Math 308 midterm exam II answers

1.

(a) Plugging in,

$$0 + \frac{1}{x} - \frac{1}{x} = 0$$

and

$$\frac{2}{x^3} - \frac{1}{x^3} - \frac{1}{x^3} = 0.$$

(b) The Wronskian is

$$W(x) = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1} \neq 0$$

for $x \neq 0$.

(c) We compute

$$v_1'(x) = -\frac{x^{-1}x^{-1}}{-2x^{-1}} = \frac{1}{2}x^{-1}$$

and

$$v_2'(x) = \frac{x^{-1}x}{-2x^{-1}} = -\frac{1}{2}x.$$

So

$$v_1(x) = \frac{1}{2}\ln|x|,$$

$$v_2(x) = -\frac{1}{4}x^2,$$

and the general solution is

$$y(x) = c_1 x + c_2 x^{-1} + \frac{1}{2}x\ln|x| - \frac{1}{4}x$$

or simply

$$y(x) = c_1 x + c_2 x^{-1} + \frac{1}{2}x\ln|x|.$$

2. The characteristic equation is

$$r^2 + 5r = 0,$$

and the general solution for the homogeneous equation is

$$y_h(x) = c_1 + c_2 e^{-5x}.$$

Using the method of undetermined coefficients, we try

$$y_p(x) = A\sin x + B\cos x.$$

Plugging in,

$$-A\sin x - B\cos x + 5(A\cos x - B\sin x) = 13\sin x.$$

Equating coefficients, we get two equations

$$-A - 5B = 13,$$

$$5A - B = 0.$$

So $B = 5A$, $-26A = 13$, $A = -\dfrac{1}{2}$ and $B = -\dfrac{5}{2}$. It follows that the general solution is

$$y(t) = c_1 + c_2 e^{-5x} - \frac{1}{2}\sin x - \frac{5}{2}\cos x.$$

The initial conditions give

$$y(0) = c_1 + c_2 - \frac{5}{2} = 1,$$
$$y'(0) = -5c_2 - \frac{1}{2}2.$$

So

$$c_2 = -\frac{1}{2},$$
$$c_1 = 1 + \frac{5}{2} + \frac{1}{2} = 4,$$

and the answer is

$$y(x) = 4 - \frac{1}{2}e^{-5x} - \frac{1}{2}\sin x - \frac{5}{2}\cos x.$$

3. The characteristic equation is

$$r^2 + 4 = 0,$$

and the fundamental solution of the homogeneous equation is $[\sin 2x, \cos 2x]$. So for a particular solution, we would take

$$y_p(t) = A\sin x + B\cos x + x(C\sin 2x + D\cos 2x) + (Ex + F)e^{2x} + Gx + H.$$

4.

(a) Since $\mathcal{L}\{ty(t)\} = -Y'(s)$, we get

$$-\frac{d}{ds}\frac{s}{s^2+4} = -\frac{(s^2+4) - s2s}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}.$$

Note: a Laplace transform of a product is **not** the product of Laplace transforms!

(b) We decompose

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+3}\right\} &= \mathcal{L}^{-1}\left\{\frac{(s-2)+2}{(s-2)^2+3}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+3}\right\} + \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s-2)^2+3}\right\} \\ &= e^{2t}\cos\sqrt{3}t + \frac{2}{\sqrt{3}}e^{2t}\sin\sqrt{3}t.\end{aligned}$$

An alternative method for part (a) I saw on one of the exams: first we compute

$$s^2\mathcal{L}\{t\cos 2t\} - 1 = \mathcal{L}\{(t\cos 2t)''\} = \mathcal{L}\{-4\sin 2t - 4t\cos 2t\} = -\frac{8}{s^2+4} - 4\mathcal{L}\{t\cos 2t\}.$$

So solving for the unknown, we get

$$\mathcal{L}\{t\cos 2t\} = \frac{1}{s^2+4}\left(1 - \frac{8}{s^2+4}\right) = \frac{(s^2+4)-8}{(s^2+4)^2}.$$