

Multivariate Stieltjes continued fractions

Michael Anshelevich

June 17, 2008

$\mu =$ probability measure on \mathbb{R} , with all moments finite.

Identify with a functional μ on $\mathbb{R}[x]$.

Moments

$$\mu[x], \mu[x^2], \mu[x^3], \dots$$

Moment generating function

$$M^\mu(z) = 1 + \mu[x]z + \mu[x^2]z^2 + \mu[x^3]z^3 + \dots$$

Has a (Stieltjes) continued fractions expansion

$$\begin{aligned} M^\mu(z) &= 1 + \mu[x]z + \mu[x^2]z^2 + \mu[x^3]z^3 + \dots \\ &= \frac{1}{1 - \beta_0 z - \frac{\gamma_1 z^2}{1 - \beta_1 z - \frac{\gamma_2 z^2}{1 - \beta_2 z - \frac{\gamma_3 z^2}{1 - \dots}}}}. \end{aligned}$$

Coefficients: Jacobi parameters.

For $\{P_n(x)\}$ monic orthogonal polynomials for μ ,

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x).$$

MULTIVARIATE CASE

$\varphi =$ positive linear functional on $\mathbb{R}\langle x_1, x_2, \dots, x_d \rangle$, polynomials in *non-commuting* variables. Unital (= probability). A “state”.

MULTIVARIATE CASE

$\varphi =$ positive linear functional on $\mathbb{R}\langle x_1, x_2, \dots, x_d \rangle$, polynomials in *non-commuting* variables. Unital (= probability). A “state”.

Where do these appear?

MULTIVARIATE CASE

φ = positive linear functional on $\mathbb{R}\langle x_1, x_2, \dots, x_d \rangle$, polynomials in *non-commuting* variables. Unital (= probability). A “state”.

Where do these appear?

Let X_1, X_2, \dots, X_d be symmetric (better, self-adjoint) operators on a Hilbert space \mathcal{H} , with common invariant dense domain containing a unit vector Ω . Then their joint distribution in the vector state Ω is the functional

$$\varphi [P(x_1, x_2, \dots, x_d)] = \langle \Omega, P(X_1, X_2, \dots, X_d)\Omega \rangle .$$

MULTIVARIATE CASE

φ = positive linear functional on $\mathbb{R}\langle x_1, x_2, \dots, x_d \rangle$, polynomials in *non-commuting* variables. Unital (= probability). A “state”.

Where do these appear?

Let X_1, X_2, \dots, X_d be symmetric (better, self-adjoint) operators on a Hilbert space \mathcal{H} , with common invariant dense domain containing a unit vector Ω . Then their joint distribution in the vector state Ω is the functional

$$\varphi [P(x_1, x_2, \dots, x_d)] = \langle \Omega, P(X_1, X_2, \dots, X_d)\Omega \rangle .$$

Moment generating function

$$M^\varphi(z_1, z_2, \dots, z_d) = 1 + \sum_{i=1}^d \varphi [x_i] z_i + \sum_{i,j=1}^d \varphi [x_i x_j] z_i z_j + \dots$$

Continued fraction expansion?

$$\begin{aligned}
 M^\mu(z) &= 1 + \mu[x]z + \mu[x^2]z^2 + \mu[x^3]z^3 + \dots \\
 &= \frac{1}{1 - \beta_0 z - \frac{\gamma_1 z^2}{1 - \beta_1 z - \frac{\gamma_2 z^2}{1 - \beta_2 z - \frac{\gamma_3 z^2}{1 - \dots}}}}
 \end{aligned}$$

can be done by induction, breaks down if some $\gamma = 0$. Multivariate:

$$\begin{aligned}
 1 + \sum a_i z_i + \sum a_{ij} z_i z_j + \dots &= \frac{1}{1 - \sum b_i z_i - \sum b_{ij} z_i z_j - \sum b_{ijk} z_i z_j z_k \dots} \\
 &\neq \frac{1}{1 - \sum b_i z_i - \sum b_{ij} z_i F_{ij} z_j}
 \end{aligned}$$

if e.g. $b_{ij} = 0$, $b_{ikj} \neq 0$.

