Derivatives and trees

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All know

$$(FG)' = F'G + FG' \qquad \text{(product rule)}$$

$$(F \circ G)' = F'(G)G' \qquad \text{(chain rule)}$$

$$(F^{-1})' = \frac{1}{F'(F^{-1})} \qquad \text{(inverse function rule)}$$

What about

$$(FG)''''$$

$$(F \circ G)''''$$

$$(F^{-1})''''?$$

Use Taylor series (invented by Newton)

Once have the answer, can check it even if don’t have Taylor expansions.

$$
\begin{aligned}
F(x) &= F(0) + \frac{F'(0)}{1!}x + \frac{F''(0)}{2!}x^2 + \frac{F^{(3)}(0)}{3!}x^3 + \ldots \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,
\end{aligned}
$$

$$
a_n = \frac{F^{(n)}(0)}{n!}.
$$

Problem: find the coefficients in the power series expansions of

$$
FG, \quad F \circ G, \quad F^{-1}.
$$

PRODUCT RULE.

$$F(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

$$G(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

$$(a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + .$$

$$F(x)G(x) = \sum_{n=0}^{\infty} a_n x^n \sum_{k=0}^{\infty} b_k x^k$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n b_k x^{n+k} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} a_n b_{m-n} x^m.$$

Thus

$$F(x)G(x) = \sum_{m=0}^{\infty} c_m x^m,$$

$$c_m = \sum_{n=0}^{m} a_n b_{m-n}.$$

$$
\begin{aligned}
(FG)^{(m)} &= m! c_m = m! \sum_{n=0}^{m} \frac{F^{(n)}}{n!} \frac{G^{(m-n)}}{(m-n)!} \\
&= \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} F^{(n)} G^{(m-n)}
\end{aligned}
$$

$$
(FG)^{(m)} = \sum_{n=0}^{m} \binom{m}{n} F^{(n)} G^{(m-n)}.
$$

Binomial coefficients:

$$
(s+t)^m = \sum_{n=0}^{m} \binom{m}{n} s^n t^{m-n}.
$$

CHAIN RULE

Assume $a_0 = b_0 = 0$,

$$F(x) = a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$G(x) = b_1 x + b_2 x^2 + b_3 x^3 \dots$$

$$\begin{aligned} F(G(x)) &= a_1 G + a_2 G^2 + a_3 G^3 + \dots \\ &= a_1 \sum_u b_u x^u + a_2 \sum_u b_u x^u \sum_v b_v x^v + \dots \\ &= a_1 \sum b_u x^u + a_2 \sum b_{u_1} b_{u_2} x^{u_1+u_2} \\ &\quad + a_3 \sum b_{u_1} b_{u_2} b_{u_3} x^{u_1+u_2+u_3} + \dots . \end{aligned}$$

$$\begin{aligned} c_1 &= a_1 b_1, \\ c_2 &= a_1 b_2 + a_2 b_1 b_1, \\ c_3 &= a_1 b_3 + a_2 b_1 b_2 + a_2 b_2 b_1 + a_3 b_1 b_1 b_1, \\ c_m &= \sum_{n=1}^{m} a_n \sum_{\substack{u_1, u_2, \dots, u_n \geq 1 \\ u_1 + u_2 + \dots + u_n = m}} b_{u_1} b_{u_2} \dots b_{u_n}. \end{aligned}$$

$$
\begin{aligned}
(F \circ G)^{(m)} &= m! \sum_{n=1}^{m} \frac{F^{(n)}}{n!} \sum \frac{G^{(u_1)}}{u_1!} \frac{G^{(u_2)}}{u_2!} \cdots \frac{G^{(u_n)}}{u_n!} \\
&= \sum_{n=1}^{m} F^{(n)} \sum \frac{1}{n!} \frac{m!}{u_1! u_2! \ldots u_n!} G^{(u_1)} G^{(u_2)} \ldots G^{(u_n)} \\
&= \sum_{n=1}^{m} F^{(n)} \sum \frac{1}{n!} \binom{m}{u_1, u_2, \ldots, u_n} G^{(u_1)} \ldots G^{(u_n)}.
\end{aligned}
$$

