Polynomials, simple and complex

Michael Anshelevich

Texas A&M University

November 16, 2011
Part I.
Parameters: \(a, b, c \in \mathbb{R}\) and \(t \geq 0\).
Meixner polynomials.

Parameters: $a, b, c \in \mathbb{R}$ and $t \geq 0$.

Polynomials $\{P_n : n \geq 0\}$: defined by $P_0 = 1$, $P_1 = x - at$, and the recursion

$$P_{n+1}(x) = xP_n(x) - (at + (b + c)n)P_n(x) - n(t + bc(n - 1))P_{n-1}(x).$$
Meixner polynomials.

Parameters: \(a, b, c \in \mathbb{R}\) and \(t \geq 0\).

Polynomials \(\{P_n : n \geq 0\}\): defined by \(P_0 = 1\), \(P_1 = x - at\), and the recursion

\[
P_{n+1}(x) = xP_n(x) - (at + (b + c)n)P_n(x) - n(t + bc(n - 1))P_{n-1}(x).
\]

Linear functional \(\varphi : \mathbb{C}[x] \rightarrow \mathbb{C}\): defined by

\[
\varphi[1] = 1, \quad \varphi[P_n(x)] = 0 \text{ for } n > 0.
\]
Moments.

Facts.

1. \( \varphi \) arises from a (probability) measure \( \mu \):

\[
\varphi[Q(x)] = \int_{\mathbb{R}} Q(x) \, d\mu(x).
\]

Measures \( \mu_{a,b,c,t} \) look quite different.
Facts.

1. \( \varphi \) arises from a (probability) measure \( \mu \):

\[
\varphi[Q(x)] = \int_{\mathbb{R}} Q(x) \, d\mu(x).
\]

Measures \( \mu_{a,b,c,t} \) look quite different.

2. We have not only \( \varphi[P_n] = 0 \) but

\[
\varphi[P_nP_k] = 0 \text{ for } n \neq k
\]

(polynomials are \textbf{orthogonal}).
Moments.

Facts.

1. $\varphi$ arises from a (probability) measure $\mu$:

$$\varphi[Q(x)] = \int_{\mathbb{R}} Q(x) \, d\mu(x).$$

Measures $\mu_{a,b,c,t}$ look quite different.

2. We have not only $\varphi[P_n] = 0$ but

$$\varphi[P_n P_k] = 0 \text{ for } n \neq k$$

(polynomials are orthogonal).

3. The $n$’th moment of $\varphi$

$$\varphi[x^n] = \text{Polynomial in } a, b, c, t \text{ with positive integer coefficients.}$$
\[ \pi \in S(n) = \text{permutation}. \] Write it in the cycle notation. Let \( 1 \leq i \leq n \).

\[ i = \begin{cases} 
\text{fixed point of } \pi & \text{if } \pi(i) = i, \\
\text{ascent of } \pi & \text{if } \pi(i) > i, \\
\text{descent of } \pi & \text{if } \pi(i) < i.
\end{cases} \]
Permutation statistics.

\[ \pi \in S(n) = \text{permutation}. \text{ Write it in the cycle notation. Let } 1 \leq i \leq n. \]

\[
i = \begin{cases} 
\text{fixed point of } \pi & \text{if } \pi(i) = i, \\
\text{ascent of } \pi & \text{if } \pi(i) > i, \\
\text{descent of } \pi & \text{if } \pi(i) < i.
\end{cases}
\]

**Theorem.** (Kim, Zeng 2001)

For the functional \( \varphi \) from before,

\[
\varphi[x^n] = \sum_{\pi \in S(n)} a^\#\text{fixed points} b^\#\text{ascents} c^\#\text{descents} t^\#\text{cycles}.
\]
Set \( a = b = c = 1 \). Recursion

\[
P_{n+1}(x) = xP_n(x) - (t + 2n)P_n(x) - n(t + (n - 1))P_{n-1}(x).
\]

Laguerre polynomials. Orthogonality measure: Gamma distribution

\[
d\mu(x) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} 1_{[0,\infty)}(x) \, dx.
\]
Example: Gamma.

Set \( a = b = c = 1 \). Recursion

\[
P_{n+1}(x) = xP_n(x) - (t + 2n)P_n(x) - n(t + (n - 1))P_{n-1}(x).
\]

Laguerre polynomials. Orthogonality measure:
Gamma distribution

\[
d\mu(x) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} 1_{[0,\infty)} \, dx.
\]

Thus

\[
\varphi[x^n] = \sum_{\pi \in S(n)} t^{\# \text{cycles}}.
\]
Example: Gamma.

