Math 409, Homework 5

due March 8.

Section 500.

3.2. 1(a-c), 2(a-c), 3(a,b).

3.3. 2(a-d), 5, 7(a).

Problem 7. This is Problem 3.3.8 of the textbook, but with extra hints.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x+y) = f(x) + f(y)$$

for each $x, y \in \mathbb{R}$. Note that linear functions have this property. Are they the only ones? This problem answers that question.

(a) Show that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. (Comment: to be completely rigorous, one should use induction in the proof of parts (a-c); you can just say "by induction" and do not have to write out the complete argument.)

(b) Show that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

(c) Prove that $f(\frac{1}{m}x) = \frac{1}{m}f(x)$ for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$.

(d) Prove that $f(qx) = qf(x)$ for all $x \in \mathbb{R}$ and $q \in \mathbb{Q} = \left\{\frac{n}{m} | n \in \mathbb{Z}, m \in \mathbb{N}\right\}$.

(e) Prove that f is continuous at 0 if and only if f is continuous everywhere on $\mathbb{R}$. Hint: $\lim_{x\to a} f(x) = \lim_{x \to 0} f(x+a)$.

(f) Prove that if f is continuous at 0, then there is a $k \in \mathbb{R}$ such that $f(x) = kx$ for all $x \in \mathbb{R}$. Note that in this case, $k = f(1)$. Hint: use the fact that the rational numbers are dense.

Problem 8. Let

$$f(x) = \frac{x^2 - 4x - 5}{x - 5}$$

for $x \neq 5$. How should $f(5)$ be defined so that f is continuous at 5?

Section 200.

3.2. 1(a-c), 2(a-c).

3.3. 2(a-d), 5.

Problem 5. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f has a limit at a if and only if $a = 0$.

Problem 6. Consider the formula

$$f(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n}.$$

Calculate $f(x)$ for all x for which it is defined, write down the formula for f (you will need to consider a few cases depending on the value of x), and draw the graph of f.

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