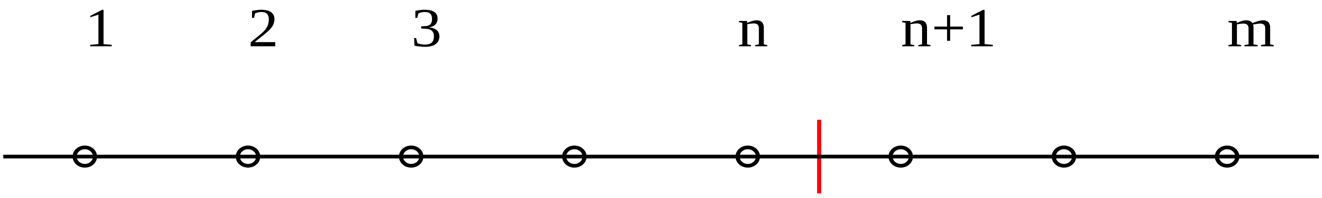
Multinomial coefficients:

$$
\begin{aligned}
&(s_1 + s_2 + \ldots + s_n)^m \\
&\quad = \sum_{\substack{u_1, u_2, \ldots, u_n \geq 0 \\ u_1 + u_2 + \ldots + u_n = m}} \binom{m}{u_1, u_2, \ldots, u_n} s_1^{u_1} s_2^{u_2} \ldots s_n^{u_n}.
\end{aligned}
$$

Ugly! Reformulate.

Back to product rule.

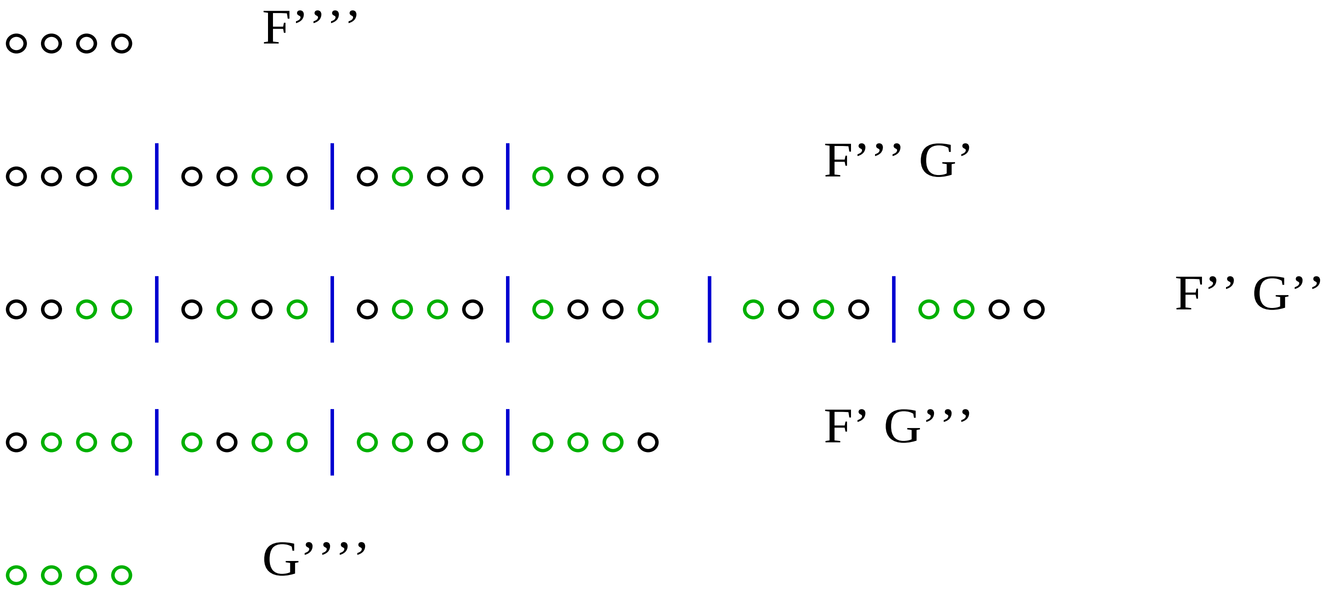
$$c_m = \sum_{n=0}^{m} a_n b_{m-n}.$$



$$(FG)^{(m)} = \sum_{n=0}^{m} \binom{m}{n} F^{(n)} G^{(m-n)}.$$

$\binom{m}{n}$ = # of n-element subsets in an m-element set.

