Math 423 final exam answers

1. (a) False. This is only true if \( V = W_1 \oplus W_2 \), in other words if \( W_1 \cap W_2 = \{0\} \).
(b) False. In this case, \( AB \) is self-adjoint if and only if \( A \) and \( B \) commute.
(c) True. Homework problem.
(d) False. In fact, \( \det(cA) = c^n \det A \).
(e) True. By the dimension theorem, it follows that \( T \) is also one-to-one.

2. The columns of the matrix are the coordinates of \( T(1) \) and \( T(x) \) in the basis \( \gamma \). So
\[
T(1) = 2
\]
and
\[
T(x) = 1 + (2 + x) = 3 + x.
\]
Therefore \( T(a + bx) = a(2) + b(3 + x) = (2a + 3b) + bx \).

3. (a) \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).
(b) \( iI = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \).
(c) \( \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \) is diagonalizable, with eigenvectors \((1, 0)\) and \((1, 1)\), but these are not orthogonal, and the matrix is not normal.
(d) \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) is a Jordan block, and is not diagonalizable.

4. On a finite dimensional vector space, \( T \) is invertible if and only if it is one-to-one, in other words if \( N(T) = \{0\} \). But the null space of \( T \) is exactly the eigenspace for the eigenvalue 0. So \( T \) has no eigenvectors with eigenvalue 0 if and only if \( N(T) = \{0\} \).

Method II. Since the determinant is the product of eigenvalues, \( T \) has 0 as an eigenvalue if and only if \( \det T = 0 \), if and only if \( T \) is not invertible.

5. The matrix is upper-triangular, so its eigenvalues are 2 with multiplicity 2 and 1 with multiplicity 1. We compute
\[
N(A - 2I) = N \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} = N \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} (e_1)
\]
and
\[
N((A - 2I)^2) = N \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} = N \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} (e_1, e_2),
\]
so we take as the end vector $e_2$, and compute 
$(A - 2I)e_2 = e_1$. Also,

$N(A - I) = N \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = N \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} \left( \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right) = \text{Span} \left( -e_1 + 2e_2 + e_3 \right)$. 

So in the Jordan basis $\{e_1, e_2, -e_1 + 2e_2 + e_3\}$, the matrix has the form

$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

6. $\dim(R(T)) = 2$. Therefore by the dimension theorem, $\dim(N(T)) = 2 - 2 = 0$. It follows that $T$ is one-to-one.

7. We can choose an orthonormal basis

$\beta_1 = \{v_1, v_2, \ldots, v_k\}$

for $W_1$, and complete it to an orthonormal basis

$\gamma_1 = \{v_1, v_2, \ldots, v_n\}$

for $V$. Similarly, we can choose an orthonormal basis

$\beta_2 = \{u_1, u_2, \ldots, u_k\}$

for $W_2$, and complete it to an orthonormal basis

$\gamma_2 = \{u_1, u_2, \ldots, u_n\}$

for $V$. Note that we have the same $k$ since $\dim(W_1) = \dim(W_2)$. Now define the operator $T$ by $T(v_i) = u_i$, for $1 \leq i \leq n$. Then $T$ is well-defined, orthogonal, and maps $W_1$ to $W_2$.

8. 

(a) We compute

$\langle a, b \rangle = 0$

but

$\|a\| = \sqrt{3}$

and

$\|b\| = \sqrt{6}$

(b) From part (a), we know that an orthonormal basis for this subspace is

$\left\{ \frac{1}{\sqrt{3}} a, \frac{1}{\sqrt{6}} b \right\}$.

So the projection is

$\left\langle x, \frac{1}{\sqrt{3}} a \right\rangle \frac{1}{\sqrt{3}} a + \left\langle x, \frac{1}{\sqrt{6}} b \right\rangle \frac{1}{\sqrt{6}} b = \frac{1}{3} a + \frac{1}{6} b = 2a + \frac{1}{2} b = \left( \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right)^T$. 