# Free convolution

Michael Anshelevich

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$\mu, \nu$ probability measures on $\mathbb{R}$. Their convolution

$$\mu * \nu(S) = \int_{\mathbb{R}} \mu(S - x)\, d\nu(x).$$

Symmetrically,

$$\int_{\mathbb{R}} f(z)\, d(\mu * \nu)(z) = \iint_{\mathbb{R}\times\mathbb{R}} f(x+y)\, d\mu(x)\, d\nu(y).$$

Other definitions below.

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## EQUIVALENT DEFINITIONS OF FREE CONVOLUTION

1) Let $\Gamma_1, \Gamma_2$ be discrete groups ($\Gamma_1 = \Gamma_2 = \mathbb{Z}$ works).

$\Gamma_1 * \Gamma_1 =$ their free product.

$L(\Gamma_1), L(\Gamma_2), L(\Gamma_1 * \Gamma_2) =$ group von Neumann algebras.

All with distinguished (tracial) states.

For a bounded self-adjoint operator $g$, its distribution with respect to a state $\varphi =$ unique probability measure $\mu$ such that

$$\varphi\left[g^n\right] = \int_{\mathbb{R}} x^n \, \mu(x).$$

Equivalently,

$$\mu(S) = \varphi\left[E(S)\right],$$

where $E =$ spectral measure of $g$. Works for unbounded.

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Let $g_1 \in L(\Gamma_1)$ have distribution $\mu$, $g_2 \in \Gamma_2$ have distribution $\nu$.

**Theorem.** (DVV ’86) The distribution of $g_1 + g_2 \in L(\Gamma_1 * \Gamma_2)$ depends only on $\mu, \nu$. Call it the free convolution $\mu \boxplus \nu$ of $\mu, \nu$.

Can take $g_1, g_2$ in any freely independent subalgebras. Can obtain any $\mu, \nu$ as distributions.

2) Let $\{A_n\}_{n=1}^{\infty}$ be $n \times n$ diagonal matrices, with diagonal entries $\left\{\lambda_{i,n}\right\}$ such that

$$\frac{1}{n}\left(\delta_{\lambda_{1,n}} + \delta_{\lambda_{2,n}} + \ldots + \delta_{\lambda_{n,n}}\right) \xrightarrow{w} \mu.$$

Let $\{B_n\}$ similarly approximate $\nu$.

Let $\{U_n\}$ be random unitary matrices, uniformly distributed according to Haar measure on $\mathcal{U}_n$.

$A_n + U_n B_n U_n^{-1}$ = random Hermitian matrix,
$\left\{\rho_{i,n}\right\}$ = its random eigenvalues,

$$\frac{1}{n}\left(\delta_{\rho_{1,n}} + \delta_{\rho_{2,n}} + \ldots + \delta_{\rho_{n,n}}\right) = \text{random measure.}$$

**Theorem.** (DVV '91, Benaych-Georges '03) These random measures converge almost surely to a fixed measure $\mu \boxplus \nu$.

3) For $z \in \mathbb{C}_+$, let

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x)$$

be the Cauchy transform of $\mu$. Has a local inverse with respect to composition. Define

$$R_\mu(z) = G_\mu^{-1}(z) - \frac{1}{z}$$

on some domain in $\mathbb{C}$.

**Theorem.** (DVV, HB '93)

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z).$$

Can recover $G$ from $R$, and $\mu \boxplus \nu$ from $G_{\mu \boxplus \nu}$ via Stieltjes inversion formula.

4) Using free cumulants and non-crossing partitions.

5) Using operators on the full Fock space.

## SIMILARITY.

Corresponding definitions for classical convolution.

1,2) If $X, Y$ are independent random variables with distributions $\mu, \nu$, then the distribution of $X + Y$ is $\mu * \nu$.

3) For $\mathcal{F}_\mu =$ Fourier transform of $\mu$,

$$\log \mathcal{F}_\mu(z) + \log \mathcal{F}_\nu(z) = \log \mathcal{F}_{\mu*\nu}(z).$$

4) Using classical cumulants, all partitions.