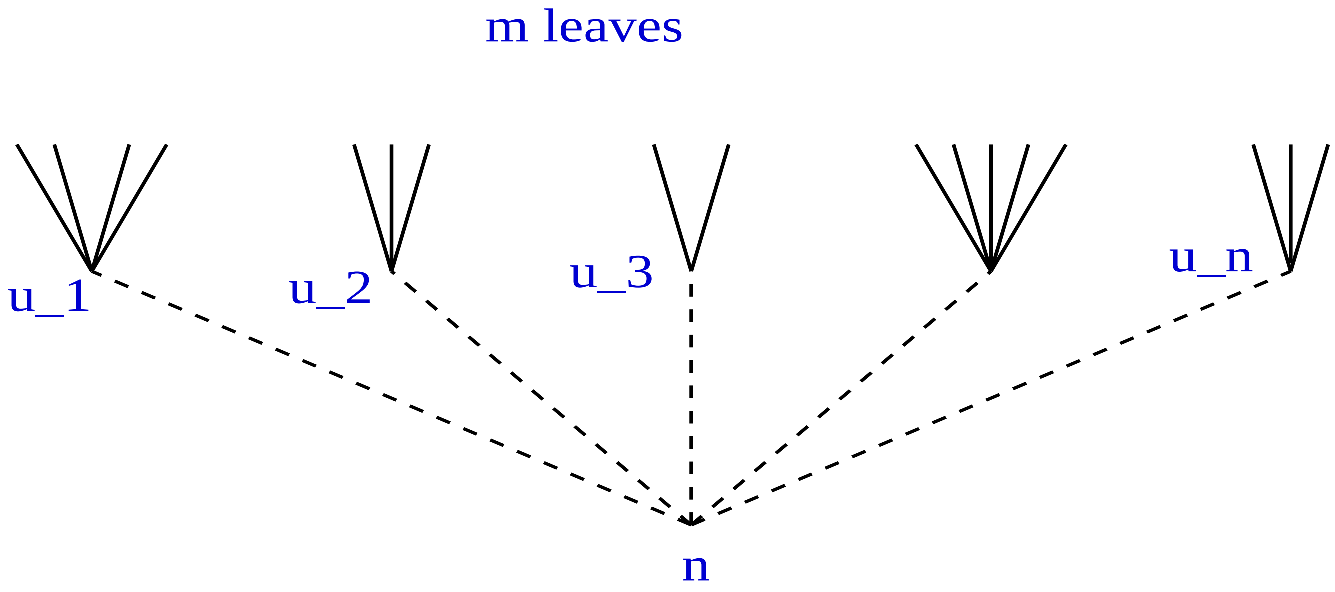
$$(FG)^{(m)} = \sum_{S \subset [m]} F^{|S|} G^{m-|S|}.$$


F’’’’
F’’’ G’
F’’ G’’
F’ G’’’
G’’’’

What is the combinatorial structure for the chain rule?

$$c_m = \sum_{n=1}^{m} a_n \sum_{\substack{u_1,u_2,\ldots,u_n \geq 1 \\ u_1+u_2+\ldots+u_n=m}} b_{u_1} b_{u_2} \ldots b_{u_n}.$$

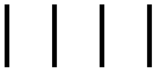
bushes.



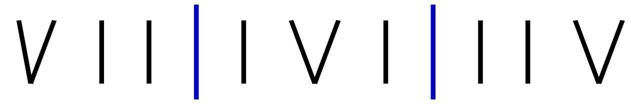
For a shrubbery $\mathcal{P}$

$$A(\mathcal{P}) = a_{\#\text{ bushes}} \prod_{\text{bushes}} b_{\#\text{ leaves on a bush}}$$

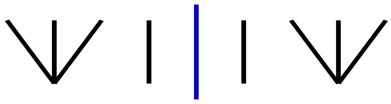
$$c_m = \sum_{\text{shrubberies with } m \text{ leaves}} A(\mathcal{P})$$



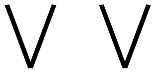
a_4 b_1 b_1 b_1 b_1



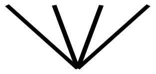
a_3 b_1 b_1 b_2



a_2 b_1 b_3

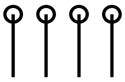


a_2 b_2 b_2

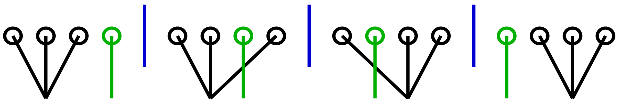


a 1 b 4

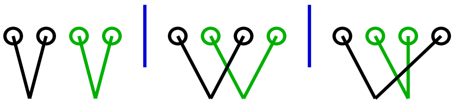
$$(F \circ G)^{(m)} = \sum_{\text{ordered shrubberies}} = \sum_{\text{partitions}} .$$



