Free Meixner distributions and random matrices

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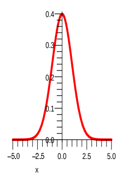
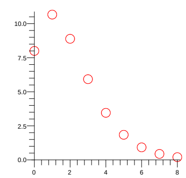
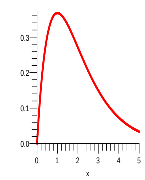
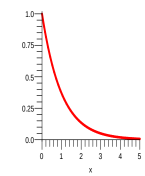
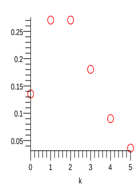
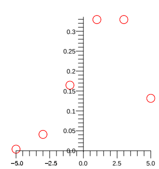
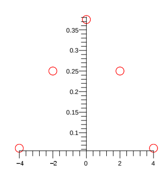
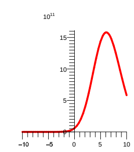
July 13, 2006

Some common distributions first...

Gaussian	Negative binomial Pascal geometric	Gamma chi-square exponential
$\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$	$\sum_{n=0}^{\infty} \binom{t+n-1}{t-1} (1-\mu)^t \mu^n \delta_n$	$\frac{1}{\Gamma(t)} e^{-x} x^{t-1} \mathbf{1}_{[0,\infty)}(x)$
		 
$(-\infty, \infty)$	$\{0, 1, 2, \ldots\}$	$[0, \infty)$
Poisson	**Binomial**	**Hyperbolic tangent**
$e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta_n$	$\sum_{n=0}^{T} \binom{T}{n} p^n (1-p)^{T-n} \delta_{2n-T}$	$C_{t,\zeta} e^{(\pi-2\zeta)x} \lvert\Gamma(t+ix)\rvert^2$
	 	
$\{0, 1, 2, \ldots\}$	$\{-T, -T+2, \ldots, T-2, T\}$	$(-\infty, \infty)$

all shifted to have mean zero.

Meixner family of probability distributions.

What do these have in common?

Look at their orthogonal polynomials.

For the Gaussian distribution, Hermite polynomials are orthogonal:

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} H_n(x,t) H_k(x,t) e^{-x^2/2t}\, dx = 0 \text{ if } n \neq k.$$

$$H_1 = x, \qquad H_2 = x^2 - t, \qquad H_3 = x^3 - 3xt, \ldots$$

Distributions	Polynomials
Gaussian	Hermite
Poisson	Charlier
Gamma (chi-square, exponential)	Laguerre
Negative binomial (geometric)	Meixner
Binomial	Krawtchouk
Hyperbolic tangent	Meixner-Pollaczek

Generating functions for polynomials

$$F(x,t,z) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x,t) z^n.$$

Polynomials	Generating function
Hermite	$\exp(xz - tz^2/2)$
Charlier	$(1+z)^{x+t}e^{-tz}$
Laguerre	$(1+z)^{-t}\exp -\left(\frac{x+t}{1+z}\right)$
Meixner	$(1-\mu)^{(\mu^{1/2}-\mu^{-1/2})^2}((1-\mu)+z)^{x-(1-\mu)t}$ $\times((1-\mu)+\mu z)^{-x+\frac{1-\mu}{\mu}t}$
Krawtchouk	$(1+(1-p)z)^{pT+x}(1-pz)^{(1-p)T-x}$
Meixner-Pollaczek	$\left(1-\frac{ze^{i\zeta}}{2\sin\zeta}\right)^{-t+ix}\left(1-\frac{ze^{-i\zeta}}{2\sin\zeta}\right)^{-t-ix}$

All of the form $e^{xu(z)-tv(z)}$.

Meixner distributions μ have orthogonal polynomials with generating functions $e^{xu(z)-tv(z)}$.

Theorem. (Meixner 1934) Up to re-scalings, translations, they are the only ones.

Another way to see: recursion relations.

Hermite

$$xH_n = H_{n+1} + ntH_{n-1}.$$

Charlier

$$xC_n = C_{n+1} + nC_n + ntC_{n-1}.$$

Laguerre

$$xL_n = L_{n+1} + 2nL_n + n(t + (n-1))L_{n-1}.$$

Meixner

$$xM_n = M_{n+1} + \frac{1+\mu}{1-\mu}nM_n + n\left(t + \frac{\mu}{(1-\mu)^2}(n-1)\right)M_{n-1}.$$

Krawtchouk

$$xK_n = K_{n+1} + (1-2p)nK_n + n(T - p(1-p)(n-1))K_{n-1}.$$

Meixner-Pollaczek

$$xP_n(x,t) = P_{n+1}(x,t) + 2(\cos\zeta)nP_n(x,t) + n(t + (n-1))P_{n-1}(x,t).$$

Thus all of the Meixner form

$$xM_n(x,t) = M_{n+1}(x,t) + bnM_n(x,t) + n(t + c(n-1))M_{n-1}(x,t).$$

Generating function

$$H(x,z,t) = \exp\left(x\int \frac{dz}{1+bz+cz^2} - t\int \frac{z\,dz}{1+bz+cz^2}\right).$$

More details:

Wim Schoutens, Stochastic Processes and Orthogonal Polynomials, Springer 2000.

