Linearization coefficients, orthogonal polynomials, and free probability

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$\{H_n(x)\}$ = (monic) Hermite polynomials, orthogonal with respect to the normal distribution

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

$$\int_{\mathbb{R}} H_n(x)\, d\mu(x) = \delta_{n0},$$

$$\int_{\mathbb{R}} H_n(x) H_k(x)\, d\mu(x) = \delta_{nk} n!$$

What is

$$c_{n,k,l} = \int_{\mathbb{R}} H_n(x) H_k(x) H_l(x)\, d\mu(x)?$$

Why care:

$$f = \sum a_n H_n, \qquad g = \sum b_k H_k; \qquad fg = \sum \cdot H_l.$$

$$H_n H_k = \sum_{l=0}^{n+k} c_{n,k,l} \frac{1}{l!} H_l.$$

Linearization coefficients.

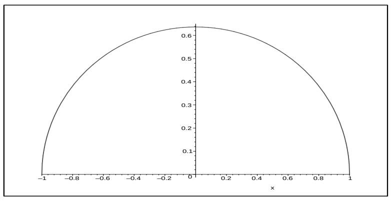
Answer: positive integers!

$$c_{n,k,l} = \begin{cases} \dfrac{n!k!l!}{\left(\frac{n+k-l}{2}\right)!\left(\frac{n+l-k}{2}\right)!\left(\frac{k+l-n}{2}\right)!}, & n+k+l \text{ even}, \\ & n+k \geq l, \\ & n+l \geq k, \\ & k+l \geq n, \\ 0, & \text{otherwise.} \end{cases}$$

Integers "count something".

$\{U_n\}$ Chebyshev polynomials of the 2nd kind, orthogonal with respect to the semicircle law

$$d\mu(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{[-2,2]}\, dx.$$



Linearization coefficients are

$$c_{n,k,l} = \begin{cases} 1, & \text{same conditions,} \\ 0, & \text{otherwise.} \end{cases}$$

A very general family: $\{P_{n,q,\alpha}(x)\}$ continuous big q-Hermite polynomials. Can write down the orthogonality measure (basic hypergeometric function), easier to use the recursion relation

$$P_{n+1} = xP_n - \alpha[n]_q P_n - [n]_q P_{n-1},$$

where

$$[n]_q = 1 + q + \ldots + q^{n-1}$$

are the q-integers.

$\alpha = 0$, $q = 1$: Hermite.

$\alpha = 0$, $q = 0$: Chebyshev.

$\alpha = 1$, $q = 1$: Charlier (orthogonal wrt Poisson).

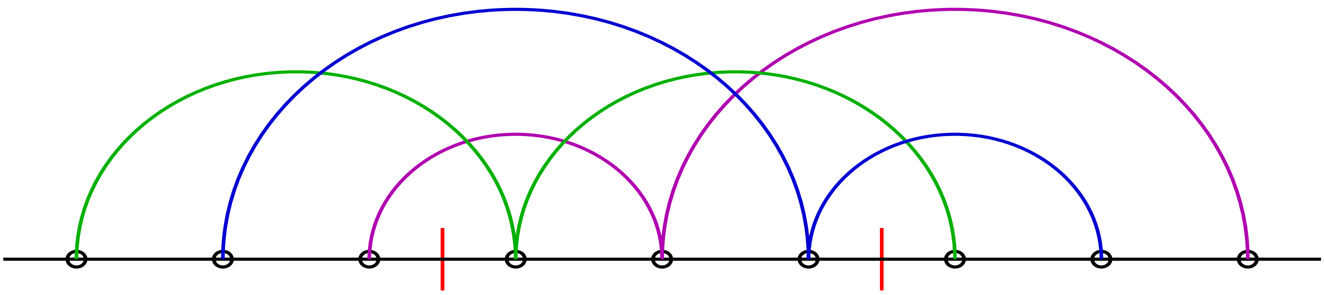
Proposition. (M.A. '04)

$$c_{n,k,l} = \sum_{\pi \in \mathcal{P}(n,k,l)} q^{\mathrm{rc}(\pi)} \alpha^{n+k+l-2|\pi|}.$$

where

$\mathcal{P}(n,k,l) =$ inhomogeneous (n,k,l)-partitions,

$\mathrm{rc}\,(\pi) =$ number of restricted crossings of π.



