Polynomials, simple and complex

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Part I.
Meixner polynomials.

Parameters: $a, b, c \in \mathbb{R}$ and $t \geq 0$.

Polynomials $\{P_n : n \geq 0\}$: defined by $P_0 = 1$, $P_1 = x - at$, and the recursion

$$P_{n+1}(x) = xP_n(x) - (at + (b + c)n)P_n(x) - n(t + bc(n - 1))P_{n-1}(x).$$

Linear functional $\varphi : \mathbb{C}[x] \rightarrow \mathbb{C}$: defined by

$$\varphi[1] = 1, \quad \varphi[P_n(x)] = 0 \text{ for } n > 0.$$
Moments.

Facts.

1. \( \varphi \) arises from a (probability) measure \( \mu \):

\[
\varphi[Q(x)] = \int_{\mathbb{R}} Q(x) \, d\mu(x).
\]

Measures \( \mu_{a,b,c,t} \) look quite different.

2. We have not only \( \varphi[P_n] = 0 \) but

\[
\varphi[P_n P_k] = 0 \quad \text{for} \ n \neq k
\]

(polynomials are orthogonal).

3. The \( n \)’th moment of \( \varphi \)

\[
\varphi[x^n] = \text{Polynomial in } a, b, c, t \text{ with positive integer coefficients.}
\]
Permutation statistics.

\[ \pi \in S(n) = \text{permutation}. \] Write it in the cycle notation. Let \( 1 \leq i \leq n \).

\[
i = \begin{cases} 
\text{fixed point of } \pi & \text{if } \pi(i) = i, \\
\text{ascent of } \pi & \text{if } \pi(i) > i, \\
\text{descent of } \pi & \text{if } \pi(i) < i.
\end{cases}
\]

**Theorem.** (Kim, Zeng 2001)

For the functional \( \varphi \) from before,

\[
\varphi[x^n] = \sum_{\pi \in S(n)} a^{\# \text{fixed points}} b^{\# \text{ascents}} c^{\# \text{descents}} t^{\# \text{cycles}}.
\]
Set $a = b = c = 1$. Recursion

$$P_{n+1}(x) = xP_n(x) - (t + 2n)P_n(x) - n(t + (n - 1))P_{n-1}(x).$$

Laguerre polynomials. Orthogonality measure: Gamma distribution

$$d\mu(x) = \frac{1}{\Gamma(t)}x^{t-1}e^{-x}\mathbf{1}_{[0,\infty)} \, dx.$$ 

Thus

$$\varphi[x^n] = \sum_{\pi \in S(n)} t^\#\text{cycles}.$$ 

In particular, for $t = 1$, $\int_0^\infty x^n e^{-x} \, dx = n!$. 

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Example: derangements.

Set instead \( a = 0, \ b = c = 1, \ t = 1 \). Recursion

\[
P_{n+1}(x) = xP_n(x) - 2n \ P_n(x) - n^2 \ P_{n-1}(x).
\]

Shifted Laguerre polynomials. Orthogonality measure: shifted Gamma distribution

\[
d\mu(x) = e^{-x-1} 1_{[-1, \infty)} \ dx.
\]

\[\pi \in S(n) \text{ with no fixed points} \iff \text{Derangements } \pi \in D(n).\]

Thus

\[
\int_{-1}^{\infty} x^n e^{-x-1} \ dx = \int_{0}^{\infty} (x - 1)^n e^{-x} \ dx = |D(n)|.
\]

Note: no formula for this number.
Example: Poisson.

Set $a = b = 1$, $c = 0$. Recursion

$$P_{n+1}(x) = xP_n(x) - (t + n)P_n(x) - nt P_{n-1}(x).$$

Charlier polynomials. Orthogonality measure: Poisson distribution

$$d\mu(x) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta_k(x).$$

$$\pi \in S(n) \text{ with (# descents = #cycles)} \leftrightarrow \text{Partitions } \pi \in P(n).$$

Thus

$$\varphi[x^n] = \sum_{\pi \in P(n)} t^{\# \text{subsets}}.$$
Example: Gaussian.

Set $a = b = c = 0$. Recursion

$$P_{n+1}(x) = xP_n(x) - nt P_{n-1}(x).$$

Hermite polynomials. Orthogonality measure: Gaussian (or normal) distribution

$$d\mu(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \, dx.$$
Example: Gaussian.

\(( \pi \in S(n) \text{ with no fixed points and } \# \text{ ascents} = \# \text{ descents} = \# \text{cycles} ) \leftrightarrow \text{Pairings } \pi \in \mathcal{P}_2(n) \).

Thus

\[ \varphi[x^n] = \sum_{\pi \in \mathcal{P}_2(n)} t^{\#\text{pairs}} \]

Zero if \( n \) odd.

\[ \varphi[x^{2n}] = t^n |\mathcal{P}_2(n)| = (2n - 1)(2n - 3) \cdots 3 \cdot 1 \cdot t^n. \]
A new wrinkle: Instead of using $n \in \mathbb{N}$, use

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q}.$$ 

$q$-integers (Ramanujan, quantum groups, etc.)

