Thus
$$y(x, t) = B \sin \frac{m\pi x}{2} \cos 2m\pi t \tag{4}$$
is a solution. Since this solution is bounded, the condition $|y(x, t)| < M$ is automatically satisfied.

In order to satisfy the last condition, $y(x, 0) = 5 \sin \pi x - 3 \sin 4\pi x$, we first use the principle of superposition to obtain the solution
$$y(x, t) = B_1 \sin \frac{m_1 \pi x}{2} \cos 2m_1\pi t + B_2 \sin \frac{m_2 \pi x}{2} \cos 2m_2\pi t \tag{5}$$
Then putting $t = 0$ we arrive at
$$y(x, 0) = B_1 \sin \frac{m_1\pi x}{2} + B_2 \sin \frac{m_2 \pi x}{2}$$
$$= 6 \sin \pi x - 3 \sin 4\pi x$$
This is possible if and only if $B_1 = 6$, $m_1 = 2$, $B_2 = -3$, $m_2 = 8$. Thus the required solution *(5)* is
$$y(x, t) = 6 \sin \pi x \cos 4\pi t - 3 \sin 4\pi x \cos 16\pi t \tag{6}$$

This boundary value problem can be interpreted physically in terms of the vibrations of a string. The string has its ends fixed at $x = 0$ and $x = 2$ and is given an initial shape $f(x) = 6 \sin \pi x - 3 \sin 4\pi x$. It is then released so that its initial velocity is zero. Then *(6)* gives the displacement of any point x of the string at any later time t.

1.25. Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $0 < x < 3$, $t > 0$, given that $u(0, t) = u(3, t) = 0$, $u(x, 0) = f(x)$, $|u(x, t)| < M$.

This problem differs from Problem 1.23 only in the condition $u(x, 0) = f(x)$. In seeking to satisfy this last condition we see that taking a finite number of terms, as in *(1)* of Problem 1.23, will be insufficient for arbitrary $f(x)$. Thus we are led to assume that infinitely many terms are taken, i.e.
$$u(x, t) = \sum_{m=1}^{\infty} B_m e^{-2m^2\pi^2 t/9} \sin \frac{m\pi x}{3}$$
The condition $u(x, 0) = f(x)$ then leads to
$$f(x) = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{3}$$
or the problem of expansion of a function into a sine series. Such trigonometric expansions, or *Fourier series,* will be considered in detail in the next chapter.

Supplementary Problems

MATHEMATICAL FORMULATION OF PHYSICAL PROBLEMS

1.26. If a taut, horizontal string with fixed ends vibrates in a vertical plane under the influence of gravity, show that its equation is
$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} - g$$
where g is the acceleration due to gravity.

1.27. A thin bar located on the x-axis has its ends at $x = 0$ and $x = L$. The initial temperature of the bar is $f(x)$, $0 < x < L$, and the ends $x = 0, x = L$ are maintained at constant temperatures T_1, T_2 respectively. Assuming the surrounding medium is at temperature u_0 and that Newton's law of cooling applies, show that the partial differential equation for the temperature of the bar at any point at any time is given by
$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} - \beta(u - u_0)$$
and write the corresponding boundary conditions.

1.28. Write the boundary conditions in Problem 1.27 if (*a*) the ends $x = 0$ and $x = L$ are insulated, (*b*) the ends $x = 0$ and $x = L$ radiate into the surrounding medium according to Newton's law of cooling.

1.29. The gravitational potential v at any point (x, y, z) outside of a mass m located at the point (X, Y, Z) is defined as the mass m divided by the distance of the point (x, y, z) from (X, Y, Z). Show that v satisfies Laplace's equation $\nabla^2 v = 0$.

1.30. Extend the result of Problem 1.29 to a solid body.

1.31. A string has its ends fixed at $x = 0$ and $x = L$. It is displaced a distance h at its midpoint and then released. Formulate a boundary value problem for the displacement $y(x, t)$ of any point x of the string at time t.

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

1.32. Determine whether each of the following partial differential equations is linear or nonlinear, state the order of each equation, and name the dependent and independent variables.