If $\alpha = 0$, $n + k + l = 2\,|\pi|$, so $\pi =$ pair partition.

If $q = 0$, $\mathrm{rc}\,(\pi) = 0$, $\pi =$ non-crossing partition.

Hermite: pair partitions.

Chebyshev: non-crossing pair partitions.

FREE PROBABILITY.

Let μ, ν be two probability measures. Will define their free convolution.

Let $\{A_n\}_{n=1}^{\infty}$ be $n \times n$ diagonal matrices, with diagonal entries $\left\{\lambda_{i,n}\right\}$ such that

$$\frac{1}{n}\left(\delta_{\lambda_{1,n}} + \delta_{\lambda_{2,n}} + \ldots + \delta_{\lambda_{n,n}}\right) \xrightarrow{w} \mu.$$

Let $\{B_n\}$ similarly approximate ν.

Instead, let $\{U_n\}$ be random unitary matrices, uniformly distributed according to Haar measure on $\mathcal{U}_n$.

$$A_n + U_n B_n U_n^{-1} = \text{ random Hermitian matrix,}$$

$\left\{\rho_{i,n}\right\}$ = its random eigenvalues,

$$\frac{1}{n}\left(\delta_{\rho_{1,n}} + \delta_{\rho_{2,n}} + \ldots + \delta_{\rho_{n,n}}\right) = \text{random measure.}$$

Theorem. (Voiculescu '91, Benaych-Georges '03) These random measures converge almost surely to a fixed measure $\mu \boxplus \nu$, the free convolution of μ and ν.

How to calculate it: for $z \in \mathbb{C}_+$, let

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x)$$

be the Cauchy transform of μ. Has a local inverse with respect to composition. Define

$$R_\mu(z) = G_\mu^{-1}(z) - \frac{1}{z}.$$

R-transform of μ, analytic function on some domain in $\mathbb{C}$.

Theorem. (Bercovici, Voiculescu '93)

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z).$$

Can recover G from R, and $\mu \boxplus \nu$ from $G_{\mu \boxplus \nu}$ via Stieltjes inversion formula.

Moreover, can calculate all mixed moments such as

$$\lim_{n\to\infty} \mathrm{tr}\left[A_n(U_nB_nU_n^{-1})A_n(U_nB_nU_n^{-1})^2\right]$$

as follows: $A_n, U_nB_nU_n^{-1}$ are asymptotically freely independent.

If X, Y independent, cannot have

$$\mathrm{E}[XYXY] \neq \mathrm{E}[XXYY].$$

Definition. (Voiculescu) Operators a, b in a (von Neumann) algebra are freely independent with respect to a state φ if whenever

$$\varphi\left[p_1(a)\right] = \ldots = \varphi\left[p_n(a)\right] = 0,$$

and

$$\varphi\left[q_1(b)\right] = \ldots = \varphi\left[q_n(b)\right] = 0,$$

then

$$\varphi\left[p_1(a)q_1(b)\ldots p_n(a)q_n(b)\right] = 0.$$

Free probability: non-commutative probability theory, dealing with operators, based on free independence.

Freely independent operators appear in group algebras of free groups.

Also as operators on the full Fock space.

Many parallels to the usual theory.

Many differences.

