Measures, orthogonal polynomials, and continued fractions

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MEASURES AND ORTHOGONAL POLYNOMIALS.
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A linear functional \( \mu[P] = \int P(x) \, d\mu(x) \),

\[ \mu : \mathbb{R}[x] \rightarrow \mathbb{R}. \]
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Gram-Schmidt \( \{1, x, x^2, x^3, \ldots\} \Rightarrow \)
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Gram-Schmidt \( \{1, x, x^2, x^3, \ldots \} \Rightarrow \)

\( \Rightarrow \) monic orthogonal polynomials

\[ \{P_0 = 1, P_1, P_2, P_3, \ldots \}. \]
Theorem. (Favard, Stone, etc.) For some $\beta_i \in \mathbb{R}$, $\gamma_i \geq 0$

$$xP_n = P_{n+1} + \beta_n P_n + \gamma_n P_{n-1}.$$
**Theorem.** (Favard, Stone, etc.) For some $\beta_i \in \mathbb{R}$, $\gamma_i \geq 0$

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2nd order recursion relation.
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2nd order recursion relation.

Two independent solutions $\{P_n, Q_n\}$.

Initial conditions

$$P_{-1} = 0, \quad P_0 = 1, \quad P_1 = x - \beta_0,$$
$$Q_0 = 0, \quad Q_1 = 1, \quad Q_2 = x - \beta_1.$$
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**Exercise.**

\[ Q_n(x) = (I \otimes \mu) \left[ \frac{P_n(x) - P_n(y)}{x - y} \right]. \]
\[ \mu \leftrightarrow \left\{ (\beta_0, \beta_1, \beta_2, \ldots) \right\} \equiv \left\{ (\gamma_1, \gamma_2, \gamma_3, \ldots) \right\} \text{ equivalent.} \]
\[ \mu \leftrightarrow \left\{ \left( \beta_0, \beta_1, \beta_2, \ldots \right), \left( \gamma_1, \gamma_2, \gamma_3, \ldots \right) \right\} \] equivalent.

More explicit relation?

If know \[ \left\{ \left( \beta_0, \beta_1, \beta_2, \ldots \right), \left( \gamma_1, \gamma_2, \gamma_3, \ldots \right) \right\} \], how to recover \( \mu \)?

Without going through \( \{P_n\} \).
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---

Cauchy transform

\[
G_\mu(z) = \int \frac{1}{z-x} d\mu(x) = \mu \left[ \frac{1}{z-x} \right]
\]

\[
= \frac{\mu[1]}{z} + \frac{\mu[x]}{z^2} + \frac{\mu[x^2]}{z^3} + \frac{\mu[x^3]}{z^4} + \ldots
\]
\[ G_\mu(z) = \frac{\mu[1]}{z} + \frac{\mu[x]}{z^2} + \frac{\mu[x^2]}{z^3} + \frac{\mu[x^3]}{z^4} + \ldots \]

**Theorem.** Also

\[ G_\mu(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \frac{\gamma_2}{z - \beta_2 - \frac{\gamma_3}{z - \ldots}}}} \]

Same coefficients as in the recursion.
\[ G_{\mu}(z) = \frac{\mu[1]}{z} + \frac{\mu[x]}{z^2} + \frac{\mu[x^2]}{z^3} + \frac{\mu[x^3]}{z^4} + \ldots. \]

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Same coefficients as in the recursion.

**Note:** \( \mu \geq 0 \iff \text{all } \gamma \geq 0 \), no determinants.
\[ G_\mu(z) = \frac{\mu[1]}{z} + \frac{\mu[x]}{z^2} + \frac{\mu[x^2]}{z^3} + \frac{\mu[x^3]}{z^4} + \ldots. \]

**Theorem.** Also

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Same coefficients as in the recursion.

