Linearization coefficients for orthogonal polynomials

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$P_n =$ monic polynomials of degree $n = 0, 1, \ldots$.

$\{P_n\} =$ basis for the polynomials in 1 variable.

Linearization coefficients: coefficients in the expansion of

$$P_{n_1} P_{n_2} \ldots P_{n_k}$$

in the basis $\{P_n\}$.

For many classical families of orthogonal polynomials, these are positive. For some examples, in fact positive integers.

Let μ be a positive measure on $\mathbb{R}$ such that its moments

$$m_n(\mu) = \int_{\mathbb{R}} x^n d\mu(x) < \infty.$$

$\{P_n\}_{n=0}^{\infty} =$ Gram-Schmidt orthogonalization of $\{x^n\}_{n=0}^{\infty}$. These are, up to normalization, the orthogonal polynomials for μ,

$$\int P_n(x) P_k(x) d\mu(x) = \langle P_n P_k \rangle = 0$$

if $n \neq k$.

Theorem (Favard's theorem). *$\{P_n\}_{n=0}^{\infty}$ are the monic orthogonal polynomials for some positive functional iff they satisfy a three-term recursion relation*

$$xP_n = P_{n+1} + \alpha_{n+1} P_n + \beta_n P_{n-1},$$

where $P_{-1} = 0$, $\alpha_n \in \mathbb{R}$, $\beta_n \geq 0$.

When $\{P_n\}$ are orthogonal,

$$P_{n_1} P_{n_2} \dots P_{n_k} = \sum_{m=0}^{\infty} \frac{1}{\left\langle P_m^2 \right\rangle} \langle P_{n_1} P_{n_2} \dots P_{n_k} P_m \rangle \, P_m,$$

where $\langle \cdot \rangle$ is the expectation with respect to the orthogonality measure,

$$\langle F \rangle = \int F(x) d\mu(x).$$

Thus their linearization coefficients are, up to normalization,

$$\langle P_{n_1} P_{n_2} \dots P_{n_k} \rangle \, .$$

Generalizations of moments $\langle x^n \rangle$ if $P_1(x) = x$.

Hermite polynomials:

$$xH_n(x,t) = H_{n+1}(x,t) + ntH_{n-1}(x,t).$$

Orthogonal with respect to the Gaussian distribution

$$d\mu_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx.$$

Moments

$$m_n(\mu_t) = \begin{cases} 0, & n \text{ odd}, \\ t^{n/2}(n-1)!!, & n \text{ even} \end{cases}$$
$$= t^{n/2} \left| \{\text{Pair set partitions of } n\} \right|.$$

Linearization coefficients

$$\left\langle \prod_{j=1}^{k} H_{n_k}(x,t) \right\rangle = t^{n/2} \left| \mathcal{P}_2(n_1, n_2, \ldots, n_k) \right|.$$

Here $\mathcal{P}_2(n_1, n_2, \ldots, n_k)$ = inhomogeneous pair partitions.

Centered **Charlier** polynomials:

$$xC_n(x,t) = C_{n+1}(x,t) + nC_n(x,t) + tnC_{n-1}(x,t).$$

Orthogonal with respect to the Poisson distribution

$$d\mu_t(x) = e^{-t} \sum_{j=0}^{\infty} \frac{t^j}{j!} \delta_j(x)$$

shifted by t. Moments: for $t = 1$, related to Bell numbers. More generally,

$$m_n(\mu_t) = \sum_{\substack{\pi \in \mathcal{P}(n), \\ s(\pi)=0}} t^{|\pi|}.$$

Here the sum is over all set partitions with no singletons, and $|\pi| =$ number of classes of the partition.

Linearization coefficients

$$\left\langle \prod_{j=1}^{k} C_{n_k}(x,t) \right\rangle = \sum_{\substack{\pi \in \mathcal{P}(n_1, n_2, \ldots, n_k), \\ s(\pi)=0}} t^{|\pi|}.$$

(Zeng '90)

Centered **Laguerre** polynomials:

$$xL_n(x,t) = L_{n+1}(x,t) + 2nL_n(x,t) + \\ n(t+n-1)L_{n-1}(x,t).$$

Orthogonal with respect to the Gamma distribution

$$d\mu_t(x) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} dx$$

shifted by t. Moments: for $t = 1$, $n!$. Suggests statistics over permutations. General moments:

$$m_n(\mu_t) = \sum_{\sigma \in \mathcal{D}(n)} t^{\mathrm{cyc}(\sigma)}.$$

Here cyc (σ) = the number of cycles of σ, $\mathcal{D}(n)$ = derangements. Linearization coefficients

$$\left\langle \prod_{j=1}^{k} L_{n_k}(x,t) \right\rangle = \sum_{\sigma \in \mathcal{D}(n_1, n_2, \ldots, n_k)} t^{\mathrm{cyc}(\sigma)}.$$

(Foata-Zeilberger ’88)

Here $\mathcal{D}(n_1, n_2, \ldots, n_k)$ = “inhomogeneous permutations” = generalized derangements.

