

echelon or staircase-shaped forms. These echelon forms will be studied in the next section. They will also be used for $m \times n$ systems, where $m \neq n$.

EXERCISES

1. Use back substitution to solve each of the following systems of equations.

(a) $x_1 - 3x_2 = 2$

$$2x_2 = 6$$

(b) $x_1 + x_2 + x_3 = 8$

$$2x_2 + x_3 = 5$$

$$3x_3 = 9$$

(c) $x_1 + 2x_2 + 2x_3 + x_4 = 5$

$$3x_2 + x_3 - 2x_4 = 1$$

$$-x_3 + 2x_4 = -1$$

$$4x_4 = 4$$

(d) $x_1 + x_2 + x_3 + x_4 + x_5 = 5$

$$2x_2 + x_3 - 2x_4 + x_5 = 1$$

$$4x_3 + x_4 - 2x_5 = 1$$

$$x_4 - 3x_5 = 0$$

$$2x_5 = 2$$

2. Write out the coefficient matrix for each of the systems in Exercise 1.

3. In each of the following systems, interpret each equation as a line in the plane. For each system, graph the lines and determine geometrically the number of solutions.

(a) $x_1 + x_2 = 4$

$$x_1 - x_2 = 2$$

(c) $2x_1 - x_2 = 3$

$$-4x_1 + 2x_2 = -6$$

(b) $x_1 + 2x_2 = 4$

$$-2x_1 - 4x_2 = 4$$

(d) $x_1 + x_2 = 1$

$$x_1 - x_2 = 1$$

$$-x_1 + 3x_2 = 3$$

4. Write an augmented matrix for each of the systems in Exercise 3.

5. Write out the system of equations that corresponds to each of the following augmented matrices.

(a) $\left[\begin{array}{cc|c} 3 & 2 & 8 \\ 1 & 5 & 7 \end{array} \right]$

(b) $\left[\begin{array}{ccc|c} 5 & -2 & 1 & 3 \\ 2 & 3 & -4 & 0 \end{array} \right]$

(c) $\left[\begin{array}{ccc|c} 2 & 1 & 4 & -1 \\ 4 & -2 & 3 & 4 \\ 5 & 2 & 6 & -1 \end{array} \right]$

(d) $\left[\begin{array}{cccc|c} 4 & -3 & 1 & 2 & 4 \\ 3 & 1 & -5 & 6 & 5 \\ 1 & 1 & 2 & 4 & 8 \\ 5 & 1 & 3 & -2 & 7 \end{array} \right]$

6. Solve each of the following systems.

(a) $x_1 - 2x_2 = 5$

$$3x_1 + x_2 = 1$$

(c) $4x_1 + 3x_2 = 4$

$$\frac{2}{3}x_1 + 4x_2 = 3$$

(e) $2x_1 + x_2 + 3x_3 = 1$

$$4x_1 + 3x_2 + 5x_3 = 1$$

$$6x_1 + 5x_2 + 5x_3 = -3$$

(g) $\frac{1}{3}x_1 + \frac{2}{3}x_2 + 2x_3 = -1$

$$x_1 + 2x_2 + \frac{3}{2}x_3 = \frac{3}{2}$$

$$\frac{1}{2}x_1 + 2x_2 + \frac{12}{5}x_3 = \frac{1}{10}$$

(b) $2x_1 + x_2 = 8$

$$4x_1 - 3x_2 = 6$$

(d) $x_1 + 2x_2 - x_3 = 1$

$$2x_1 - x_2 + x_3 = 3$$

$$-x_1 + 2x_2 + 3x_3 = 7$$

(f) $3x_1 + 2x_2 + x_3 = 0$

$$-2x_1 + x_2 - x_3 = 2$$

$$2x_1 - x_2 + 2x_3 = -1$$

(h) $x_2 + x_3 + x_4 = 0$

$$3x_1 + 3x_3 - 4x_4 = 7$$

$$x_1 + x_2 + x_3 + 2x_4 = 6$$

$$2x_1 + 3x_2 + x_3 + 3x_4 = 6$$

7. The two systems

$$2x_1 + x_2 = 3$$

$$4x_1 + 3x_2 = 5$$

and

$$2x_1 + x_2 = -1$$

$$4x_1 + 3x_2 = 1$$

have the same coefficient matrix but different right-hand sides. Solve both systems simultaneously by eliminating the first entry in the second row of the augmented matrix

$$\left(\begin{array}{cc|cc} 2 & 1 & 3 & -1 \\ 4 & 3 & 5 & 1 \end{array} \right)$$

and then performing back substitutions for each of the columns corresponding to the right-hand sides.