**Note:** \( \mu \geq 0 \Leftrightarrow \) all \( \gamma \geq 0 \), no determinants.

**Proof II.** Flajolet (1980): lattice paths.
Proof I. Look at

\[ G(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{\gamma_2}} \frac{1}{z - \beta_1 - \frac{\gamma_2}{\gamma_3}} \frac{1}{z - \beta_2 - \frac{\gamma_3}{\gamma_nH}} ... \]
Proof 1. Look at

$$G(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{\gamma_1}} \frac{\gamma_2}{\gamma_2} \frac{\gamma_3}{\gamma_3} \ldots \frac{z - \beta_{n-1} - \gamma_n H}{z - \beta_n - \gamma_n H}$$

$$G = \frac{\text{polynomial}}{\text{polynomial}}.$$
Proof I. Look at

\[
G(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \frac{\gamma_2}{z - \beta_2 - \frac{\gamma_3}{\ldots}}}} - \beta_{n-1} - \gamma_n H
\]

\[
G = \frac{\text{polynomial}}{\text{polynomial}}.
\]

Claim.

\[
G = \frac{Q_n - \gamma_n Q_{n-1} H}{P_n - \gamma_n P_{n-1} H},
\]

where \(\{P, Q\}\) with recursion \(\{\beta_i, \gamma_i\}\).
Proof. By induction.
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\[
G = \frac{1}{z - \beta_0 - \gamma_1 H_0} = \frac{Q_1 - \gamma_1 Q_0 H_0}{P_1 - \gamma_1 P_0 H_0}.
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Proof. By induction.

\[ G = \frac{1}{z - \beta_0 - \gamma_1 H_0} = \frac{Q_1 - \gamma_1 Q_0 H_0}{P_1 - \gamma_1 P_0 H_0}. \]

Assume

\[ G = \frac{Q_n - \gamma_n Q_{n-1} H}{P_n - \gamma_n P_{n-1} H} \]
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G = \frac{Q_n - \gamma_n Q_{n-1} H}{P_n - \gamma_n P_{n-1} H}
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and say

\[
H = \frac{1}{z - \beta_n - \gamma_{n+1} K}.
\]
**Proof.** By induction.

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Then

\[ G = \frac{Q_n - \gamma_n Q_{n-1} \frac{1}{z - \beta_n - \gamma_{n+1} K}}{P_n - \gamma_n P_{n-1} \frac{1}{z - \beta_n - \gamma_{n+1} K}}. \]
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\[ G = \frac{Q_n - \gamma_n Q_{n-1}}{P_n - \gamma_n P_{n-1}} \frac{1}{z - \beta_n - \gamma_{n+1} K} \]

\[ = \frac{zQ_n - \beta_n Q_n - \gamma_{n+1} Q_n K - \gamma_n Q_{n-1}}{zP_n - \beta_n P_n - \gamma_{n+1} P_n K - \gamma_n P_{n-1}}. \]
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\[
G = \frac{Q_n - \gamma_n Q_{n-1} \frac{1}{z - \beta_n - \gamma_{n+1} K}}{P_n - \gamma_n P_{n-1} \frac{1}{z - \beta_n - \gamma_{n+1} K}} = \frac{zQ_n - \beta_n Q_n - \gamma_{n+1} Q_n K - \gamma_n Q_{n-1}}{zP_n - \beta_n P_n - \gamma_{n+1} P_n K - \gamma_n P_{n-1}} = \frac{Q_{n+1} - \gamma_{n+1} Q_n K}{P_{n+1} - \gamma_{n+1} P_n K}.
\]
FINITE CONTINUED FRACTIONS.
Finite continued fractions.

Let $G_n = G$ cut off at level $n$.

$$G_n(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \frac{\gamma_2}{z - \beta_2 - \frac{\gamma_3}{\ldots}}}}$$

$H = 0.$
Let $G_n = G$ cut off at level $n$.