Meixner and **Meixner-Pollaczek** polynomials:

$$\begin{aligned} xP_n(x,t) = P_{n+1}(x,t) &+ (\alpha+\beta)nP_n(x,t) \\ &+ n(t+\alpha\beta(n-1))P_{n-1}(x,t), \end{aligned}$$

depending on whether α, β are real or $\beta = \bar{\alpha}$.

Appell polynomials: exponential generating function

$$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(x,t) z^n = f(z)^t e^{xz}.$$

The only orthogonal ones are the Hermite polynomials.

Sheffer polynomials: exponential generating function

$$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(x,t) z^n = f(z)^t e^{xu(z)}.$$

The only orthogonal ones are precisely the classes above.

Linearization coefficients:

$$\left\langle \prod_{j=1}^{k} P_{n_k}(x,t) \right\rangle$$
$$= \sum_{\sigma \in \mathcal{D}(n_1,n_2,\ldots,n_k)} \alpha^{\text{dec}(\sigma)-\text{cyc}(\sigma)} \beta^{\text{exc}(\sigma)-\text{cyc}(\sigma)} t^{\text{cyc}(\sigma)}.$$

(Zeng ’90, Kim-Zeng ’01)

Here dec(σ), resp. exc(σ) are the numbers of the descents, resp. ascents in the permutation σ, i.e. the number of i such that $\sigma(i) > i$, resp. $\sigma(i) < i$.

Note: recover Laguerre for $\alpha = \beta = 1$, Charlier for $\alpha = 1$, $\beta = 0$, Hermite for $\alpha = \beta = 0$.

Chebyshev polynomials of the second kind:

$$xU_n(x,t) = U_{n+1}(x,t) + tU_{n-1}(x,t).$$

Orthogonal with respect to

$$d\mu_t(x) = \frac{1}{2\pi t}\sqrt{4t - x^2}\,\mathbf{1}_{[-2\sqrt{t},2\sqrt{t}]}(x)dx.$$

Moments

$$m_n(\mu_t) = \begin{cases} 0, & n \text{ odd}, \\ t^{n/2} \times n\text{'th Catalan number}, & n \text{ even} \end{cases}$$

$$= t^{n/2}\left|\{\text{Non-crossing pair set partitions of } n\}\right|.$$

Linearization coefficients

$$\left\langle \prod_{j=1}^{k} H_{n_k}(x,t) \right\rangle = t^{n/2}\left|NC_2(n_1, n_2, \ldots, n_k)\right|.$$

(≈ de Sainte-Catherine, Viennot ’85)

Here $NC_2(n_1, n_2, \ldots, n_k) =$ inhomogeneous non-crossing pair partitions.

Expect to have

$$\left\langle \prod_{j=1}^{k} C_{0,n_k}(x,t) \right\rangle = \sum_{\substack{\pi \in NC(n_1,n_2,\ldots,n_k) \\ s(\pi)=0}} t^{|\pi|}$$

for some polynomials. Indeed have this for the free Charlier polynomials

$$\begin{aligned} xC_{0,0}(x,t) &= C_{0,1}(x,t) \qquad (+1), \\ xC_{0,m}(x,t) &= C_{0,m+1}(x,t) + C_{0,m}(x,t) \\ &\quad + tC_{0,m-1}(x,t). \end{aligned}$$

These are orthogonal with respect to

$$\begin{aligned} d\mu_t(x) = \\ &\frac{1}{2\pi x}\sqrt{4t-(x-t-1)^2}\,\mathbf{1}_{[c_1,c_2]}(x)dx \\ &\qquad + \max(1-t,0)\delta_0(x), \end{aligned}$$

where $c_1 = t+1-2\sqrt{t}$, $c_2 = t+1+2\sqrt{t}$.

Continuous (Rogers) q**-Hermite** polynomials:

$$xH_{q,n}(x,t) = H_{q,n+1}(x,t) + t[n]_q H_{q,n-1}(x,t).$$

Here q is (say) in $(-1,1)$, and

$$[n]_q = \sum_{j=0}^{n-1} q^j = \frac{1-q^n}{1-q}.$$

Orthogonal with respect to

$$d\mu_{t,q}(x) = \frac{1}{\pi\sqrt{t}}\sqrt{1-q}\,\sin(\theta)(q;q)_\infty \left|(qe^{2i\theta};q)_\infty\right|^2 dx,$$

for $x = \frac{2}{\sqrt{1-q}}\sqrt{t}\cos(\theta)$, $\theta \in [0,\pi]$, and

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

This is a probability measure supported on the interval $[-2\sqrt{t}/\sqrt{1-q}, 2\sqrt{t}/\sqrt{1-q}]$.