8. Solve the two systems

$$x_1 + 2x_2 - 2x_3 = 1$$

$$2x_1 + 5x_2 + x_3 = 9$$

$$x_1 + 3x_2 + 4x_3 = 9$$

$$x_1 + 2x_2 - 2x_3 = 9$$

$$2x_1 + 5x_2 + x_3 = 9$$

$$x_1 + 3x_2 + 4x_3 = -2$$

by doing elimination on a 3×5 augmented matrix and then performing two back substitutions.

9. Given a system of the form

$$-m_1x_1 + x_2 = b_1$$

$$-m_2x_1 + x_2 = b_2$$

where m_1 , m_2 , b_1 , and b_2 are constants:

(a) Show that the system will have a unique solution if $m_1 \neq m_2$.

(b) If $m_1 = m_2$, show that the system will be consistent only if $b_1 = b_2$.

(c) Give a geometric interpretation to parts (a) and (b).

10. Consider a system of the form

$$a_{11}x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + a_{22}x_2 = 0$$

The reduced row echelon form of the augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

There is one free variable x_3 . Setting $x_3 = 3$, we obtain the solution $(5, 3, 3)$, and the general solution consists of all multiples of $(5, 3, 3)$. It follows that the variables x_1, x_2, x_3 should be assigned values in the ratio

$$x_1 : x_2 : x_3 = 5 : 3 : 3$$

This simple system is an example of the closed Leontief input-output model. Leontief's models are fundamental to our understanding of economic systems. Modern applications would involve thousands of industries and lead to very large linear systems. The Leontief models will be studied in greater detail later in Section 8 of Chapter 6.

EXERCISES

1. Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{pmatrix}$ (h) $\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2. In each of the following, the augmented matrix is in row echelon form. For each case, indicate whether the corresponding linear system is consistent. If the system has a unique solution, find it.

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right]$ (b) $\left[\begin{array}{ccc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$ (c) $\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$

(d) $\left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$ (e) $\left[\begin{array}{ccc|c} 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array} \right]$ (f) $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$

3. In each of the following, the augmented matrix is in reduced row echelon form. In each case, find the solution set to the corresponding linear system.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array} \right]$ (b) $\left[\begin{array}{ccc|c} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$

(c)
$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

(d)
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{array} \right]$$

(e)
$$\left[\begin{array}{cccc|c} 1 & 5 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

(f)
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

4. For each of the systems in Exercise 3, make a list of the lead variables and a second list of the free variables.
5. For each of the following systems of equations, use Gaussian elimination to obtain an equivalent system whose coefficient matrix is in row echelon form. Indicate whether the system is consistent. If the system is consistent and involves no free variables, use back substitution to find the unique solution. If the system is consistent and there are free variables, transform it to reduced row echelon form and find all solutions.