PRODUCT-TYPE EXAMPLES

Distributions with fixed marginals. For simplicity, restrict to the symmetric case:
 $\mu^{(1)}, \mu^{(2)},$

$$x_i P_n^{(i)}(x_i) = P_{n+1}^{(i)}(x_i) + \gamma_n^{(i)} P_{n-1}^{(i)}(x_i), \quad i = 1, 2.$$

Commutative case: many measures μ on \mathbb{R}^2 with marginals $\mu^{(1)}, \mu^{(2)}$. Canonical one: product measure

$$\mu[x_1^k x_2^n] = \mu[x_1^k] \mu[x_2^n],$$

x_1, x_2 independent with respect to μ , orthogonal polynomials

$$\left\{ P_i^{(1)}(x_1) P_j^{(2)}(x_2) \right\}.$$

Non-commutative case: want φ on $\mathbb{R}\langle x_1, x_2 \rangle$ with

$$\varphi [x_1^n] = \mu^{(1)} [x_1^n], \quad \varphi [x_2^n] = \mu^{(2)} [x_2^n].$$

Again many choices, more than one canonical choice.

Non-commutative case: want φ on $\mathbb{R}\langle x_1, x_2 \rangle$ with

$$\varphi [x_1^n] = \mu^{(1)} [x_1^n], \quad \varphi [x_2^n] = \mu^{(2)} [x_2^n].$$

Again many choices, more than one canonical choice.

Example.

$$1 + M(z_1, z_2) = \frac{1}{1 - \frac{\gamma_1^1 z_1^2}{1 - \frac{\gamma_2^1 z_1^2}{1 - \frac{\gamma_3^1 z_1^2}{1 - \dots}}}} - \frac{\gamma_1^2 z_2^2}{1 - \frac{\gamma_2^2 z_2^2}{1 - \frac{\gamma_3^2 z_2^2}{1 - \dots}}}$$

Corresponds to the rule

$$\varphi \left[x_1^{u(1)} x_2^{v(1)} x_1^{u(2)} x_2^{v(2)} \dots \right] = \varphi \left[x_1^{u(1)} \right] \varphi \left[x_2^{v(1)} \right] \varphi \left[x_1^{u(2)} \right] \varphi \left[x_2^{v(2)} \right] \dots$$

(recall x_1, x_2 do not commute).

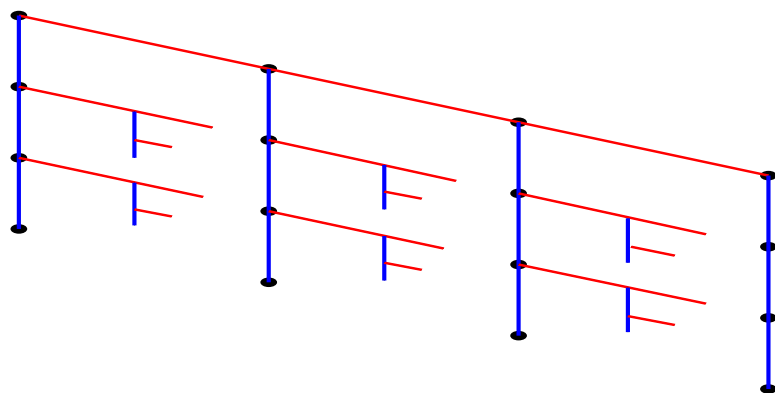
Boolean product. Not very natural since very degenerate.