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G_n(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \frac{\gamma_2}{z - \beta_2 - \frac{\gamma_3}{\ldots}}}}
$$

$$
H = 0.
$$

$$
G_n = \frac{Q_n - \gamma_n Q_{n-1} H}{P_n - \gamma_n P_{n-1} H} = \frac{Q_n}{P_n}.
$$
\[ G_n = \frac{Q_n}{P_n}. \]

\( P_n \) monic, \( n \) real roots
\[ G_n = \frac{Q_n}{P_n}. \]

\[ P_n \text{ monic, } n \text{ real roots} \]

\[ P_n(x) = \prod_{i=1}^{n} (x - x_i). \]
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\[ G_n(z) = \int \frac{1}{z - x} d\mu_n(x) \]
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\[ G_n(z) = \int \frac{1}{z - x} d\mu_n(x) = \frac{Q_n(z)}{\prod(z - x_i)} \]
\[ G_n = \frac{Q_n}{P_n}. \]

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\[ P_n(x) = \prod_{i=1}^{n} (x - x_i). \]

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\[ = \text{(partial fractions)} = \sum \frac{a_i}{z - x_i}. \]
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= (partial fractions) = \[ \sum \frac{a_i}{z - x_i}. \]

\[ G_n = \text{Cauchy transform of} \]

\[ \mu_n = \sum a_i \delta_{x_i}, \]
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\[ \mu_n = \sum a_i \delta_{x_i}, \]

\[ x_i = \text{roots of } P_n, \quad a_i = \frac{Q_n(x_i)}{P_n'(x_i)}. \]
\[ G_n(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1} - \frac{\gamma_2}{z - \beta_2} - \frac{\gamma_3}{\cdots} - \frac{\beta_{n-1}}{z - \beta_{n-1}} - 0} \]

\( G_n \) approximate \( G \).
\[ G_n(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \frac{\gamma_2}{z - \beta_2 - \frac{\gamma_3}{\cdots}}}} \]

\[ G_n \text{ approximate } G. \]

\[ G_n = G_{\mu_n}, \quad \mu_n = \sum_{i=1}^{n} a_i \delta_{x_i}. \]
\[ G_n(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1} - \frac{\gamma_2}{z - \beta_2} - \frac{\gamma_3}{z - \beta_3} - \ldots - \frac{\gamma_{n-1}}{z - \beta_{n-1}} - 0} \]

\( G_n \) approximate \( G \).

\[ G_n = G_{\mu_n}, \quad \mu_n = \sum_{i=1}^{n} a_i \delta_{x_i}. \]

Do \( \mu_n \) approximate \( \mu \)?
$$G_n(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{\gamma_2} \frac{\gamma_2}{\gamma_3} \cdots \frac{\gamma_{n-1}}{\gamma_n} - 0}$$

$G_n$ approximate $G$.

$$G_n = G_{\mu_n}, \quad \mu_n = \sum_{i=1}^{n} a_i \delta x_i.$$ 

Do $\mu_n$ approximate $\mu$?

Yes.
\[ G_n(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \frac{\gamma_2}{z - \beta_2 - \frac{\gamma_3}{\cdots}}} - \beta_{n-1} - 0} \]

\( G_n \) approximate \( G \).

\[ G_n = G_{\mu_n}, \quad \mu_n = \sum_{i=1}^{n} a_i \delta_{x_i}. \]

Do \( \mu_n \) approximate \( \mu \)?

Yes. In fact,

\[ \mu_n[P(x)] = \mu[P(x)] \text{ for } \deg P \leq 2n - 1. \]
GAUSSIAN QUADRATURE.
Gaussian quadrature.

Want to evaluate

\[ \int f(x) \, d\mu(x) \approx \sum_{i=1}^{n} a_i f(x_i). \]
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“Riemann sums”.

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Want $\int P(x) \, d\mu(x) = \sum_{i=1}^{n} a_i P(x_i)$ for $P$ of low degree.
Gaussian quadrature.

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“Riemann sums”.

Want $$\int P(x) \, d\mu(x) = \sum_{i=1}^{n} a_i P(x_i)$$ for $$P$$ of low degree.

How to choose $$a_i, x_i$$?
Gaussain Quadrature.

