MATH 311 HANDOUT
CHECKING THE SOLUTION OF THE BESSEL EQUATION

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We want to check that the first Bessel function

$$J_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{2k+1} k!(k+1)!} \, x^{2k+1}$$

is the solution of the Bessel equation for $n = 1$,

$$y'' + \frac{1}{x} y' + \left(1 - \frac{1}{x^2}\right) y = 0.$$

Differentiating term-by-term, we calculate

$$J_1'(x) = \sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{2^{2k+1} k!(k+1)!} \, x^{2k}$$

and

$$J_1''(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)(2k)}{2^{2k+1} k!(k+1)!} \, x^{2k-1} = \sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)(2k)}{2^{2k+1} k!(k+1)!} \, x^{2k-1},$$

Thus

$$\frac{1}{x} J_1'(x) = \sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{2^{2k+1} k!(k+1)!} \, x^{2k-1},$$

$$\frac{1}{x^2} J_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{2k+1} k!(k+1)!} \, x^{2k-1},$$

and we also need to re-write, substituting $k - 1$ for k,

$$J_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{2k+1} k!(k+1)!} \, x^{2k+1} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2^{2(k-1)+1} (k-1)!((k-1)+1)!} \, x^{2(k-1)+1}$$
$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2^{2k-1} (k-1)! k!} \, x^{2k-1}.$$

Note that we have now expressed all the four terms we need as power series with the power of x^{2k-1}. Therefore

$$J_1'' + \frac{1}{x} J_1' + J_1 - \frac{1}{x^2} J_1 = \sum_k \left((-1)^k \frac{(2k+1)(2k)}{2^{2k+1} k!(k+1)!} + (-1)^k \frac{2k+1}{2^{2k+1} k!(k+1)!} \right.$$
$$\left. + (-1)^{k-1} \frac{1}{2^{2k-1} (k-1)! k!} - (-1)^k \frac{1}{2^{2k+1} k!(k+1)!} \right) x^{2k-1}.$$

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This looks complicated, but notice that each of these four terms is a multiple of the expression

$$(-1)^k \frac{1}{2^{2k-1}(k-1)!k!}.$$

Factoring out this expression from the coefficient of x^{2k-1}, we are left with

$$\frac{(2k+1)(2k)}{2^2 k(k+1)} + \frac{2k+1}{2^2 k(k+1)} - \frac{1}{1} - \frac{1}{2^2 k(k+1)} = \frac{(2k+1)2k + (2k+1) - 4k(k+1) - 1}{4k(k+1)}$$

$$= \frac{4k^2 + 2k + 2k + 1 - 4k^2 - 4k - 1}{4k(k+1)} = 0.$$

Therefore all the coefficients in the power series expansion of

$$J_1'' + \frac{1}{x} J_1' + J_1 - \frac{1}{x^2} J_1$$

are zero.