(*a*) $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (*c*) $\phi\dfrac{\partial \phi}{\partial x} = \dfrac{\partial^3 \phi}{\partial y^3}$ (*e*) $\dfrac{\partial z}{\partial r} + \dfrac{\partial z}{\partial s} = \dfrac{1}{z^2}$

(*b*) $(x^2 + y^2)\dfrac{\partial^4 T}{\partial z^4} = \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2}$ (*d*) $\dfrac{\partial^2 y}{\partial t^2} - 4\dfrac{\partial^2 y}{\partial x^2} = x^2$

1.33. Classify each of the following equations as elliptic, hyperbolic or parabolic.

(*a*) $\dfrac{\partial^2 \phi}{\partial x^2} - \dfrac{\partial^2 \phi}{\partial y^2} = 0$

(*b*) $\dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial x\,\partial y} = 4$

(*c*) $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = x + 3y$

(*d*) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$

(*e*) $(x^2 - 1)\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + (y^2 - 1)\dfrac{\partial^2 u}{\partial y^2} = x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$

(*f*) $(M^2 - 1)\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0, \quad M > 0$

SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

1.34. Show that $z(x, y) = 4e^{-3x}\cos 3y$ is a solution to the boundary value problem

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \quad z(x, \pi/2) = 0, \quad z(x, 0) = 4e^{-3x}$$

1.35. (*a*) Show that $v(x, y) = xF(2x + y)$ is a general solution of $x\dfrac{\partial v}{\partial x} - 2x\dfrac{\partial v}{\partial y} = v$.

(*b*) Find a particular solution satisfying $v(1, y) = y^2$.

1.36. Find a partial differential equation having general solution $u = F(x - 3y) + G(2x + y)$.

1.37. Find a partial differential equation having general solution

$$(a)\quad z = e^x f(2y - 3x), \qquad (b)\quad z = f(2x + y) + g(x - 2y)$$

GENERAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

1.38. (*a*) Solve $x\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial z}{\partial y} = 0$.

(*b*) Find the particular solution for which $z(x, 0) = x^5 + x - \dfrac{68}{x}$, $z(2, y) = 3y^4$.

1.39. Find general solutions of each of the following.

(a) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ (b) $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 3u$ (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(d) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\,\partial y} - 3\frac{\partial^2 z}{\partial y^2} = 0$ (e) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

1.40. Find general solutions of each of the following.

(a) $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = x$ (c) $\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^3\,\partial y} = 4$

(b) $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} + 12t^2$ (d) $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x\,\partial y} + 2\frac{\partial^2 z}{\partial y^2} = x \sin y$

1.41. Solve $\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^4} = 16.$

1.42. Show that a general solution of $\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}$ is $v = \frac{F(r-ct) + G(r+ct)}{r}$.

SEPARATION OF VARIABLES

1.43. Solve each of the following boundary value problems by the method of separation of variables.

(a) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0, \quad u(x,0) = 4e^{-x}$

(b) $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u, \quad u(x,0) = 3e^{-5x} + 2e^{-3x}$

(c) $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad u(\pi,t) = 0, \quad u(x,0) = 2\sin 3x - 4\sin 5x$

(d) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u_x(0,t) = 0, \quad u(2,t) = 0, \quad u(x,0) = 8\cos\frac{3\pi x}{4} - 6\cos\frac{9\pi x}{4}$

(e) $\frac{\partial u}{\partial t} = 3\frac{\partial u}{\partial x}, \quad u(x,0) = 8e^{-2x}$

(f) $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u, \quad u(x,0) = 10e^{-x} - 6e^{-4x}$

(g) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad u(4,t) = 0, \quad u(x,0) = 6\sin\frac{\pi x}{2} + 3\sin \pi x$

1.44. Solve and give a physical interpretation to the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}, \quad y(0,t) = y(5,t) = 0, \quad y(x,0) = 0, \quad y_t(x,0) = f(x) \qquad (0 < x < 5,\ t > 0)$$

if (a) $f(x) = 5\sin \pi x$, (b) $f(x) = 3\sin 2\pi x - 2\sin 5\pi x$.

1.45. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u$ if $u(0,t) = 0, \quad u(3,t) = 0, \quad u(x,0) = 2\sin \pi x - \sin 4\pi x.$

1.46. Suppose that in Problem 1.24 we have $y(x,0) = f(x)$, where $0 < x < 2$. Show how the problem can be solved if we know how to expand $f(x)$ in a series of sines.

1.47. Suppose that in Problem 1.25 the boundary conditions are $u_x(0,t) = 0$, $u(3,t) = 0$, $u(x,0) = f(x)$. Show how the problem can be solved if we know how to expand $f(x)$ in a series of cosines. Give a physical interpretation of this problem.