In all problems where \( p = \infty \) is allowed, do not forget to consider that case!

1. Let \( m(E) < \infty \). Show that if \( 1 \leq p \leq \infty \), then \( L_p(E) \subset L_1(E) \).

2. Exercise 19.43 (page 348).


4. Let \( 1 \leq p \leq \infty \), and \( f_k, f \in L_p(\mathbb{R}) \). We say that \( f_k \to f \) weakly if for all \( g \in L_q(\mathbb{R}) \),

(a) Prove that if \( f_k \to f \) in \( L_p \), then \( f_k \to f \) weakly.

(b) Show that the converse is false for \( p = \infty \). (Hint: Riemann-Lebesgue Lemma).

Quiz 10.

(a) Let \( 1 < p < \infty \). Given \( f \in L_p(E) \), describe conditions on \( g \in L_q(E) \) under which we have equality in Hölder’s inequality,

\[
\left| \int_E fg \right| = \|f\|_p \|g\|_q.
\]

Note that the condition in Exercise 24 is necessary but not sufficient. Hint: first write \( f(x) = |f(x)| \text{sign}(f(x)) \), where \( \text{sign}(u) = -1, 0, 1 \).

(b) Let \( g \in L_q(E) \). Prove that

\[
\|g\|_q = \sup_{\|f\|_p \leq 1} \int_E fg.
\]

(c) Recall that for each \( g \) as above, we have a linear functional \( F_g : L_p(E) \to \mathbb{R} \),

\[
F_g(f) = \int_E fg,
\]

and the norm of this functional is defined to be the smallest constant \( C \) such that \( |F_g f| \leq C \|f\|_p \) for all \( f \). Show that \( \|F_g\| = \|g\|_q \).
Math 447, Honors Homework 10

In all problems where \( p = \infty \) is allowed, do not forget to consider that case!

1. Let \( m(E) < \infty \). Show that if \( 0 < p_1 < p_2 \leq \infty \), then \( L^p_2(E) \subset L^p_1(E) \). Hint: for practice, you may want to solve the non-honors version of the problem first.

2. Exercise 19.43 (page 348).


4. Let \( 1 \leq p \leq \infty \), and \( f_k, f \in L^p(\mathbb{R}) \). We say that \( f_k \to f \) weakly if for all \( g \in L^q(\mathbb{R}) \), \( \int f_k g \to \int f g \).

   (a) Prove that if \( f_k \to f \) in \( L^p \), then \( f_k \to f \) weakly.
   (b) Show that the converse is false for \( p = \infty \). (Hint: Riemann-Lebesgue Lemma). Does it hold for finite \( p \)? This last question is to think about, not to turn in.

Honors quiz 10.

(a) Let \( 1 \leq p < \infty \). Given \( f \in L^p(E) \), describe conditions on \( g \in L^q(E) \) under which we have equality in Hölder’s inequality,

\[
\left| \int_E fg \right| = \|f\|_p \|g\|_q.
\]

Note that the condition in Exercise 24 is necessary but not sufficient. Hint: first write \( f(x) = |f(x)| \text{sign}(f(x)) \), where \( \text{sign}(u) = -1, 0, 1 \). Do not forget the case \( p = 1 \)!

(b) Let \( g \in L^q(E), 1 < q \leq \infty \). Prove that

\[
\|g\|_q = \sup_{\|f\|_p \leq 1} \int_E fg.
\]

(c) Recall that for each \( g \) as above, we have a linear functional \( F_g : L^p(E) \to \mathbb{R} \),

\[
F_g(f) = \int_E fg,
\]

and the norm of this functional is defined to be the smallest constant \( C \) such that \( |F_g f| \leq C \|f\|_p \) for all \( f \). Show that \( \|F_g\| = \|g\|_q \).