Set $a = b = c = 1$. Recursion

$$P_{n+1}(x) = xP_n(x) - (t + 2n)P_n(x) - n(t + (n - 1))P_{n-1}(x).$$

Laguerre polynomials. Orthogonality measure: Gamma distribution

$$d\mu(x) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} \mathbf{1}_{[0,\infty)} dx.$$ 

Thus

$$\varphi[x^n] = \sum_{\pi \in S(n)} t^{\#\text{cycles}}.$$ 

In particular, for $t = 1$, $\int_0^\infty x^n e^{-x} dx = n!.$
Example: derangements.

Set instead $a = 0$, $b = c = 1$, $t = 1$. Recursion

$$P_{n+1}(x) = xP_n(x) - 2n P_n(x) - n^2 P_{n-1}(x).$$

Shifted Laguerre polynomials. Orthogonality measure: shifted Gamma distribution

$$d\mu(x) = e^{-x-1}1_{[-1,\infty)} \, dx.$$
Example: derangements.

Set instead \(a = 0, b = c = 1, t = 1\). Recursion

\[
P_{n+1}(x) = xP_n(x) - 2n P_n(x) - n^2 P_{n-1}(x).
\]

Shifted Laguerre polynomials. Orthogonality measure: shifted Gamma distribution

\[
d\mu(x) = e^{-x-1}1_{[-1,\infty)} \, dx.
\]

\(\pi \in S(n)\) with no fixed points \(\leftrightarrow\) Derangements \(\pi \in \mathcal{D}(n)\).
Example: derangements.

Set instead $a = 0, b = c = 1, t = 1$. Recursion

$$P_{n+1}(x) = xP_n(x) - 2nP_n(x) - n^2 P_{n-1}(x).$$

Shifted Laguerre polynomials. Orthogonality measure: shifted Gamma distribution

$$d\mu(x) = e^{-x-1}1_{[-1,\infty)} \, dx.$$ 

$$\pi \in S(n) \text{ with no fixed points} \iff \text{Derangements } \pi \in \mathcal{D}(n).$$

Thus

$$\int_{-1}^{\infty} x^n e^{-x-1} \, dx = \int_{0}^{\infty} (x - 1)^n e^{-x} \, dx = |\mathcal{D}(n)|.$$ 

Note: no formula for this number.
Example: Poisson.

Set $a = b = 1$, $c = 0$. Recursion

$$P_{n+1}(x) = xP_n(x) - (t + n)P_n(x) - nt P_{n-1}(x).$$

Charlier polynomials. Orthogonality measure: Poisson distribution

$$d\mu(x) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta_k(x).$$
Example: Poisson.

Set \( a = b = 1, \ c = 0 \). Recursion

\[
P_{n+1}(x) = xP_n(x) - (t + n)P_n(x) - nt \ P_{n-1}(x).
\]

Charlier polynomials. Orthogonality measure: Poisson distribution

\[
d\mu(x) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta_k(x).
\]

\( \pi \in S(n) \) with (\# descents = \#cycles) \( \leftrightarrow \) Partitions \( \pi \in \mathcal{P}(n) \).

Thus

\[
\varphi[x^n] = \sum_{\pi \in \mathcal{P}(n)} t^{\# \text{subsets}}.
\]
Example: Gaussian.

Set \( a = b = c = 0 \). Recursion

\[
P_{n+1}(x) = xP_n(x) - nt P_{n-1}(x).
\]

Hermite polynomials. Orthogonality measure: Gaussian (or normal) distribution

\[
d\mu(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx.
\]
Example: Gaussian.

Set \( a = b = c = 0 \). Recursion

\[
P_{n+1}(x) = xP_n(x) - nt\ P_{n-1}(x).
\]

Hermite polynomials. Orthogonality measure: Gaussian (or normal) distribution

\[
d\mu(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \, dx.
\]

\[
\left( \pi \in S(n) \text{ with no fixed points and} \atop \# \text{ ascents} = \# \text{ descents} = \# \text{cycles} \right) \leftrightarrow \text{Pairings } \pi \in P_2(n).
\]
Example: Gaussian.

\[
\left( \pi \in S(n) \text{ with no fixed points and} \right) \quad \left( \# \text{ ascents} = \# \text{ descents} = \# \text{cycles} \right) \quad \leftrightarrow \quad \text{Pairings } \pi \in \mathcal{P}_2(n).
\]

Thus

\[
\varphi[x^n] = \sum_{\pi \in \mathcal{P}_2(n)} t^{\# \text{pairs}}
\]

Zero if \( n \) odd.

\[
\varphi[x^{2n}] = t^n |\mathcal{P}_2(n)| = (2n - 1)(2n - 3) \ldots 3 \cdot 1 \cdot t^n.
\]
A new wrinkle: Instead of using $n \in \mathbb{N}$, use

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q}. $$

$q$-integers (Ramanujan, quantum groups, etc.)
A new wrinkle: Instead of using $n \in \mathbb{N}$, use

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q}.$$ 

$q$-integers (Ramanujan, quantum groups, etc.)