5) Using operators on the symmetric Fock space.

## DIFFERENCE.

$\boxplus$ commutative, associative. Not distributive:

$$\mu \boxplus (\nu + \sigma) \neq \mu \boxplus \nu + \mu \boxplus \sigma.$$

Equivalently: the operator $C_\mu : \nu \mapsto \mu \boxplus \nu$ non-linear.

Causes many difficulties...

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Different behavior of atoms (DVV, HB '98).

If $\mu$ has an atom $\alpha\delta_a$, $\nu$ has an atom $\beta\delta_b$, then $\mu * \nu$ has an atom $(\alpha + \beta)\delta_{a+b}$.

But: $\mu \boxplus \nu$ has an atom at $a + b$ only if $\alpha + \beta - 1 > 0$. Thus usually lose atoms.

## SIMILARITY.

A convolution semigroup $\{\mu_t, t \in [0,\infty)\}$ such that

$$\mu_s * \mu_t = \mu_{s+t}.$$

Describe all convolution semigroups of probability measures on $\mathbb{R}$. Any $\mu$ included in a convolution semigroup is infinitely divisible: can break it up as

$$\mu = \underbrace{\mu_{1/k} * \mu_{1/k} * \cdots * \mu_{1/k}}_{k \text{ parts}}$$

**Theorem.** (Lévy-Khinchine) Any such $\mu$ is $\mu^*_{\gamma,\sigma}$ for

$$\mathcal{F}_{\mu^*_{\gamma,\sigma}}(\theta)$$
$$= \exp\left[i\gamma\theta + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2}\right) \frac{1+x^2}{x^2}\, d\sigma(x)\right]$$

for $\sigma$ a finite measure.

A free convolution semigroup $\{\mu_t, t \in [0,\infty)\}$ such that

$$\mu_s \boxplus \mu_t = \mu_{s+t}.$$

Any $\mu$ included in a free convolution semigroup is freely infinitely divisible.

**Theorem.** (DVV, HB '93)

$$R_{\mu^{\boxplus}_{\gamma,\sigma}}(z) = \gamma + \int_{\mathbb{R}} \frac{z+x}{1-zx} d\sigma(x).$$

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Thus get a formal (Bercovici-Pata) bijection

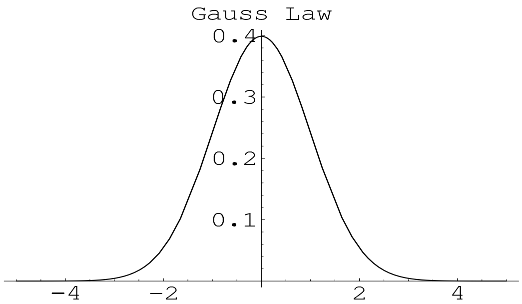
$$\Lambda : \mu^{*}_{\gamma,\sigma} \mapsto \mu^{\boxplus}_{\gamma,\sigma}.$$

Is a weakly continuous homomorphism.

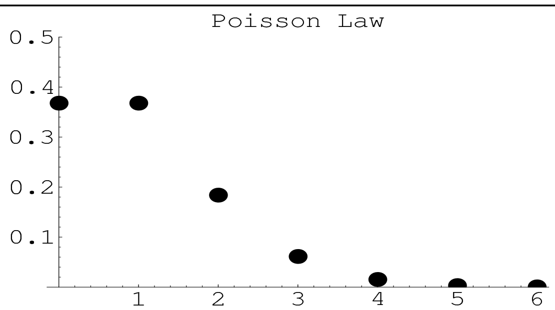
# Basic Laws

Continuous lines = Densities, dots = Dirac Measures.

