Math 423 final exam answers

1.

(a) False. This is only true if $V = W_1 \oplus W_2$, in other words if $W_1 \cap W_2 = \{0\}$.
(b) False. In this case, AB is self-adjoint if and only if A and B commute.
(c) True. Homework problem.
(d) False. In fact, $\det(cA) = c^n \det A$.
(e) True. By the dimension theorem, it follows that T is also one-to-one.

2. The columns of the matrix are the coordinates of $T(1)$ and $T(x)$ in the basis γ. So

$$T(1) = 2$$

and

$$T(x) = 1 + (2 + x) = 3 + x.$$

Therefore $T(a + bx) = a(2) + b(3 + x) = (2a + 3b) + bx$.

3.

(a) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
(b) $iI = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.
(c) $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ is diagonalizable, with eigenvectors $(1, 0)$ and $(1, 1)$, but these are not orthogonal, and the matrix is not normal.
(d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a Jordan block, and is not diagonalizable.

4. On a finite dimensional vector space, T is invertible if and only if it is one-to-one, in other words if $N(T) = \{0\}$. But the null space of T is exactly the eigenspace for the eigenvalue 0. So T has no eigenvectors with eigenvalue 0 if and only if $N(T) = \{0\}$.

Method II. Since the determinant is the product of eigenvalues, T has 0 as an eigenvalue if and only if $\det T = 0$, if and only if T is not invertible.

5. The matrix is upper-triangular, so its eigenvalues are 2 with multiplicity 2 and 1 with multiplicity 1. We compute

$$N(A - 2I) = N \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} = N \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left(\mathbf{e}_1\right)$$

and

$$N((A - 2I)^2) = N \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} = N \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left(\mathbf{e}_1, \mathbf{e}_2\right),$$

so we take as the end vector $\mathbf{e}_2$, and compute $(A-2I)\mathbf{e}_2 = \mathbf{e}_1$. Also,

$$N(A-I) = N\begin{pmatrix}1 & 1 & -1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{pmatrix} = N\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}\right) = \operatorname{Span}\left(-\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3\right).$$

So in the Jordan basis $\{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3\}$, the matrix has the form

$$\begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

6. $\dim(R(T)) = 2$. Therefore by the dimension theorem, $\dim(N(T)) = 2 - 2 = 0$. It follows that T is one-to-one.

7. We can choose an orthonormal basis

$$\beta_1 = \{v_1, v_2, \ldots, v_k\}$$

for W_1, and complete it to an orthonormal basis

$$\gamma_1 = \{v_1, v_2, \ldots, v_n\}$$

for V. Similarly, we can choose an orthonormal basis

$$\beta_2 = \{u_1, u_2, \ldots, u_k\}$$

for W_2, and complete it to an orthonormal basis

$$\gamma_2 = \{u_1, u_2, \ldots, u_n\}$$

for V. Note that we have the same k since $\dim(W_1) = \dim(W_2)$. Now define the operator T by $T(v_i) = u_i$, for $1 \leq i \leq n$. Then T is well-defined, orthogonal, and maps W_1 to W_2.

8.

(a) We compute

$$\langle \mathbf{a}, \mathbf{b}\rangle = 0$$

but

$$\|\mathbf{a}\| = \sqrt{3}$$

and

$$\|\mathbf{b}\| = \sqrt{6}$$

(b) From part (a), we know that an orthonormal basis for this subspace is

$$\left\{\frac{1}{\sqrt{3}}\mathbf{a}, \frac{1}{\sqrt{6}}\mathbf{b}\right\}.$$

So the projection is

$$\left\langle \mathbf{x}, \frac{1}{\sqrt{3}}\mathbf{a}\right\rangle \frac{1}{\sqrt{3}}\mathbf{a} + \left\langle \mathbf{x}, \frac{1}{\sqrt{6}}\mathbf{b}\right\rangle \frac{1}{\sqrt{6}}\mathbf{b} = \frac{1}{3}6\mathbf{a} + \frac{1}{6}3\mathbf{b} = 2\mathbf{a} + \frac{1}{2}\mathbf{b} = \left(\frac{3}{2}, \frac{3}{2}, 3\right)^T.$$