Central Limit Theorem. Let $X_1, X_2, \ldots$ be independent centered random variables with common distribution μ. Then under certain conditions on μ (variance function slowly varying)

$$\text{dist}\, \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}$$

$$= (\mu * \mu * \ldots * \mu) \circ D_{1/\sqrt{n}} \longrightarrow \text{ normal distribution.}$$

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Free Central Limit Theorem. Let $Y_1, Y_2, \ldots$ be freely independent centered operators with common distribution μ. Then under the same conditions on μ

$$\begin{aligned} &\text{dist}\, \frac{Y_1 + Y_2 + \ldots + Y_n}{\sqrt{n}} \\ &= (\mu \boxplus \mu \boxplus \ldots \boxplus \mu) \circ D_{1/\sqrt{n}} \\ &\longrightarrow \text{ semicircular distribution.} \end{aligned}$$

DIFFERENCE.

For

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$$

Bernoulli,

$$\mu * \mu = \frac{1}{4}\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{4}\delta_2$$

atomic with 3 atoms.

In contrast:

$$\mu \boxplus \mu = \frac{1}{\pi\sqrt{x(2-x)}}\mathbf{1}_{[0,2]}(x)\,dx$$

absolutely continuous!

Combinatorics.

Consider measures all of whose moments are finite.

$$\int_{\mathbb{R}} e^{xz}\, d\mu(x) = \sum_{n=0}^{\infty} \frac{1}{n!} m_n(\mu) z^n,$$

$$m_n = \int_{\mathbb{R}} x^n\, d\mu(x) = \text{ moments.}$$

Have another sequence of cumulants (semi-invariants).

Yet another sequence: free cumulants

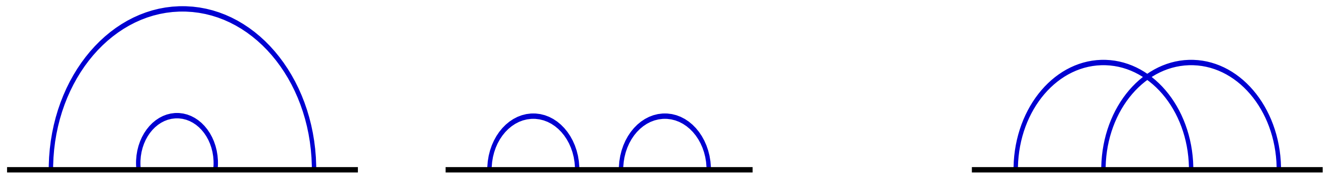
$$R_\mu(z) = \sum_{n=1}^{\infty} r_n z^{n-1}.$$

$$r_n(X+Y) = r_n(X) + r_n(Y)$$

if X, Y freely independent.

Combinatorial relation between moments and free cumulants:

Let $\pi \in NC(n)$ be a non-crossing partition of the set of n elements.



$$r_\pi = \prod_{B \in \pi} r_{|B|}.$$

Then (Speicher)

$$m_n = \sum_{\pi \in NC(n)} r_\pi.$$

Similar relation for classical cumulants and all partitions.

Proof of the free central limit theorem (combinatorial case):

$$\begin{aligned} r_k(Y_1 + \ldots + Y_n) &= r_k(Y_1) + \ldots + r_k(Y_n) \\ &= n r_k(Y). \end{aligned}$$

$$r_k(Y/\sqrt{n}) = r_k(Y)/\sqrt{n}^k = \frac{1}{n^{k/2}} r_k(Y).$$

Thus

$$r_k\left(\frac{Y_1 + Y_2 + \ldots + Y_n}{\sqrt{n}}\right) = \frac{1}{n^{(k-2)/2}} r_k(Y),$$

so in the limit $r_1 = 0$, $r_2 = r_2(Y)$, $r_k = 0$ for $k > 2$. So

$$\begin{aligned} m_n(\mu) &= \#\{\text{non-crossing pair partitions}\} \\ &= \begin{cases} 0, & n \text{ odd}, \\ n\text{'th Catalan number } \frac{1}{n/2+1}\binom{n}{n/2}, & n \text{ even}. \end{cases} \end{aligned}$$

Then $\mu =$ semicircle law.