As before define $\{P_n : n \geq 0\}$ by

$$P_{n+1}(x) = x P_n(x) - (at + (b + c)[n]_q) P_n(x) - [n]_q (t + bc[n - 1]_q) P_{n-1}(x).$$

Again orthogonal polynomials for a measure $\mu = \mu_{a,b,c,t,q}$. 
A new wrinkle: Instead of using $n \in \mathbb{N}$, use

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q}.$$ 

$q$-integers (Ramanujan, quantum groups, etc.)

As before define $\{P_n : n \geq 0\}$ by

$$P_{n+1}(x) = xP_n(x) - (at + (b + c)[n]_q)P_n(x) - [n]_q(t + bc[n - 1]_q)P_{n-1}(x).$$

Again orthogonal polynomials for a measure $\mu = \mu_{a,b,c,t,q}$.

Moments of $\mu$? Answer seems to involve crossings.
Crossings.

$q$-Gaussian case [Ismail, Stanton, Viennot 1987]:

\[ P_{n+1}(x) = xP_n(x) - [n]_q t P_{n-1}(x). \]

($q$-Hermite polynomials, Rogers (1894)). Moments:

\[ \varphi[x^{2n}] = t^n \sum_{\pi \in \mathcal{P}_2(2n)} q^{\# \text{crossings in the pairing}}. \]
Crossings.

\(q\)-Gaussian case [Ismail, Stanton, Viennot 1987]:

\[ P_{n+1}(x) = x P_n(x) - [n]_q t P_{n-1}(x). \]

\((q\text{-Hermite polynomials, Rogers (1894))}. \) Moments:

\[ \varphi[x^{2n}] = t^n \sum_{\pi \in \mathcal{P}_2(2n)} q^{\# \text{crossings in the pairing}}. \]

\(q\)-Poisson case [A 2005], [Kim, Stanton, Zeng 2006]:

\[ P_{n+1}(x) = x P_n(x) - (t + [n]_q) P_n(x) - [n]_q t P_{n-1}(x). \]

Moments:

\[ \varphi[x^n] = \sum_{\pi \in \mathcal{P}(n)} t^{\# \text{subsets}} q^{\# \text{crossings in a partition}}. \]
Full case: unknown. Difficulty: how to count crossings of a permutation.

Have an approach (linked partitions), want to start work soon.
Full case: unknown. Difficulty: how to count crossings of a permutation.

Have an approach (linked partitions), want to start work soon.

More general question: Linearization coefficients

\[ \varphi[P_{n_1} P_{n_2} \ldots P_{n_k}] . \]

Again expect combinatorial interpretations!
Why do analysts care?

\(a, b, c\) some parameters.

\[ t = \text{convolution parameter} = \text{time parameter}. \]
Why do analysts care?

\(a, b, c\) some parameters.

\[ t = \text{convolution parameter} = \text{time parameter}. \]

Recall: Gaussian, Poisson, Gamma

\[
\begin{align*}
\text{d}\mu_t(x) &= \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \, dx, \\
&= e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!}\delta_k(x), \\
&= x^{t-1} e^{-x} 1_{[0,\infty)} \, dx.
\end{align*}
\]
Why do analysts care?

$a, b, c$ some parameters.

\[ t = \text{convolution parameter} = \text{time parameter}. \]

Recall: Gaussian, Poisson, Gamma

\[
d\mu_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \, dx, \quad e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta_k(x), \quad x^{t-1} e^{-x} 1_{[0, \infty)} \, dx.
\]

In each case, form a \textbf{convolution semigroup}:

\[
\mu_t * \mu_s = \mu_{t+s},
\]

where (roughly)

\[
d(\mu * \nu)(x) = \int_{\mathbb{R}} \mu(y-x) \, d\nu(y).
\]

Thus Combinatorics comes from Analysis.
Independence.

Even more: if $X, Y$ are independent random variables, with distributions $\mu_X, \mu_Y$, the distribution of the sum is the convolution:

$$\mu_{X+Y} = \mu_X * \mu_Y.$$ 

So a convolution semigroup of measures $\{\mu_t : t \geq 0\}$

comes from a process with independent increments $\{X(t) : t \geq 0\}$.