Moments

$$m_n(\mu_{t,q}) = \sum_{\pi \in \mathcal{P}_2(n)} q^{\mathsf{rc}(\pi)}.$$

Here $\mathsf{rc}\,(\pi)$ is the number of crossings of the pair partition π.

Linearization coefficients

$$\left\langle \prod_{j=1}^{k} H_{n_k}(x,t) \right\rangle = \sum_{\pi \in \mathcal{P}_2(n_1, n_2, \ldots, n_k)} q^{\mathsf{rc}(\pi)}.$$

(Ismail, Stanton, Viennot ’87)

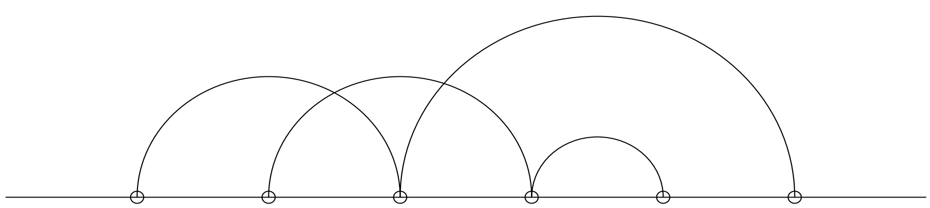
Centered **continuous big q-Hermite** polynomials, which in our context are q-analogs of the Charlier polynomials. Recursion relations

$$
\begin{aligned}
& xC_{q,n}(x,t) \\
& = C_{q,n+1}(x,t) + [n]_q C_{q,n}(x,t) + t[n]_q C_{q,n-1}(x,t).
\end{aligned}
$$

Moments

$$
m_n(\mu_{t,q}) = \sum_{\substack{\pi \in \mathcal{P}(n) \\ s(\pi)=0}} q^{\mathrm{rc}(\pi)} t^{|\pi|},
$$

where $\mathrm{rc}(\pi) =$ number of restricted crossings of the partition π.



Linearization coefficients

$$
\left\langle \prod_{j=1}^{k} C_{q,n_k}(x,t) \right\rangle = \sum_{\substack{\pi \in \mathcal{P}(n_1,n_2,\ldots,n_k) \\ s(\pi)=0}} q^{\mathrm{rc}(\pi)} t^{|\pi|}.
$$

Hermite	Charlier	Laguerre / Meixner	$f(z)^t e^{xu(z)}$
q-Hermite	big q-Hermite	Al-Salam-Chihara?	$F(z)\prod_{k=0}^{\infty}\frac{1}{1-u(q^k z)x}$?
Chebyshev	free Charlier	free Meixner	$\frac{1}{1+tf(z)-u(z)x}$

PROCESSES ON A q-DEFORMED FULL FOCK SPACE

Consider the Hilbert space $H = L^2(\mathbb{R}_+, dx)$. Let

$$\mathcal{F}_{\mathsf{alg}}(H) = \bigoplus_{k=0}^{\infty} H^{\otimes k}$$

be its algebraic Fock space. Here the 0'th component is spanned by the vacuum vector Ω.

$$\langle f_1 \otimes \ldots \otimes f_k, g_1 \otimes \ldots \otimes g_n \rangle_0 = \delta_{kn} \langle f_1, g_1 \rangle \ldots \langle f_k, g_k \rangle$$

is an inner product.

Let

$$P_q(f_1 \otimes \ldots \otimes f_n) = \sum_{\sigma \in \mathsf{Sym}(n)} q^{i(\sigma)} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)},$$

where $\mathsf{Sym}(n)$ is the permutation group and $i(\sigma)$ is the number of inversions of σ.
(Bożejko, Speicher 91) $\Rightarrow P_q \geq 0$ for $-1 < q < 1$.

$$\langle \xi, \eta \rangle_q = \langle \xi, P_q \eta \rangle_0 .$$

another inner product, $\mathcal{F}_q(L^2(\mathbb{R}_+)) =$ the completion of $\mathcal{F}_{\text{alg}}(L^2(\mathbb{R}_+)$ with respect to the corresponding norm, the q**-deformed full Fock space**.

On $\mathcal{F}_q(L^2(\mathbb{R}_+))$, define **creation**, **annihilation**, and **gauge** operators $a^*(t), a(t), p(t)$.

The non-commutative stochastic process

$$X(t) = a^*(t) + a(t)$$

is the q**-Brownian motion**, and the process

$$X(t) = a^*(t) + a(t) + p(t)$$

is the centered q**-Poisson process**.

Corresponding distribution:

$$\langle \Omega, f(X(t))\Omega \rangle = \int f(x) d\mu_t(x).$$

For the degenerate case $q = 1$, get the corresponding classical processes.