Example. Free product, free independence. Rule for

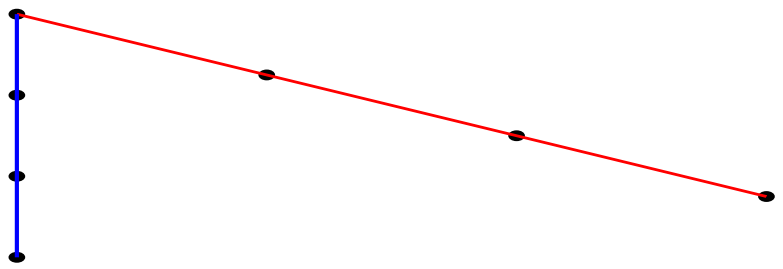
$$\varphi \left[x_1^{u(1)} x_2^{v(1)} x_1^{u(2)} x_2^{v(2)} \dots \right]$$

complicated, but appears in applications, e.g. random matrices.

$$\begin{array}{c}
 1 \\
 \hline
 1 - \frac{\gamma_1^1 z_1 | z_1}{1 - \frac{\gamma_2^1 z_1 | z_1}{1 - \frac{\gamma_3^1 z_1 | z_1}{1 - \dots}} - \frac{\gamma_1^2 z_2 | z_2}{1 - \dots}} - \frac{\gamma_1^2 z_2 | z_2}{1 - \frac{\gamma_1^1 z_1 | z_1}{1 - \frac{\gamma_2^1 z_1 | z_1}{1 - \dots}} - \frac{\gamma_2^2 z_2 | z_2}{1 - \dots}} - \frac{\gamma_1^2 z_2 | z_2}{1 - \dots}
 \end{array}$$



$$\begin{array}{c}
 1 \\
 \hline
 1 - \frac{\gamma_1^1 z_1^2}{1 - \frac{\gamma_2^1 z_1^2}{1 - \frac{\gamma_3^1 z_1^2}{1 - \dots}}} - \frac{\gamma_1^2 z_2^2}{1 - \frac{\gamma_2^2 z_2^2}{1 - \frac{\gamma_3^2 z_2^2}{1 - \dots}}}
 \end{array}$$



For a special but large class of φ , have a **matricial** continued fraction expansion. Namely, when φ has **monic** multivariate orthogonal polynomials:

$$\{P_{\vec{u}} = x_{\vec{u}} + \dots\},$$

$$\varphi [P_{\vec{u}}^*(x_1, \dots, x_d) P_{\vec{v}}(x_1, \dots, x_d)] = 0$$

for $\vec{u} \neq \vec{v}$.

Do not always have them. Existence equivalent to special (graded) Hilbert space \mathcal{H} and operators X_i .

Theorem. Let φ be a state with a monic orthogonal polynomials. There exist matrices

$$\mathcal{C}^{(k)} = \text{diagonal non-negative } d^k \times d^k \text{ matrix, } k = 1, 2, \dots$$

and

$$\mathcal{T}_i^{(k)} = d^k \times d^k \text{ matrix, } k = 0, 1, \dots, i = 1, 2, \dots, d$$

such that

$$1 + M(\mathbf{z}) =$$

$$\frac{1}{1 - \sum_{i_0} z_{i_0} \mathcal{T}_{i_0}^{(0)} - \frac{\sum_{j_1} z_{j_1} E_{j_1} \mathcal{C}^{(1)} | \sum_{k_1} E_{k_1} z_{k_1}}{1 - \sum_{i_1} z_{i_1} \mathcal{T}_{i_1}^{(1)} - \frac{\sum_{j_2} z_{j_2} E_{j_2} \mathcal{C}^{(2)} | \sum_{k_2} E_{k_2} z_{k_2}}{1 - \dots}}}$$

Here for matrices

$$A, B \in M_{d^k \times d^k} \simeq M_{d \times d} \otimes M_{d \times d} \otimes \dots \otimes M_{d \times d},$$

we use the notation

$$\frac{E_i A | E_j}{B} = \langle e_i \otimes I \otimes \dots \otimes I, AB^{-1}(e_j \otimes I \otimes \dots \otimes I) \rangle \in M_{d^{k-1} \times d^{k-1}}.$$

Example.

$$E_1 \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} E_2 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$