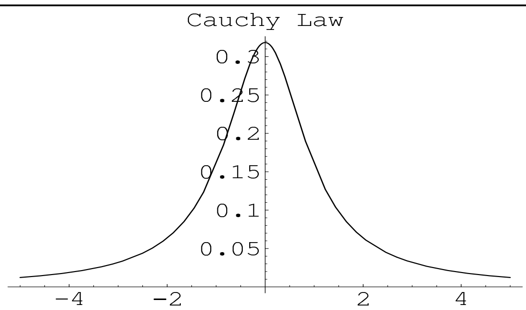
## Classical Probability




$$c\, e^{-a x^2}\, \mathrm{dx}$$

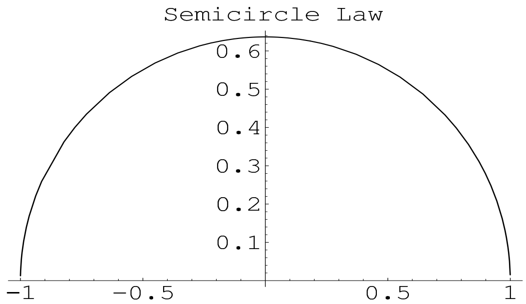



$$\sum_{n\geq 0} e^{-a}\, \frac{a^n}{n!}\, \delta_n$$

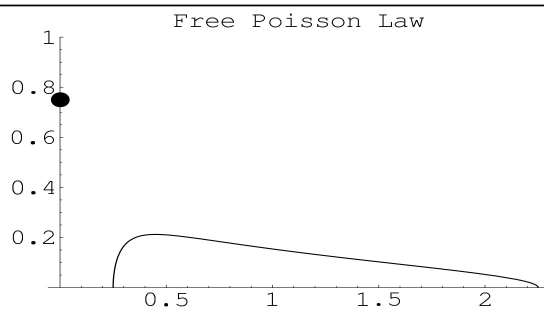



$$c\, \frac{1}{x^2 + a^2}$$

## Free Probability




$$c\, \sqrt{R^2 - x^2}\, \mathrm{dx}, \quad -R \leq x \leq R$$

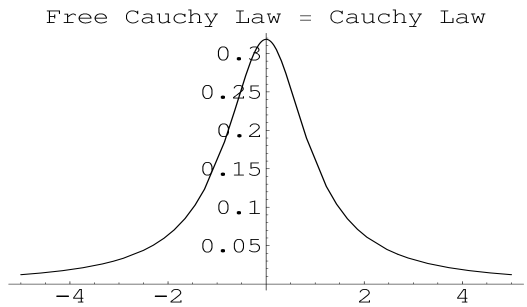



$(1-a)\,\delta_0 + \nu$, if $0 \leq a \leq 1$

$\nu$, if $a > 1$

$$\mathrm{d}\nu = (2\pi \mathrm{x})^{-1}\, (4a - (x - (1+a))^2)^{1/2}$$

$$(1 - a^{1/2})^2 \leq x \leq (1 + a^{1/2})^2$$




$$c\, \frac{1}{x^2 + a^2}$$

## Difference.

Cannot extend $\Lambda$ as a homomorphism to all probability measures.

a) (DVV, HB '95) Can write

$$\text{Semicircular} = \mu \boxplus \nu$$

with $\mu, \nu$ not semicircular. Cannot do this for Gaussian.

b) (Nica, Speicher '96) Any $\mu$ can be included as $\mu_1$ into a free convolution semigroup $\{\mu_t : t \geq 1\}$.

## SIMILARITY.

Let $\{\nu_n\}$ be any probability measures. Suppose

$$\lim_{n\to\infty} \underbrace{\nu_n * \nu_n * \ldots * \nu_n}_{k_n \text{ times}} = \mu.$$

Then $\mu$ necessarily infinitely divisible.

**Theorem.** (HB, Pata '99)

$$\lim_{n\to\infty} \underbrace{\nu_n \boxplus \nu_n \boxplus \ldots \boxplus \nu_n}_{k_n \text{ times}} = \Lambda(\mu).$$

FURTHER RESULTS.

1) Operators

$$C_t : \nu \mapsto \mu_t \boxplus \nu$$

non-linear. What are their linearizations (derivatives)?