Thus for $\mu =$ Gaussian distribution,

$$\int H_1 H_1 \dots H_1 \, d\mu = \text{ number of all pair partitions.}$$

For $\mu =$ semicircular distribution,

$$\int U_1 U_1 \dots U_1 \, d\mu = \text{ number of non-crossing pair partitions.}$$

How to get the higher-degree polynomials?

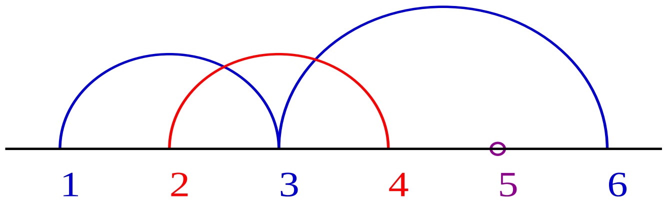
Combinatorial stochastic measures.

(Rota, Wallstrom '97)

$\{X(t)\}$ = operator-valued stochastic process, stationary wrt $\varphi[\cdot]$, freely independent increments.

For a set partition π, define the stochastic measure St_π.

$$\pi = \{(1,3,6)(2,4)(5)\}.$$



$$\mathrm{St}_\pi(t) = \int_{[0,t)^3} dX(s_1)dX(s_2)dX(s_1)dX(s_2)dX(s_3)dX(s_1).$$

$$\psi_n(t) = \int_{\substack{[0,t)^n \\ \text{all } s_i\text{'s distinct}}} dX(s_1)dX(s_2)\cdots dX(s_n)$$

the usual stochastic integrals.

Precise definition of $\mathtt{St}_\pi$: approximate by Riemann sums.

Convergence shown under various conditions.

Proposition. (M.A. ’00)

a) $\mathtt{St}_\pi$ exists as a limit in the operator norm.

b) If π has crossings, $\mathtt{St}_\pi = 0$.

Proposition. [Linearization $\Leftrightarrow$ Itô formula]

$$\prod_{j=1}^{k} \psi_{n_k} = \sum_{\pi \in NC(n_1, n_2, \ldots, n_k)} \mathrm{St}_\pi.$$

Analogy: $\psi_n \leftrightarrow P_n$. In our examples

$$\psi_n(t) = P_n(X(t), t).$$

$$\int_{\mathbb{R}} P_n(x) P_k(x) P_l(x) \, d\mu(x) = \varphi\left[\psi_n \psi_k \psi_l\right]$$
$$= \sum_{\pi \in NC(n,k,l)} \varphi\left[\mathrm{St}_\pi\right] = \sum_{\pi \in NC(n,k,l)} r_\pi.$$

For the Brownian motion,

$$\psi_n(t) = H_n(X(t), t),$$

Hermite polynomials.

$$\int_{\mathbb{R}} H_n(x) H_k(x) H_l(x) \, d\mu(x) = \left|\mathcal{P}_2(n, k, l)\right|.$$

For the centered Poisson process,

$$\psi_n(t) = C_n(X(t), t),$$

centered Charlier polynomials.

$$\int_{\mathbb{R}} C_n(x) C_k(x) C_l(x) \, d\mu(x) = |\mathcal{P}(n, k, l)| \, .$$

For the free Brownian motion,

$$\psi_n(t) = U_n(X(t), t),$$

Chebyshev polynomials of the 2nd kind.

$$\int_{\mathbb{R}} U_n(x) U_k(x) U_l(x) \, d\mu(x) = |NC_2(n, k, l)| \, .$$

For the q-Hermite, use processes on the q-deformed full Fock space, with a lot more work (M.A. '01, '05).

For the centered q-Poisson process,

$$\psi_n(t) = P_n(X(t), t),$$

centered continuous big q-Hermite polynomials.

$$\int_{\mathbb{R}} P_n(x) P_k(x) P_l(x) \, d\mu(x) = \sum_{\pi \in \mathcal{P}(n,k,l)} q^{\mathsf{rc}(\pi)}.$$