As before define $\{P_n : n \geq 0\}$ by

$$P_{n+1}(x) = xP_n(x) - (at + (b + c)[n]_q)P_n(x) - [n]_q(t + bc[n - 1]_q)P_{n-1}(x).$$

Again orthogonal polynomials for a measure $\mu = \mu_{a,b,c,t,q}$.

Moments of $\mu$? Answer seems to involve crossings.
Crossings.

$q$-Gaussian case [Ismail, Stanton, Viennot 1987]:

\[ P_{n+1}(x) = xP_n(x) - [n]_q tP_{n-1}(x). \]

($q$-Hermite polynomials, Rogers (1894)). Moments:

\[
\varphi[x^{2n}] = t^n \sum_{\pi \in \mathcal{P}_2(2n)} q^{\# \text{crossings in the pairing}}.
\]

$q$-Poisson case [A 2005], [Kim, Stanton, Zeng 2006]:

\[ P_{n+1}(x) = xP_n(x) - (t + [n]_q)P_n(x) - [n]_q tP_{n-1}(x). \]

Moments:

\[
\varphi[x^n] = \sum_{\pi \in \mathcal{P}(n)} t^{\# \text{subsets}} q^{\# \text{crossings in a partition}}.
\]
Full case: unknown. Difficulty: how to count crossings of a permutation.

Have an approach (*linked partitions*), want to start work soon.

More general question: **Linearization coefficients**

\[ \varphi[P_{n_1} P_{n_2} \cdots P_{n_k}] \]

Again expect combinatorial interpretations!
Why do analysts care?

$a, b, c$ some parameters.

\[ t = \text{convolution parameter} = \text{time parameter}. \]

Recall: Gaussian, Poisson, Gamma

\[
d\mu_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \, dx, \quad e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta_k(x), \quad x^{t-1} e^{-x} 1_{[0,\infty)} \, dx.
\]

In each case, form a convolution semigroup:

\[ \mu_t * \mu_s = \mu_{t+s}, \]

where (roughly)

\[ d(\mu * \nu)(x) = \int_{\mathbb{R}} \mu(y - x) \, d\nu(y). \]

Thus Combinatorics comes from Analysis.
Independence.

Even more: if \( X, Y \) are independent random variables, with distributions \( \mu_X, \mu_Y \), the distribution of the sum is the convolution:

\[
\mu_{X+Y} = \mu_X * \mu_Y.
\]

So a convolution semigroup of measures

\[
\{\mu_t : t \geq 0\}
\]

comes from a process with independent increments

\[
\{X(t) : t \geq 0\}.
\]

Thus Analysis comes from Probability.
Stochastic integrals.

Fancier relations:

\[ H_n(X(t)) \]

(Hermite polynomial of a Brownian motion)

\[ = \int_{0\leq t_1\leq t_2\leq\ldots\leq t_n\leq t} dX(t_1) dX(t_2) \ldots dX(t_n) \]

(multiple stochastic integral)

\[ = W\left( a(t) + a^*(t), a(t) + a^*(t), \ldots, a(t) + a^*(t) \right) \]

(Wick product on a Fock space).
Gaussian, Poisson, Gamma come from Probability. $q = 1$ case. Well understood.

Another well-understood case: $q = 0$. $t$ again a time parameter, but with respect to the operation of free convolution $\boxplus$. Arises from free independence. All in the context of Free Probability (Voiculescu 1986).

General $q$? “$q$-deformed” probability? Gaussian, Poisson case well understood [Bożejko, Kümmerer, Speicher 1997] [A 2001], used in proofs above. In general, no. Processes with $q$-independent increments?

My approach (Analysis / Probability): full Meixner case $q = 1$, moments only (unpublished).

General $q$, linearization coefficients, Gaussian and Poisson only (2005).

Part II.
Matrix-valued polynomials.

Polynomials: spanned by monomials $Ax^n$, for $A \in \mathbb{R}$ or $\mathbb{C}$.
What if $A \in \mathcal{A}$, for

$$\mathcal{A} = \text{algebra, matrix algebra, } C^*\text{-algebra}, \text{ etc.?}$$

So that

$$(Ax^n)(Bx^k) = (AB)x^{n+k} \neq (BA)x^{n+k}.$$ 

Orthogonal polynomials?

Interesting theory already for $\mathcal{A} = M_{2 \times 2}$, connected to random walks.

Difficulty: Gram-Schmidt does not always work.

$$P_2(x) = x^2 - \frac{\varphi[x^2P_1(x)]}{\varphi[P_1(x)^2]} P_1(x) - \varphi[x^2]$$

but may have $\varphi[P_1(x)^2] \in M_{2 \times 2}$ non-zero but not invertible.
Operator-valued polynomials.

\( \mathcal{A} \) non-commutative, but \( Ax = xA \). Free probability: *maximally* non-commutative. Thus instead of \( \mathcal{A}[x] \) want \( \mathcal{A}\langle x \rangle \) spanned by

\[ A_0 x A_1 x A_2 \ldots A_{n-1} x A_n. \]

To even prove *existence* of orthogonal polynomials, one needs free probability (Fock space constructions). Gram-Schmidt does not even make sense. New objects, little known about them. Both algebraic and analytic difficulties. Joint work with Belinschi, Popa, etc.