**Theorem.** (M.A. '03)

$$(D_{\mu_s} C_t)(P_n^*(s)) = P_n^*(s+t),$$

where $\{P_n\}$ is an explicit family of polynomials.

Denote by $S_r$ the scaling operator.

**Corollary.** (M.A. '99) Consider the operator

$$C : \nu \mapsto (\nu \boxplus \nu) \circ S_{1/\sqrt{2}}.$$

Corresponds to

$$X \mapsto \frac{X_1 + X_2}{\sqrt{2}}.$$

On centered probability measures with unit variance, it has the semicircular distribution as its only, attracting, fixed point (Free Central Limit Theorem).

Its derivative is a compact operator

eigenfunctions $T_n^*$, Chebyshev functions of the 1st kind.

eigenvalues $2^{1-n/2}$, less than 1 for $n > 2$.

2) Realize the Bercovici-Pata bijection as a deformation.

a) Using representations on $q$-deformed Fock spaces, can define a $q$-deformed convolution operations $*_q$ interpolating between $* = *_1$ and $\boxplus = *_0$. (M.A. '01)

b) Using random matrix representations. (Cabanal-Duvillard, Benaych-Georges '04)

## FULL FOCK SPACE

Start with $H = L^2(\mathbb{R}_+, dx) \otimes V$, $\xi \in V$ , $T$ an operator on $V$, with compatibility conditions. Construct

$$\mathcal{F}_{\text{alg}}(H) = \mathbb{C}\Omega \oplus H \oplus H^{\otimes 2} \oplus H^{\otimes 3} \oplus \ldots .$$

Define an inner product

$$\langle \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_k \rangle_0 \\ = \delta_{nk} \langle \xi_1, \eta_1 \rangle \ldots \langle \xi_n, \eta_n \rangle ,$$

Completing, get the full Fock space $\mathcal{F}_0(H)$.

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For $\zeta \in H$, define creation and annihilation operators on $\mathcal{F}_0(H)$ by

$$a^*(\zeta)(\eta_1 \otimes \ldots \otimes \eta_n) = \zeta \otimes \eta_1 \otimes \ldots \otimes \eta_n,$$
$$a(\zeta)(\eta_1 \otimes \ldots \otimes \eta_n) = \langle \zeta, \eta_1 \rangle \eta_2 \otimes \ldots \otimes \eta_n.$$

$a(\zeta)$ and $a^*(\zeta)$ are adjoints of each other. Moreover,

$$a(\zeta)a^*(\eta) = \langle \zeta, \eta \rangle \, \text{Id}.$$

Also, for $S$ an operator on $H$, define the operator $p(S)$ on $\mathcal{F}_0(H)$ by

$$p(S)(\eta_1 \otimes \ldots \otimes \eta_n) = (S\eta_1) \otimes \eta_2 \otimes \ldots \otimes \eta_n.$$

$p(S)$ is essentially self-adjoint if $S$ is.

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Let

$$\begin{aligned} a(t) &= a(\mathbf{1}_{[0,t)} \otimes \xi), \\ a^*(t) &= a^*(\mathbf{1}_{[0,t)} \otimes \xi), \\ p(t) &= p(\mathbf{1}_{[0,t)} \otimes T), \end{aligned}$$

$$X(t) = a(t) + a^*(t) + p(t).$$

If $\mu_{t,0}$ = distribution of $X(t)$ with respect to the vacuum state $\varphi[A] = \langle \Omega, A\Omega \rangle$, $\left\{\mu_{t,0}\right\}$ is a free convolution semigroup.

## Symmetric Fock space

On $\mathcal{F}_{\mathrm{alg}}(H)$, define a symmetrized inner product

$$\begin{aligned}&\langle \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_k \rangle_1 \\ &\qquad = \delta_{nk} \sum_{\sigma \in \mathrm{Sym}(n)} \left\langle \xi_{\sigma(1)}, \eta_1 \right\rangle \ldots \left\langle \xi_{\sigma(n)}, \eta_n \right\rangle ,\end{aligned}$$

Degenerate; quotienting out and completing, get the symmetric Fock space $\mathcal{F}_1(H)$.