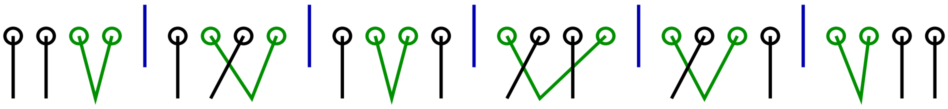
F’’’’ G’ G’ G’ G’



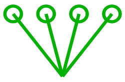
F’’ G’ G’’’



F’’ G’’ G’’



F’’’ G’ G’ G’’



F’ G’’’’

$$(F^{-1})^{(n)}.$$

$$F^{-1} \circ F = x$$

$$(F^{-1} \circ F)' = 1, \qquad (F^{-1} \circ F)^{(n)} = 0.$$

$$\begin{aligned} 1 &= a_1 b_1 \\ 0 &= a_1 b_2 + a_2 b_1 b_1 \\ 0 &= a_1 b_3 + a_2 b_1 b_2 + a_2 b_2 b_1 + a_3 b_1 b_1 b_1. \end{aligned}$$

Let $b_1 = 1$.

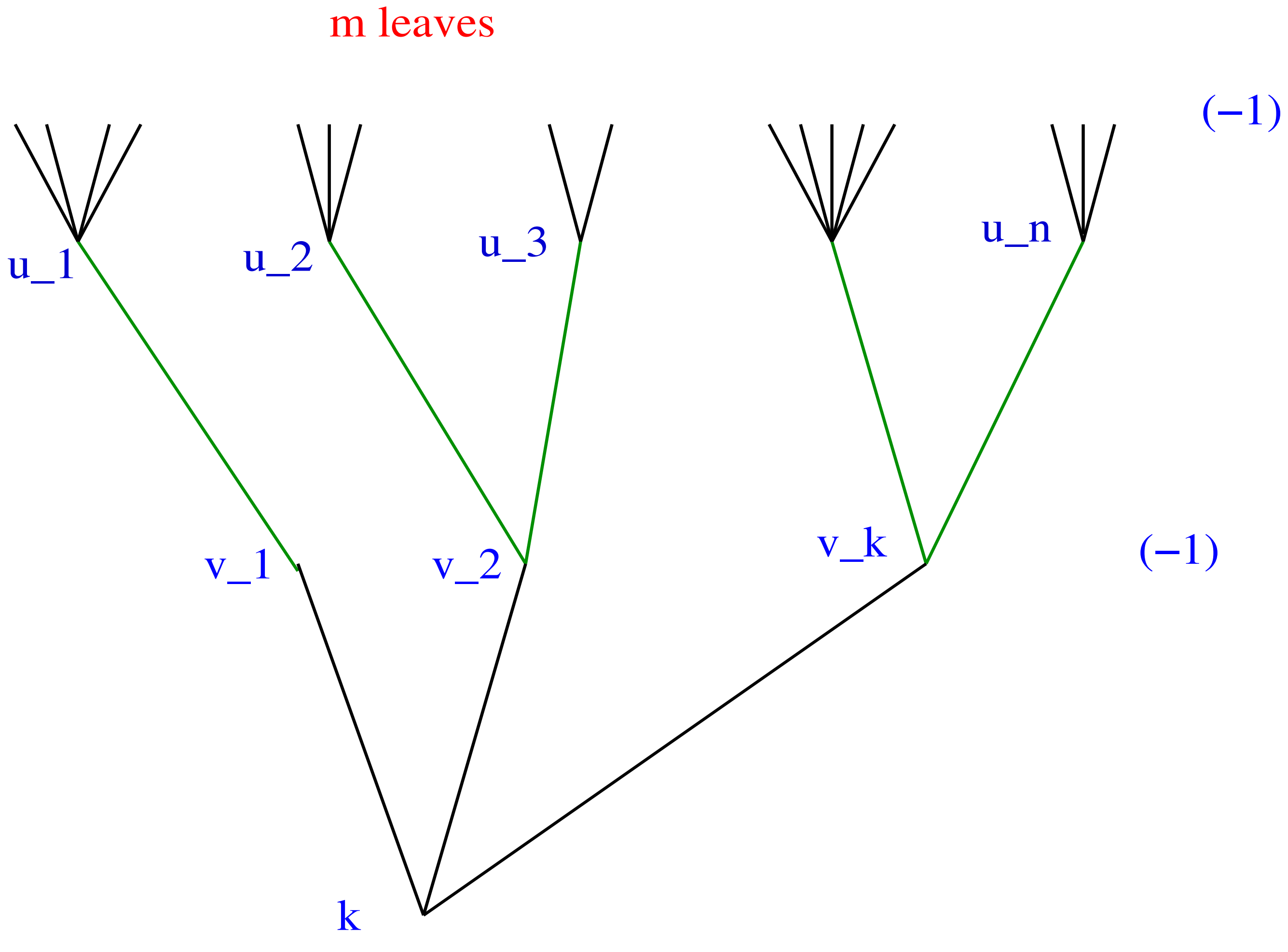
$$\begin{aligned} a_1 &= 1 \\ a_2 &= -a_1 b_2 \\ a_3 &= -a_1 b_3 - a_2 b_1 b_2 - a_2 b_2 b_1 \end{aligned}$$