Thus Analysis comes from Probability.
Stochastic integrals.

Fancier relations:
Stochastic integrals.

Fancier relations:

\[ H_n(X(t)) \]

(Hermite polynomial of a Brownian motion)
Stochastic integrals.

Fancier relations:

\[ H_n(X(t)) \]

(Hermite polynomial of a Brownian motion)

\[ = \int_{0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq t} dX(t_1) \, dX(t_2) \ldots \, dX(t_n) \]

(multiple stochastic integral)
Stochastic integrals.

Fancier relations:

\[ H_n(X(t)) \]

(Hermite polynomial of a Brownian motion)

\[ = \int_{0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq t} dX(t_1) dX(t_2) \ldots dX(t_n) \]

(multiple stochastic integral)

\[ = W \left( a(t) + a^*(t), a(t) + a^*(t), \ldots, a(t) + a^*(t) \right) \]

(Wick product on a Fock space).
Probability theories.

Gaussian, Poisson, Gamma come from Probability. $q = 1$ case. Well understood.
Gaussian, Poisson, Gamma come from Probability. $q = 1$ case. Well understood.

Another well-understood case: $q = 0$. $t$ again a time parameter, but with respect to the operation of free convolution $\boxplus$. Arises from free independence. All in the context of Free Probability (Voiculescu 1986).
Probability theories.

Gaussian, Poisson, Gamma come from Probability. $q = 1$ case. Well understood.

Another well-understood case: $q = 0$. $t$ again a time parameter, but with respect to the operation of free convolution $\boxplus$. Arises from free independence. All in the context of Free Probability (Voiculescu 1986).

General $q$? “$q$-deformed” probability? Gaussian, Poisson case well understood [Bożejko, Kümmnerer, Speicher 1997] [A 2001], used in proofs above. In general, no. Processes with $q$-independent increments?

My approach (Analysis / Probability): full Meixner case $q = 1$, moments only (unpublished).

General $q$, linearization coefficients, Gaussian and Poisson only (2005).

My approach (Analysis / Probability): full Meixner case $q = 1$, moments only (unpublished).

General $q$, linearization coefficients, Gaussian and Poisson only (2005).

Part II.
Matrix-valued polynomials.

Polynomials: spanned by monomials $Ax^n$, for $A \in \mathbb{R}$ or $\mathbb{C}$.
Matrix-valued polynomials.

Polynomials: spanned by monomials $Ax^n$, for $A \in \mathbb{R}$ or $\mathbb{C}$.

What if $A \in \mathcal{A}$, for

$$\mathcal{A} = \text{algebra, matrix algebra, } C^*-\text{algebra, etc.}$$

So that

$$(Ax^n)(Bx^k) = (AB)x^{n+k} \neq (BA)x^{n+k}.$$ 

Orthogonal polynomials?
Matrix-valued polynomials.

Polynomials: spanned by monomials $Ax^n$, for $A \in \mathbb{R}$ or $\mathbb{C}$.

What if $A \in \mathcal{A}$, for

$$\mathcal{A} = \text{ algebra, matrix algebra, } C^*\text{-algebra, etc.?}$$

So that

$$(Ax^n)(Bx^k) = (AB)x^{n+k} \neq (BA)x^{n+k}.$$ 

Orthogonal polynomials?

Interesting theory already for $\mathcal{A} = M_{2 \times 2}$, connected to random walks.

Difficulty: Gram-Schmidt does not always work.

$$P_2(x) = x^2 - \frac{\varphi[x^2P_1(x)]}{\varphi[P_1(x)^2]}P_1(x) - \varphi[x^2]$$

but may have $\varphi[P_1(x)^2] \in M_{2 \times 2}$ non-zero but not invertible.
Operator-valued polynomials.

\( \mathcal{A} \) non-commutative, but \( Ax = xA \). Free probability: \textit{maximally} non-commutative. Thus instead of \( \mathcal{A}[x] \) want \( \mathcal{A}\langle x \rangle \) spanned by

\[ A_0xA_1xA_2\ldots A_{n-1}xA_n. \]
Operator-valued polynomials.

$\mathcal{A}$ non-commutative, but $Ax = xA$. Free probability: \textit{maximally} non-commutative. Thus instead of $\mathcal{A}[x]$ want $\mathcal{A}\langle x \rangle$ spanned by

$$A_0 x A_1 x A_2 \ldots A_{n-1} x A_n.$$

To even prove \textit{existence} of orthogonal polynomials, one needs free probability (Fock space constructions). Gram-Schmidt does not even make sense. New objects, little known about them. Both algebraic and analytic difficulties. Joint work with Belinschi, Popa, etc.