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For $\zeta \in H$, define creation and annihilation operators on $\mathcal{F}_1(H)$ by

$$a^*(\zeta)(\eta_1 \otimes \ldots \otimes \eta_n) = \zeta \otimes \eta_1 \otimes \ldots \otimes \eta_n,$$

$$a(\zeta)(\eta_1 \otimes \ldots \otimes \eta_n) = \sum_{i=1}^{n} \langle \zeta, \eta_i \rangle \, \eta_1 \otimes \ldots \widehat{\eta_i} \otimes \ldots \otimes \eta_n.$$

$a(\zeta)$ and $a^*(\zeta)$ are again adjoints of each other. Moreover,

$$a(\zeta)a^*(\eta) - a^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \, \mathrm{Id}.$$

Also, for $S$ an operator on $H$, define the operator $p(S)$ on $\mathcal{F}_1(H)$ by

$$p(S)(\eta_1 \otimes \ldots \otimes \eta_n) = \sum_{i=1}^{n} (S\eta_i) \otimes \eta_2 \otimes \ldots \otimes \widehat{\eta}_i \otimes \ldots \otimes \eta_n.$$

$p(S)$ is essentially self-adjoint if $S$ is.

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Now, if $\mu_{t,1}$ = distribution of $X(t)$ with respect to the vacuum state $\varphi[A] = \langle \Omega, A\Omega \rangle$, $\left\{ \mu_{t,1} \right\}$ is a (classical) convolution semigroup.

Note: keeping $V$, $\xi$, $T$ fixed, $\mu_{t,0} = \Lambda(\mu_{t,1})$.

## $q$-DEFORMED FULL FOCK SPACE

On $\mathcal{F}_{\text{alg}}(H)$, define a new inner product

$$\langle \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_k \rangle_q = \delta_{nk} \sum_{\sigma \in \text{Sym}(n)} q^{i(\sigma)} \left\langle \xi_{\sigma(1)}, \eta_1 \right\rangle \ldots \left\langle \xi_{\sigma(n)}, \eta_n \right\rangle ,$$

where $i(\sigma) =$ number of inversions.

For $q \in (-1, 1)$, positive definite (Bożejko-Speicher). Completing, get the $q$-Fock space $\mathcal{F}_q(H)$.

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For $\zeta \in H$, define creation and annihilation operators on $\mathcal{F}_q(H)$ by

$$a^*(\zeta)(\eta_1 \otimes \ldots \otimes \eta_n) = \zeta \otimes \eta_1 \otimes \ldots \otimes \eta_n,$$

$$a(\zeta)(\eta_1 \otimes \ldots \otimes \eta_n) = \sum_{i=1}^{n} q^{i-1} \langle \zeta, \eta_i \rangle \, \eta_1 \otimes \ldots \widehat{\eta}_i \otimes \ldots \otimes \eta_n.$$

$a(\zeta)$ and $a^*(\zeta)$ are again adjoints of each other. Moreover,

$$a(\zeta)a^*(\eta) - q a^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \, \mathrm{Id}.$$

Also, for $S$ an operator on $H$, define the operator $p(S)$ on $\mathcal{F}_1(H)$ by

$$p(S)(\eta_1 \otimes \ldots \otimes \eta_n) = \sum_{i=1}^{n} q^{i-1} (S\eta_i) \otimes \eta_2 \otimes \ldots \otimes \widehat{\eta}_i \otimes \ldots \otimes \eta_n.$$

**Proposition.** $p(S)$ is essentially self-adjoint if $S$ is.

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The distributions $\left\{\mu_{t,q}\right\}$ interpolate between $\left\{\mu_{t,1}\right\}$ and $\left\{\mu_{t,0}\right\}$ (and also fermionic $\left\{\mu_{t,-1}\right\}$).