Moments expressed in terms of generalized cumulants:

$$m_n = \sum_{\pi \in \mathcal{P}(n)} R_\pi.$$

Proposition. *For the processes on the q-Fock space,*

$$R_\pi = q^{\mathsf{rc}(\pi)} \prod_{B \in \pi} r_{|B|}.$$

For the Brownian motion, $r_2 = t$, $r_n = 0$ for $n > 2$. For the centered Poisson process, $r_n = t$ for $n > 1$. The formulas for the moments follow.

COMBINATORIAL STOCHASTIC MEASURES

(Rota, Wallstrom 97)

$\{X(t)\} =$ operator-valued stochastic process, stationary wrt $\langle\cdot\rangle$, "independent" increments.

For a set partition $\pi = (B_1, B_2, \ldots, B_l)$, temporarily denote by $c(i)$ the number of the class $B_{c(i)}$ to which i belongs.

Stochastic measure corresponding to the partition π is

$$\mathrm{St}_\pi(t) = \int_{\substack{[0,t)^l \\ \text{all } s_i\text{'s distinct}}} dX(s_{c(1)})dX(s_{c(2)}) \cdots dX(s_{c(n)}).$$

$\Delta_n = \mathrm{St}_{\hat{1}}$ the higher diagonal measures

$$\Delta_n(t) = \int_{[0,t)} (dX(s))^n,$$

$\psi_n = \mathrm{St}_{\hat{0}}$ the full stochastic measures

$$\psi_n(t) = \int_{\substack{[0,t)^n \\ \text{all } s_i\text{'s distinct}}} dX(s_1)dX(s_2) \cdots dX(s_n).$$

$t \mapsto X(t)$ an "operator-valued measure". So imitate Lebesgue integration. More precisely,

$$\int \mathbf{1}_{[u,v)}(t)dX(t) = X(v) - X(u)$$

and approximate the integrand by simple functions. Convergence shown under various conditions.

Proposition (Linearization $\Leftrightarrow$ Itô formula).

$$\prod_{j=1}^{k} \psi_{n_k} = \sum_{\pi \in \mathcal{P}(n_1, n_2, \ldots, n_k)} \mathsf{St}_\pi.$$

Analogy: $\psi_n \leftrightarrow P_n$. In our examples

$$\psi_n(t) = P_n(X(t), t).$$

Proposition. $R_\pi = \langle \mathsf{St}_\pi \rangle$ *and* $r_n = \langle \Delta_n \rangle$.

$$\left\langle \prod_{j=1}^{k} \psi_{n_k} \right\rangle = \sum_{\pi \in \mathcal{P}(n_1, n_2, \ldots, n_k)} R_\pi . \qquad (1)$$

For a centered process, $r_1 = 0$.

$$\langle \psi_n \psi_k \rangle = 0$$

for $n \neq k$. Thus stochastic measures are analogs of orthogonal polynomials, and formula (1) describes their linearization coefficients.

In particular, for the stochastic measures on the q-Fock space

$$\left\langle \prod_{j=1}^{k} \psi_{n_k} \right\rangle = \sum_{\pi \in \mathcal{P}(n_1, n_2, \ldots, n_k)} R_\pi = \sum_{\pi \in \mathcal{P}(n_1, n_2, \ldots, n_k)} q^{\mathrm{rc}(\pi)} \prod_{B \in \pi} r_{|B|}.$$

For the q-Brownian motion,

$$\psi_m(t) = H_{q,m}(X(t), t),$$

a scaled version of the continuous (Rogers) q-Hermite polynomials.

For the centered q-Poisson process,

$$\psi_m(t) = C_{q,m}(X(t), t),$$

a scaled version of the centered continuous big q-Hermite polynomials.

The formulas for the linearization coefficients follow.

In fact, in the q-Brownian motion case, for $\pi \in \mathcal{P}_{1,2}$,

$$\mathsf{St}_\pi = q^{\mathsf{rc}(\pi)+\mathsf{sd}(\pi)} t^{s_2(\pi)} H_{q,s(\pi)}(X(t), t)$$

and they are 0 otherwise. Here singleton depth $\mathsf{sd}(\pi) =$ sum of depths,

$$d(i) = \left| \left\{ j | \exists a, b \in B_j : a < i < b \right\} \right|,$$

over all the singletons (i) of π. Thus

$$x^n = \sum_{\pi \in \mathcal{P}_{1,2}(n)} q^{\mathsf{rc}(\pi)+\mathsf{sd}(\pi)} t^{s_2(\pi)} H_{q,s_1(\pi)}(x, t)$$

and

$$H_{q,n}(x,t) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^{s_2(\pi)} q^{\mathsf{rc}(\pi)+\mathsf{sd}(\pi)} t^{s_2(\pi)} x^{s_1(\pi)}.$$