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Want \( \int P(x) \, d\mu(x) = \sum_{i=1}^{n} a_i P(x_i) \) for \( P \) of low degree.

How to choose \( a_i, x_i \)?

Answer: take \( x_i = \) roots of \( P_n \).
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How to choose \( a_i, x_i \)?

Answer: take \( x_i = \) roots of \( P_n \).

Choose \( a_i \) so that

\[ \int x^k \, d\mu(x) = \sum a_i x_i^k, \quad k = 0, 1, \ldots, n - 1 \]

(\( n \) equations, \( n \) unknowns).
GAUSSIAN QUADRATURE.

Want to evaluate

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Choose \( a_i \) so that

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\int x^k \, d\mu(x) = \sum a_i x^k_i, \quad k = 0, 1, \ldots, n - 1
\]

(\( n \) equations, \( n \) unknowns).

Our \( a_i = \frac{Q_n(x_i)}{P'_n(x_i)} \) work.
**Proof.** Lagrange interpolation: for any $P$ with $\deg P < n$,

\[
P(x) = \sum_{i=1}^{n} \frac{P(x_i)P_n(x)}{P'_n(x_i)(x - x_i)}.
\]

Note

\[
\mu \left[ \frac{P_n(x)}{x - x_i} \right] = \mu \left[ \frac{P_n(x) - P_n(x_i)}{x - x_i} \right] = Q_n(x_i)
\]

so

\[
\mu[P(x)] = \sum_{i=1}^{n} \frac{P(x_i)}{P'_n(x_i)} Q_n(x_i) = \left( \sum_{i=1}^{n} \frac{Q_n(x_i)}{P'_n(x_i)} \delta_{x_i} \right) [P]
\]
Proof. Lagrange interpolation: for any \( P \) with \( \deg P < n \),

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P(x) = \sum_{i=1}^{n} \frac{P(x_i)P_n(x)}{P_n'(x_i)(x - x_i)}.
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Note

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\mu[P(x)] = \sum_{i=1}^{n} \frac{P(x_i)}{P_n'(x_i)} Q_n(x_i) = \left( \sum_{i=1}^{n} \frac{Q_n(x_i)}{P_n'(x_i)} \delta_{x_i} \right) [P]
\]

\[
\mu[P(x)] = \left( \sum_{i=1}^{n} a_i \delta_{x_i} \right) [P(x)]
\]

\[
\mu[P(x)] = \mu_n[P(x)] \text{ for } \deg P < n.
\]
Proof. Lagrange interpolation: for any $P$ with $\text{deg } P < n$,

$$P(x) = \sum_{i=1}^{n} \frac{P(x_i)P_n(x)}{P'_n(x_i)(x - x_i)}.$$ 

Note

$$\mu \left[ \frac{P_n(x)}{x - x_i} \right] = \mu \left[ \frac{P_n(x) - P_n(x_i)}{x - x_i} \right] = Q_n(x_i)$$

so

$$\mu[P(x)] = \sum_{i=1}^{n} \frac{P(x_i)}{P'_n(x_i)} Q_n(x_i) = \left( \sum_{i=1}^{n} \frac{Q_n(x_i)}{P'_n(x_i)} \delta_{x_i} \right) [P]$$

In fact, same $x_i, a_i$ work for $k = n, n + 1, \ldots, 2n - 1$. 

\[ \mu[P(x)] = \mu_n[P(x)] \text{ for } \text{deg } P < n. \]
For $P_k$, $n \leq k \leq 2n - 1$,

$$P_k(x) = A(x)P_n(x) + B(x), \quad \deg A, B \leq n - 1.$$
For $P_k$, $n \leq k \leq 2n - 1$,

$$P_k(x) = A(x)P_n(x) + B(x), \quad \deg A, B \leq n - 1.$$

$$\mu[P_k] = \langle 1, P_k \rangle = 0.$$
For $P_k$, $n \leq k \leq 2n - 1$,

$$P_k(x) = A(x)P_n(x) + B(x), \quad \text{deg } A, B \leq n - 1.$$  

$$\mu[P_k] = \langle 1, P_k \rangle = 0.$$  

To show: $\mu_n[P_k] = 0$.  