P. Feinsilver, R. Schott, Algebraic Structures and Operator Calculus I. Representations and Probability Theory, Kluwer 1993.

Other properties:

Natural exponential families with quadratic variance function (Morris '83); generalized linear models.

Quadratic regression property (Laha, Lukacs '60).

Distributions of stochastic processes with linear conditional expectations and quadratic conditional variances (Wesołowski '93).

Polynomial basis for representations of $\mathfrak{sl}_2$.

Positive integer linearization coefficients (Kim, Zeng 2001).

Combinatorics: Rota, Sheffer, Viennot, etc.

FREE ANALOGS.

In $e^{xu(z)-tv(z)}$, replace e^z with $\frac{1}{1-z}$.

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots, \qquad \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

Distributions with orthogonal polynomials $\{P_n\}$,

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{1}{1 - xu(z) + tv(z)}.$$

Again can describe completely (M.A. 2003): up to re-scaling

$$\frac{1}{2\pi} \frac{t\sqrt{4(t+c)-(x-b)^2}}{t^2+tbx+cx^2} + \text{ zero, one, or two atoms.}$$

- $b = c = 0$ ("Gaussian") no atoms.

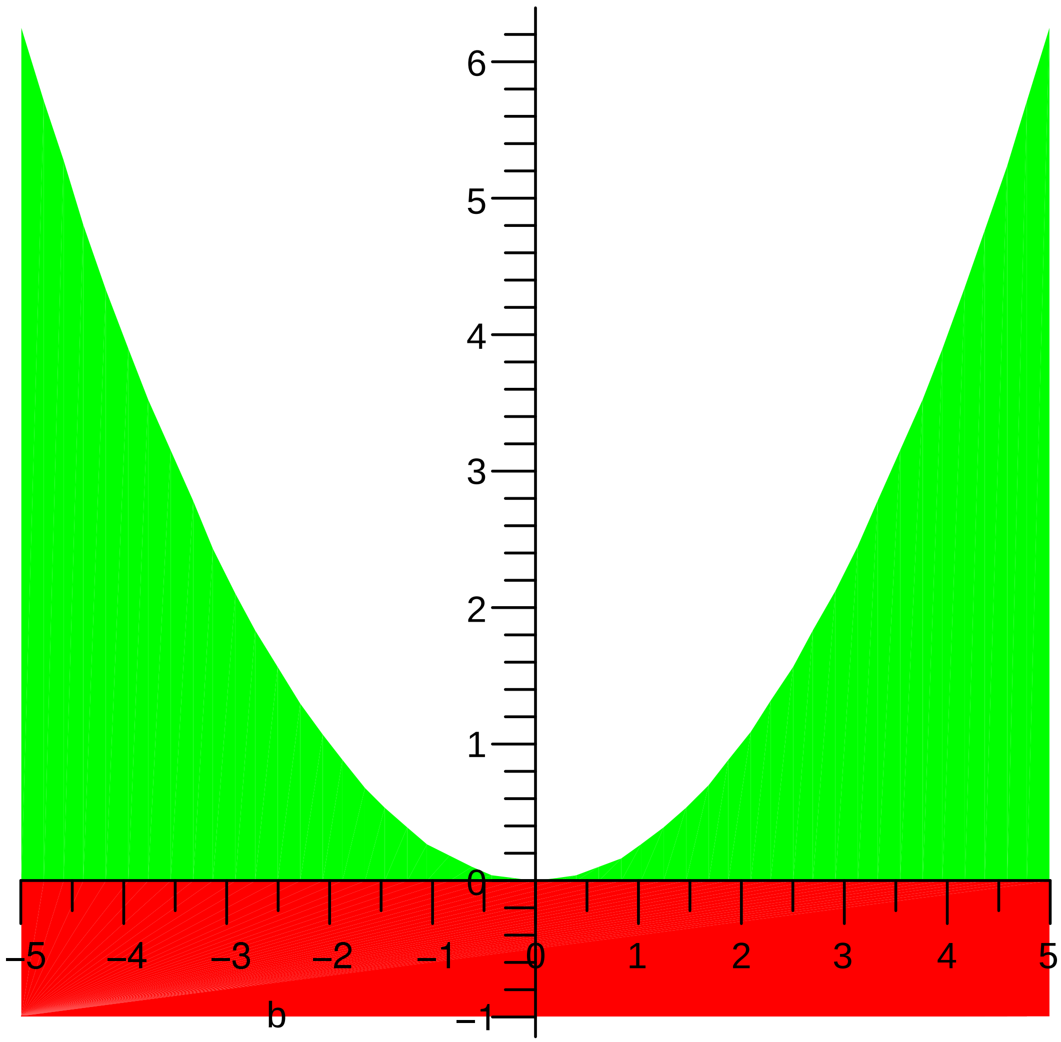
- $c = 0$ ("Poisson") at most one atom.