(a) $x_1 - 2x_2 = 3$

$2x_1 - x_2 = 9$

(c) $x_1 + x_2 = 0$

$2x_1 + 3x_2 = 0$

$3x_1 - 2x_2 = 0$

(e) $2x_1 + 3x_2 + x_3 = 1$

$x_1 + x_2 + x_3 = 3$

$3x_1 + 4x_2 + 2x_3 = 4$

(g) $x_1 + x_2 + x_3 + x_4 = 0$

$2x_1 + 3x_2 - x_3 - x_4 = 2$

$3x_1 + 2x_2 + x_3 + x_4 = 5$

$3x_1 + 6x_2 - x_3 - x_4 = 4$

(i) $-x_1 + 2x_2 - x_3 = 2$

$-2x_1 + 2x_2 + x_3 = 4$

$3x_1 + 2x_2 + 2x_3 = 5$

$-3x_1 + 8x_2 + 5x_3 = 17$

(k) $x_1 + 3x_2 + x_3 + x_4 = 3$

$2x_1 - 2x_2 + x_3 + 2x_4 = 8$

$x_1 - 5x_2 + x_4 = 5$

(b) $2x_1 - 3x_2 = 5$

$-4x_1 + 6x_2 = 8$

(d) $3x_1 + 2x_2 - x_3 = 4$

$x_1 - 2x_2 + 2x_3 = 1$

$11x_1 + 2x_2 + x_3 = 14$

(f) $x_1 - x_2 + 2x_3 = 4$

$2x_1 + 3x_2 - x_3 = 1$

$7x_1 + 3x_2 + 4x_3 = 7$

(h) $x_1 - 2x_2 = 3$

$2x_1 + x_2 = 1$

$-5x_1 + 8x_2 = 4$

(j) $x_1 + 2x_2 - 3x_3 + x_4 = 1$

$-x_1 - x_2 + 4x_3 - x_4 = 6$

$-2x_1 - 4x_2 + 7x_3 - x_4 = 1$

(l) $x_1 - 3x_2 + x_3 = 1$

$2x_1 + x_2 - x_3 = 2$

$x_1 + 4x_2 - 2x_3 = 1$

$5x_1 - 8x_2 + 2x_3 = 5$

6. Use Gauss-Jordan reduction to solve each of the following systems.

(a) $x_1 + x_2 = -1$

$4x_1 - 3x_2 = 3$

(b) $x_1 + 3x_2 + x_3 + x_4 = 3$

$2x_1 - 2x_2 + x_3 + 2x_4 = 8$

$3x_1 + x_2 + 2x_3 - x_4 = -1$

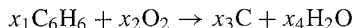
15. Let (c_1, c_2) be a solution to the 2×2 system

$$a_{11}x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + a_{22}x_2 = 0$$

Show that for any real number α the ordered pair $(\alpha c_1, \alpha c_2)$ is also a solution.

16. In Application 3 the solution $(6, 6, 6, 1)$ was obtained by setting the free variable $x_4 = 1$.
- (a) Determine the solution corresponding to $x_4 = 0$. What information, if any, does this solution give about the chemical reaction? Is the term "trivial solution" appropriate in this case?
- (b) Choose some other values of x_4 , such as 2, 4, or 5, and determine the corresponding solutions. How are these nontrivial solutions related?
17. Liquid benzene burns in the atmosphere. If a cold object is placed directly over the benzene, water will condense on the object and a deposit of soot (carbon) will also form on the object. The chemical equation for this reaction is of the form

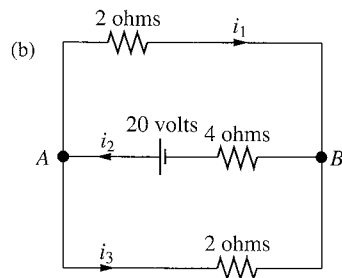
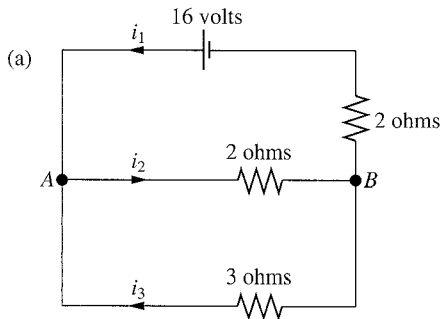


Determine values of $x_1, x_2, x_3,$ and x_4 to balance the equation.

18. Nitric acid is prepared commercially by a series of three chemical reactions. In the first reaction, nitrogen (N_2) is combined with hydrogen (H_2) to form ammonia (NH_3). Next the ammonia is combined with oxygen (O_2) to form nitrogen dioxide (NO_2) and water. Finally, the NO_2 reacts with some of the water to form nitric acid (HNO_3) and nitrogen oxide (NO). The amounts of each component of these reactions are measured in moles (a standard unit of measurement for chemical reactions). How many moles of nitrogen, hydrogen, and oxygen are necessary in order to produce 8 moles of nitric acid?
19. In Application 4, determine the relative values of $x_1, x_2,$ and x_3 if the distribution of goods is as described in the following table.

	<i>F</i>	<i>M</i>	<i>C</i>
<i>F</i>	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
<i>M</i>	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
<i>C</i>	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$

20. Determine the amount of each current for the following networks.



acquire or update as many as 10 million Web pages in a single day. Although the database matrix for pages on the Internet is extremely large, searches can be simplified dramatically since the matrices and search vectors are *sparse*; that is, most of the entries in any column are 0's.

For Internet searches, the better search engines will do simple matching searches to find all pages matching the key words, but they will not order them on the basis of the relative frequency of the key words. Because of the commercial nature of the Internet, people that want to sell products may deliberately make repeated use of key words to ensure that their Web site is highly ranked in any relative frequency search. In fact, it is easy to surreptitiously list a key word hundreds of times. If the font color of the word matches the background color of the page, then the viewer will not be aware that the word is listed repeatedly.

