Math 423 final exam answers, honors version

1.

(a) False. This is only true if $V = W_1 \oplus W_2$, in other words if $W_1 \cap W_2 = \{0\}$.
(b) True. All the eigenvalues of a unitary operator have modulus 1, and the determinant is the product of eigenvalues.
(c) False. The matrix may be complex: $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ is symmetric but not self-adjoint.
(d) False. This is not a subspace, since it does not contain the zero element.
(e) True. By the dimension theorem, it follows that T is also one-to-one.

2.

(a) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
(b) $iI = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.
(c) $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ is diagonalizable, with eigenvectors $(1, 0)$ and $(1, 1)$, but these are not orthogonal, and the matrix is not normal.
(d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a Jordan block, and is not diagonalizable.
(e) No, we cannot. All the eigenvalues of a self-adjoint operator are real, while the eigenvalues of an operator that is normal but not self-adjoint cannot all be real.

3. We compute $T(1) = 2x$ and

$$T(x) = T(-2 + (2 + x)) = -4x + x + (1 + x) = 1 - 2x.$$

Therefore

$$T(a + bx) = 2ax + b(1 - 2x) = b + (2a - 2b)x.$$

4. Let

$$\beta = \{u_1, u_2, \dots, u_k\}$$

be a basis for W, and

$$\beta' = \{u_{k+1}, \dots, u_n\}$$

be a basis for $W^\perp$. Then $Pu_i = u_i$ for $i \le k$ and $Pu_i = 0$ for $i > k$. Thus each u_i is an eigenvector of P, with eigenvalues 1 (multiplicity k) and 0 (multiplicity $n - k$).

5. The matrix is upper-triangular, so its eigenvalues are 2 with multiplicity 2 and 1 with multiplicity 1. We compute

$$N(A - 2I) = N \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} = N \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}(\mathbf{e}_1)$$

and

$$N((A-2I)^2) = N\begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} = N\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\mathbf{e}_1, \mathbf{e}_2\right),$$

so we take as the end vector $\mathbf{e}_2$, and compute $(A-2I)\mathbf{e}_2 = \mathbf{e}_1$. Also,

$$N(A-I) = N\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = N\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right) = \mathrm{Span}\left(-\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3\right).$$

So in the Jordan basis $\{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3\}$, the matrix has the form

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. We know that $\mathcal{P}_n(\mathbb{R})$ has dimension $n+1$. The set $\{p_0, p_1, \ldots, p_n\}$ has $n+1$ elements. So to show that this set is linearly dependent, it is enough to show that it is *not* spanning. Indeed, all the polynomials in $\mathrm{Span}\left(\{p_0, p_1, \ldots, p_n\}\right)$ satisfy $p(1) = 0$, and so not every polynomial is in this span.

7.

(a) Since U maps an orthonormal basis to an orthonormal basis, it preserves dimensions of subspaces. Since W is U-invariant, $U(W) \subset W$. But since $\dim U(W) = \dim W$, it follows that $U(W) = W$.

(b) Since U preserves inner products, $U(W^{\perp}) \subset U(W)^{\perp} = W^{\perp}$. So $W^{\perp}$ is U-invariant.

8.

(a) We compute

$$\langle \mathbf{a}, \mathbf{b} \rangle = 0$$

but

$$\|\mathbf{a}\| = \sqrt{3}$$

and

$$\|\mathbf{b}\| = \sqrt{6}$$

(b) From part (a), we know that an orthonormal basis for this subspace is

$$\left\{ \frac{1}{\sqrt{3}}\mathbf{a}, \frac{1}{\sqrt{6}}\mathbf{b} \right\}.$$

So the projection is

$$\left\langle \mathbf{x}, \frac{1}{\sqrt{3}}\mathbf{a} \right\rangle \frac{1}{\sqrt{3}}\mathbf{a} + \left\langle \mathbf{x}, \frac{1}{\sqrt{6}}\mathbf{b} \right\rangle \frac{1}{\sqrt{6}}\mathbf{b} = \frac{1}{3}6\mathbf{a} + \frac{1}{6}3\mathbf{b} = 2\mathbf{a} + \frac{1}{2}\mathbf{b} = \left(\frac{3}{2}, \frac{3}{2}, 3\right)^T.$$