For $P_k$, $n \leq k \leq 2n - 1$,

$$P_k(x) = A(x)P_n(x) + B(x), \quad \deg A, B \leq n - 1.$$  

$$\mu[P_k] = \langle 1, P_k \rangle = 0.$$  

To show: $\mu_n[P_k] = 0$.

$$\mu[AP_n] = \langle A, P_n \rangle = 0 \quad \deg A < n$$
For $P_k$, $n \leq k \leq 2n - 1$,

$$P_k(x) = A(x)P_n(x) + B(x), \quad \text{deg } A, B \leq n - 1.$$ 

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To show: $\mu_n[P_k] = 0$.

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Finally,

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so $\mu_n[P_k] = 0$.

So $\mu_n \to \mu$, $G_n \to G$, and therefore $G_\mu = G$. 
If know \( \{\beta_i, \gamma_i\} \), can find \( \mu \)?
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Usually hard:

\[
G_\mu(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \frac{\gamma_2}{z - \beta_2 - \frac{\gamma_3}{z - \ldots}}}}
\]

an infinite expression.
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Class of explicit examples.
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Class of explicit examples.

Semicircle law:

\[
\frac{1}{2\pi t} \sqrt{4t - x^2} 1_{[-2\sqrt{t}, 2\sqrt{t}]} \, dx.
\]
Semicircle law:

\[
\frac{1}{2\pi t} \sqrt{4t - x^2} 1_{[-2\sqrt{t}, 2\sqrt{t}]}(x) \, dx.
\]

Marchenko-Pastur distributions:

\[
\frac{1}{2\pi} \cdot \frac{\sqrt{4t - (x - 1 - t)^2}}{x} 1_{[\sqrt{1+t-2\sqrt{t}, 1+t+2\sqrt{t}}]}(x) \, dx.
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\[ + \max(1 - t, 0) \delta_0. \]
Semicircular, Marchenko-Pastur orthogonal polynomials satisfy

\[ \sum_{n=0}^{\infty} P_n(x) u^n = \frac{A(u)}{1 - B(u)x}. \]
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In general: free Meixner distributions
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In general: free Meixner distributions

\[
\frac{1}{2\pi t} \cdot \frac{\sqrt{4(t+c) - (x-b)^2}}{1 + (b/t)x + (c/t^2)x^2}^1_{[b-2\sqrt{t+c}, b+2\sqrt{t+c}]} dx
\]

+0, 1, 2 atoms.
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○ AC support an interval
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- AC support an interval
- \( \sqrt{\text{polynomial}} \)
- polynomial
- limited atoms
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In general: free Meixner distributions

\[ \frac{1}{2\pi t} \cdot \frac{\sqrt{4(t + c) - (x - b)^2}}{1 + (b/t)x + (c/t^2)x^2} \mathbf{1}_{[b-2\sqrt{t+c}, b+2\sqrt{t+c}]} \, dx \]

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PERIODIC CONTINUED FRACTIONS.
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\[ G(z) = \frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \frac{\gamma_2}{z - \beta_2 - \frac{\gamma_3}{\cdots}}}} \]

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\[ G = \frac{Q_n - \gamma_n Q_{n-1} G}{P_n - \gamma_n P_{n-1} G}. \]
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Quadratic equation!
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Quadratic equation!

\[ D = (\gamma_n Q_{n-1} + P_n)^2 - 4\gamma_n Q_n P_{n-1}. \]
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Stieltjes inversion formula:
\[ G = \frac{(\gamma_n Q_{n-1} + P_n) - \sqrt{D}}{2\gamma_n P_{n-1}} = \int \frac{1}{z - x} \, d\mu(x). \]

Stieltjes inversion formula:

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No SC.
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atoms: roots of \(P_{n-1}\), at most \((n - 1)\).
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Recall \( D = (\gamma_n Q_{n-1} + P_n)^2 - 4\gamma_n Q_n P_{n-1}. \)
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Recall \( D = (\gamma_n Q_{n-1} + P_n)^2 - 4\gamma_n Q_n P_{n-1} \).

