I_{p(m)} & \\ & 0 \end{pmatrix}$$

such that

$$\lim_{m\to\infty} \frac{m}{p(m)} = \lambda > 0, \qquad \lim_{m\to\infty} \frac{p(m)}{d(m)} = \theta \in]0,1].$$

Stationary Jacobi case

- Stationary complex Jacobi or JUE or Beta matrix (Casalis-Capitiaine, Collins, Demni)
- Θ : Haar unitary Brownian matrix.
- Cauchy transform

$$G(z) = \frac{(2-r)z + (1/\lambda - 1) + \sqrt{Az^2 - Bz + C}}{2z(z-1)}$$

where $A = r^2 = 1/(\lambda\theta)^2$, $B = 2(r + (r-2)/\lambda)$ et $C = (1 - 1/\lambda)^2$.

- $\Rightarrow$

$$\mu(dx) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi\lambda\theta x(1-x)} dx + a_0\delta_0 + a_1\delta_1$$

- $\lambda \in]0,1] \Rightarrow a_0 = 0$, $\theta \leq 1/(\lambda+1) \Rightarrow a_1 = 0$ (injectivity).

More...

- Stationary case $+\lambda = 1$, $\theta = 1/2$: Tchebycheff polynomials of the first kind martingales:

$$(M_t^k := e^{kt} T_k(2J_t - P))_{t \geq 0}, \quad T_k(x) := \cos(k \arccos(x)),\ k \geq 0.$$

- General case : non linear partial differential equation for the Cauchy transform G_t:

$$\partial_t G_t(z) = \partial_z \{[(1-2\lambda\theta)z - \theta(1-\lambda)] G_t(z) + \lambda\theta z(z-1) G_t^2(z)\}$$

- Stationary case : $G_t = G$,

$$[(1-2\lambda\theta)z - \theta(1-\lambda)] G(z) + \lambda\theta z(z-1) G^2(z) = -\lambda\theta$$

Free SDE

- W: free complex Brownian motion.
- Free Wishart (Capitaine-Donati): $\lambda \geq 1$ (injectivity),

$$dZ_t = \sqrt{Z_t}dW_t + dW_t^{\star}\sqrt{Z_t} + \lambda P dt$$

- Uniqueness of the solution for $\lambda > 1$ (invertibility).
- Free Jacobi (Demni) : $\lambda \in]0,1]$, $\theta \leq 1/(\lambda+1)$ (injectivity),

$$dJ_t = \sqrt{\lambda\theta}(\sqrt{J_t}dW_t\sqrt{P - J_t} + \sqrt{P - J_t}dW_t^{\star}\sqrt{J_t}) + (\theta P - J_t)dt$$

- Uniqueness ?