Math 308 practice midterm exam II answers

1. First, we find the general solution of the homogeneous equation. The auxiliary equation is

$$r^2 + 9 = (r - 3i)(r + 3i),$$

so this solution is

$$c_1 \cos 3x + c_2 \sin 3x.$$

Next, we use the method of undetermined coefficients to find a particular solution to the non-homogeneous equation. We try

$$y_p(x) = Ae^{3x}.$$

Plugging in, we get

$$9A + 9A = 1,$$

so $A = \frac{1}{18}$ and

$$y(x) = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{18} e^{3x}.$$

Finally, we use the initial conditions to obtain

$$y(0) = c_1 + \frac{1}{18} = 0,$$
$$y'(0) = 3c_2 + \frac{1}{6} = 1.$$

So $c_1 = -\frac{1}{18}$ and $c_2 = \frac{5}{18}$. Thus finally,

$$y(x) = -\frac{1}{18} \cos 3x + \frac{5}{18} \sin 3x + \frac{1}{18} e^{3x}.$$

2. We first find that a fundamental solution for the homogeneous equation is $[e^{2x}, xe^{2x}]$. So in the method of undetermined coefficients, we would try a particular solution of the form

$$y_p(x) = A \sin x + B \cos x + C \sin 2x + D \cos 2x + Ex^2 e^{2x} + Fx + G.$$

3.

(a) We check that

$$y_1'' - \frac{1}{x} y_1' + \frac{1}{x^2} y_1 = 0 - \frac{1}{x} 1 + \frac{1}{x^2} x = 0$$

and

$$y_2'' - \frac{1}{x} y_2' + \frac{1}{x^2} y_2 = \frac{1}{x} - \frac{1}{x}(1 + \ln x) + \frac{1}{x^2}(x \ln x) = 0.$$

(b) The Wronskian is

$$W[x, x \ln x] = x(x \ln x)' - (x)' x \ln x = x.$$

Since it is non-zero for $x > 0$ (which is where y_2 is well defined), the solutions form a fundamental set.

(c) We compute

$$v_1' = -\frac{\frac{1}{x} x \ln x}{x} = -\frac{\ln x}{x}$$

and

$$v_2' = \frac{\frac{1}{x} x}{x} = \frac{1}{x}.$$

Thus

$$v_1 = -\int \frac{\ln x}{x}\, dx = -\int u\, du = -u^2/2 = -\frac{1}{2}(\ln x)^2,$$

$v_2 = \ln x$, and

$$y_p(x) = -\frac{1}{2} x(\ln x)^2 + \ln x(x \ln x) = \frac{1}{2} x(\ln x)^2.$$

4.

(a) Answer: $\dfrac{2}{(s+2)^3}$.

(b)

$$\frac{1}{16}\mathcal{L}^{-1}\left\{\frac{2}{(s-1/2)^3}\right\} = \frac{1}{16} t^2 e^{t/2}.$$

5.

(a) From the second equation, $x = y^3$. So from the first equation, $x^4 = 16$. Therefore the two critical points are $(8, 2)$ and $(-8, -2)$.

(b) The Jacobian is

$$J = \begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}.$$

You only need to do one of the following. At $(8, 2)$, the Jacobian is

$$J = \begin{pmatrix} -2 & -8 \\ 1 & -12 \end{pmatrix}.$$

Its eigenvalues are roots of

$$\lambda^2 + 14\lambda + (24 + 8) = 0.$$

Since $14^2 - 4 \cdot 32 > 0$, the equation has two real roots

$$-7 \pm \sqrt{7^2 - 32},$$

both of which are negative. The critical point is a sink.

At $(-8, -2)$, the Jacobian is

$$J = \begin{pmatrix} 2 & 8 \\ 1 & -12 \end{pmatrix}.$$

Its eigenvalues are roots of

$$\lambda^2 + 10\lambda + (-24 - 8) = 0.$$

Since $10^2 + 4 \cdot 32 > 0$, the equation has two real roots,

$$-5 \pm \sqrt{5^2 + 32},$$

one positive, one negative. The critical point is a saddle point.