Math 447, Homework 11.

1. For each of the following functions, state whether its Fourier series on \([-\pi, \pi]\) converges to it uniformly? Pointwise? Pointwise except some points (which ones)? Justify your answer. You do not need to verify that the functions have the given Fourier series. Also, use Parseval’s identity and evaluations at 0, \(\pi/2, \pi\) to derive any non-trivial numerical identities. If we do not get to it in class, for pointwise convergence use the result on page 259 of the text.

   (a) \(|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x.\)

   (b) \(x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.\)

2. Let \(f\) be piecewise continuous on \([-\pi, \pi]\), with \(f(-\pi) = f(\pi)\), and

\[ f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \]

Let \(F\) be its indefinite integral.

   (a) Show that the Fourier series of \(F(x) - \frac{a_0}{2}x\) is obtained by term-by-term integration of the Fourier series of \(f\). Hint: recall the definition of \(a_0\).

   (b) Show that with the correct choice of the constant of integration, this converges to \(F - \frac{a_0}{2}x\) uniformly. If you are going to use a theorem, make sure to check all the conditions!

3. Let \(f_n \to f\) and \(g_n \to g\) in \(L^2\). Show that

   (a) \(\|f_n\|_2 \to \|f\|_2\).
   (b) \(\langle f_n, g_n \rangle \to \langle f, g \rangle\).
Quiz 11.

(a) Let \((V_1, \langle \cdot, \cdot \rangle_1)\) and \((V_2, \langle \cdot, \cdot \rangle_2)\) be two inner product spaces. Suppose \(U : V_1 \to V_2\) is an isometry, that is, a linear map such that for any \(v \in V_1\), \(\|Uv\| = \|v\|\) (note that the norms on two sides of this equation live in different spaces). Prove that for any \(v, w \in V_1\),

\[ \langle Uv, Uw \rangle_2 = \langle v, w \rangle_1. \]

Hint: start with \(\langle v + w, v + w \rangle_1\), and use linearity properties of the inner product and its relation to the norm.

(b) Let \(f \in L_2(-\pi, \pi)\), with

\[ f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \]

Denote \(Uf = (a_0/\sqrt{2}, b_1, a_1, b_2, a_2, \ldots)\). Prove that \(Uf \in \ell_2\), that \(U : L_2(-\pi, \pi) \to \ell_2\) is a bijection, and that \(\|Uf\|_2 = \|f\|_2\). Hint: most of this was done in class, you just need to combine the relevant results.

(c) Conclude that if \(g\) is another function in \(L_2(-\pi, \pi)\) with

\[ g \sim \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} (\alpha_k \cos kx + \beta_k \sin kx), \]

then

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} fg = \frac{a_0 \alpha_0}{2} + \sum_{k=1}^{\infty} (a_k \alpha_k + b_k \beta_k). \]