For Web searches a more sophisticated algorithm is necessary for ranking the pages that contain all of the key search words. In Chapter 6 we will study a special type of matrix model for assigning probabilities in certain random processes. This type of model is referred to as a *Markov process* or a *Markov chain*. In Section 3 of Chapter 6 we will see how to use Markov chains to model Web surfing and obtain rankings of Web pages.

REFERENCES

- Berry, Michael W., and Murray Browne, *Understanding Search Engines—Mathematical Modeling and Text Retrieval*, SIAM, Philadelphia, 1999.

EXERCISES

- If

$$A = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{bmatrix}$$

compute:

- | | | | |
|----------|-------------|---------------|-----------------------|
| (a) $2A$ | (b) $A + B$ | (c) $2A - 3B$ | (d) $(2A)^T - (3B)^T$ |
| (e) AB | (f) BA | (g) $A^T B^T$ | (h) $(BA)^T$ |

- For each of the following pairs of matrices, determine whether it is possible to multiply the first matrix times the second. If it is possible, perform the multiplication.

(a) $\begin{bmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -2 \\ 6 & -4 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 4 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{bmatrix}$

$$(e) \begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{bmatrix} \quad (f) \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 & 5 \end{bmatrix}$$

3. For which of the pairs in Exercise 2 is it possible to multiply the second matrix times the first, and what would the dimension of the product be?
4. Write each of the following systems of equations as a matrix equation.

$$(a) \begin{cases} 3x_1 + 2x_2 = 1 \\ 2x_1 - 3x_2 = 5 \end{cases} \quad (b) \begin{cases} x_1 + x_2 = 5 \\ 2x_1 + x_2 - x_3 = 6 \\ 3x_1 - 2x_2 + 2x_3 = 7 \end{cases} \quad (c) \begin{cases} 2x_1 + x_2 + x_3 = 4 \\ x_1 - x_2 + 2x_3 = 2 \\ 3x_1 - 2x_2 - x_3 = 0 \end{cases}$$

5. If

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 1 \\ 2 & 7 \end{bmatrix}$$

verify that

$$(a) 5A = 3A + 2A \quad (b) 6A = 3(2A) \quad (c) (A^T)^T = A$$

6. If

$$A = \begin{bmatrix} 4 & 1 & 6 \\ 2 & 3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -4 \end{bmatrix}$$

verify that

$$(a) A + B = B + A \quad (b) 3(A + B) = 3A + 3B \\ (c) (A + B)^T = A^T + B^T$$

7. If

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$$

verify that

$$(a) 3(AB) = (3A)B = A(3B) \quad (b) (AB)^T = B^T A^T$$

8. If

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

verify that

$$(a) (A + B) + C = A + (B + C) \quad (b) (AB)C = A(BC) \\ (c) A(B + C) = AB + AC \quad (d) (A + B)C = AC + BC$$

9. Prove the associative law of multiplication for 2×2 matrices; that is, let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

and show that

$$(AB)C = A(BC)$$

10. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Compute A^2 and A^3 . What will A^n turn out to be?

11. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Compute A^2 and A^3 . What will A^{2n} and A^{2n+1} turn out to be?

12. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Show that $A^n = O$ for $n \geq 4$.

13. Given

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

- (a) Write \mathbf{b} as a linear combination of the column vectors \mathbf{a}_1 and \mathbf{a}_2 .
 (b) Use the result from part (a) to determine a solution to the linear system $A\mathbf{x} = \mathbf{b}$. Does the system have any other solutions? Explain.
 (c) Write \mathbf{c} as a linear combination of the column vectors \mathbf{a}_1 and \mathbf{a}_2 .
14. For each of the following choices of A and \mathbf{b} , determine whether the system $A\mathbf{x} = \mathbf{b}$ is consistent by examining how \mathbf{b} relates to the column vectors of A . Explain your answers in each case.

$$(a) A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad (b) A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

15. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$A^{-1} = \frac{1}{d} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

16. Let A be a nonsingular matrix. Show that A^{-1} is also nonsingular and $(A^{-1})^{-1} = A$.
 17. Prove that if A is nonsingular then A^T is nonsingular and

$$(A^T)^{-1} = (A^{-1})^T$$

[Hint: $(AB)^T = B^T A^T$.]

18. Let A be an $n \times n$ matrix and let \mathbf{x} and \mathbf{y} be vectors in R^n . Show that if $A\mathbf{x} = A\mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then the matrix A must be singular.