$$a_m = -\sum_{n=1}^{m-1} a_n \sum_{\substack{u_1,u_2,\dots,u_n \geq 1 \\ u_1+u_2+\dots+u_n=m}} b_{u_1} b_{u_2} \dots b_{u_n}.$$

In its turn

$$a_n = -\sum_{k=1}^{n-1} a_k \sum_{v_1,v_2,\dots,v_k} b_{v_1} b_{v_2} \dots b_{v_k}.$$

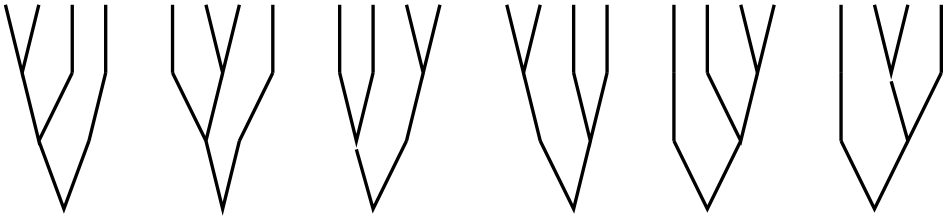
$$a_m = -\sum_{n=1}^{m-1} a_n \sum_{\substack{u_1,u_2,\ldots,u_n \geq 1 \\ u_1+u_2+\ldots+u_n=m}} b_{u_1} b_{u_2} \ldots b_{u_n}.$$



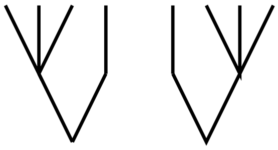
Decorated tree, on each level at least one branch splits.

$$B(\mathcal{T}) = b_k b_{v_1} \dots b_{v_k} b_{u_1} \dots b_{u_n}.$$

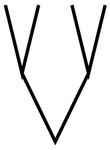
$$a_m = \sum_{\text{splitting trees with } m \text{ leaves}} (-1)^{\#\text{ levels}} B(\mathcal{T})$$



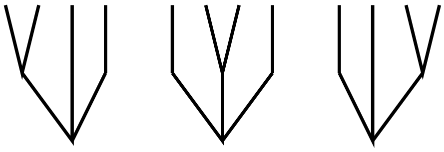
−b_2 b_2 b_2



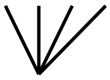
b_2 b_3



b_2 b_2 b_2



b_3 b_2



−b_4

Used combinatorics to find derivatives.

Can also use derivatives for combinatorics.

What is the sum of binomial coefficients?

Many ways. One of them:

Let $F(x) = G(x) = e^x$. Then

$$F^{(n)}(0) = G^{(n)}(0) = 1.$$

Use the product rule:

$$(FG)^{(m)} = \sum_{n=0}^{m} \binom{m}{n} F^{(n)} G^{(m-n)} = \sum_{n=0}^{m} \binom{m}{n}.$$

But $(FG)(x) = e^{2x}$ and

$$(FG)^{(m)}(0) = 2^m.$$

How many partitions of a set of n elements are there?

Called Bell numbers, no easy formula. But:

Let $F(x) = e^{ax} - 1$, $G(x) = e^{bx} - 1$.

$$F^{(n)}(0) = a^n, \qquad G^{(n)}(0) = b^n.$$

Use the chain rule:

$$(F \circ G)^{(m)} = \sum_{\text{partitions}} a^{\text{\# classes}} b^m.$$

But

$$F \circ G = e^{a(e^{bx}-1)} - 1.$$

Thus

$$\begin{aligned} &\text{\# partitions of } m \text{ with } k \text{ classes} \\ &\qquad = \text{ coefficient of } a^k b^m \text{ in } e^{a(e^{bx}-1)} - 1. \end{aligned}$$

$$\text{\# partitions of } m = \text{coefficient of } b^m \text{ in } e^{e^{bx}-1} - 1.$$