So if \( P_{n-1}(a) = 0 \),
\[ G = \frac{(\gamma_n Q_{n-1} + P_n) - \sqrt{D}}{2\gamma_n P_{n-1}} = \int \frac{1}{z-x} d\mu(x). \]

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So if \( P_{n-1}(a) = 0 \), then \( D(a) \geq 0 \).
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Atoms outside of the AC support.
Eventually **constant** continued fractions \((n = 1)\)
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\[\beta_i = \beta, \gamma_i = \gamma \text{ for } i \geq N.\]
Eventually **constant** continued fractions ($n = 1$)

\[ \beta_i = \beta, \gamma_i = \gamma \text{ for } i \geq N. \]

\[ \Leftrightarrow \frac{\sqrt{\text{polynomial}}}{\text{polynomial}} \text{ on one interval.} \]
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If \(\beta = 0, \gamma = 1\) (Peherstorfer?)
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If \(\beta = 0, \gamma = 1\) (Peherstorfer?)

Bernstein-Szegő class \(\frac{\sqrt{4 - x^2}}{\text{polynomial}}\) on \([-2, 2]\).
Eventually **constant** continued fractions \((n = 1)\)

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**Weyl’s Theorem.**
Eventually \textbf{constant} continued fractions ($n = 1$)

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\hline

\textbf{Weyl’s Theorem.} If $\beta_i \to 0, \gamma_i \to 1$, \hline
Eventually **constant** continued fractions \((n = 1)\)

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\beta_i = \beta, \gamma_i = \gamma \text{ for } i \geq N.
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\[
\Leftrightarrow \sqrt{\text{polynomial}}/\text{polynomial} \text{ on one interval.}
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**Weyl’s Theorem.** If \(\beta_i \to 0, \gamma_i \to 1\), then

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\sigma_{ess}(\mu) = [-2, 2].
\]
Eventually **constant** continued fractions \( (n = 1) \)

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\]

**Denisov-Rakhmanov Theorem.**
Eventually **constant** continued fractions \((n = 1)\)

\[ \beta_i = \beta, \gamma_i = \gamma \text{ for } i \geq N. \]

\[ \iff \sqrt{\text{polynomial}} \text{ polynomial on one interval.} \]

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Bernstein-Szegő class \(\sqrt{4 - x^2} \text{ polynomial on } [-2, 2] \).

---

**Weyl’s Theorem.** If \(\beta_i \to 0, \gamma_i \to 1\), then

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\[ \sigma_{ess}(\mu) = \text{AC support of } \mu = [-2, 2], \]
Eventually **constant** continued fractions ($n = 1$)

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Eventually periodic
Eventually periodic $\Rightarrow \sqrt{\text{polynomial}} \over \text{polynomial}$ on $n$ intervals.
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Eventually periodic $\Rightarrow \sqrt[\text{polynomial}]{\text{polynomial}}$ on $n$ intervals.

$\sqrt[\text{polynomial}]{\text{polynomial}}$ on $n$ intervals $\not\Rightarrow$ eventually periodic.

Weyl: if $\{\beta_i, \gamma_i\}$ asymptotically periodic, same essential spectrum as for actually periodic.
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Last, Simon: if $\{\beta_i, \gamma_i\}$ approaches the isospectral torus of a periodic sequence, same essential spectrum.
Eventually periodic $\Rightarrow$ polynomial polynomial on $n$ intervals.

\[ \text{polynomial polynomial on } n \text{ intervals } \not\Rightarrow \text{eventually periodic}. \]

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Damanik, Killip, Simon: converse true.
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- Connection between multi-matrix models and states with multivariate (eventually) periodic continued fractions.