- $c > 0$, $b^2 > 4c$ ("Negative binomial") at most one atom.

- $c > 0$, $b^2 = 4c$ ("Gamma") no atoms.

- $c > 0$, $b^2 < 4c$ ("Hyperbolic tangent") no atoms.

- $-1 \leq c < 0$ ("Binomial") at most two atoms.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b

$$\frac{1}{2\pi}\frac{t\sqrt{4(t+c)-(x-b)^2}}{t^2+tbx+cx^2} + \text{ zero, one, or two atoms.}$$

APPEARANCE IN RANDOM MATRIX THEORY.

1. $b = c = 0$. Semicircular distribution. Limit of spectral distribution of GUE.

2. $c = 0$. Marchenko-Pastur distribution. Limit of Wishart matrices.

3. $-1 \leq c < 0$. Limit of Jacobi or Beta (Wachter '80; Capitaine, Casalis 2002).

4. $c > 0, b^2 \geq 4c$. Wireless communications (Gaussian matrices with correlated entries)?

Jacobi(n, α, β) has distribution

$$(Z_n^{\alpha,\beta})^{-1} \det(1 - M)^{\alpha} \det(M)^{\beta} \mathbf{1}_{0 \le M \le 1} \, dM$$

Also,

$$J = (X + X')^{-1/2} X (X + X')^{-1/2}$$

for some independent Wishart matrices. Limiting distribution

$$\frac{1}{2\pi} \frac{\sqrt{(x - \lambda_-)(\lambda_+ - x)}}{x - x^2} + \max(0, 1 - \alpha)\delta_0 + \max(0, 1 - \beta)\delta_1.$$

for some $\lambda_\pm$.

These are all free Meixner distributions.

FREE PROBABILITY.

$$\text{Random } n \times n \text{ matrices } A, B \longrightarrow \text{ non-commuting objects } x, y.$$

$$\text{Expectation } \frac{1}{n} \operatorname{Tr} \mathsf{E} \longrightarrow \text{ state } \varphi.$$

A, B diagonal, U random unitary matrix with uniform (Haar) distribution.

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Tr} \mathsf{E}\left[A^2(UBU^{-1})A(UBU^{-1})^3A^2\right] = \varphi\left[x^2yxy^3x^2\right], \text{ etc.}$$

Then x, y freely independent. Explicit formulas to calculate all moments.

Free central limit theorem. Let $x_1, x_2, \ldots$ be identically distributed, mean zero, variance one, freely independent. Then

$$\text{dist}\left(\frac{x_1 + x_2 + \ldots + x_k}{\sqrt{k}}\right) \longrightarrow_{k\to\infty} \text{Semicircular.}$$

Free Poisson limit theorem. Let $\left\{x_{i,k} : 1 \leq i \leq k, k = 1, 2, \ldots\right\}$ have distributions

$$\text{dist}(x_{i,k}) = \left(1 - \frac{1}{k}\right)\delta_0 + \frac{1}{k}\delta_1$$

and be freely independent. Then

$$\text{dist}(x_{1,k} + x_{2,k} + \ldots + x_{k,k}) \longrightarrow_{k\to\infty} \text{Marchenko-Pastur.}$$

Appearance in free probability.

1. $b = c = 0$. Appears in the free central limit theorem, free analog of Gaussian.

2. $c = 0$. Appears in the free Poisson limit theorem, free analog of the Poisson distribution.

3. $-1 \leq c < 0$. Free binomial distributions.

$P_\alpha = n \times n$ diagonal matrix with αn 1's, $(1-\alpha)n$ 0's on the diagonal.

Asymptotic distribution of P_α is Bernoulli $(1-\alpha)\delta_0 + \alpha\delta_1$.

$U_1, U_2, \ldots, U_T = n \times n$ independent random unitary matrices with uniform (Haar) distribution.

$$\lim_{n\to\infty} \text{dist}\Big(U_1 P_\alpha U_1^* + U_2 P_\alpha U_2^* + \ldots + U_T P_\alpha U_T^*\Big) = \text{FreeBinomial}(\alpha, T)$$

(up to a shift).

In fact, can do this for any *real* $T \geq 1$: for $\beta = \frac{1}{T}$,

$$\text{dist}\Big(P_\beta U P_\alpha U^* P_\beta\Big) \to \text{ shift of FreeBinomial } (\alpha, T) \stackrel{d}{=} \lim \text{Jacobi}$$

True for finite matrices (Collins 2005).

4. $c > 0, b^2 \geq 4c$. S-transform is a rational function.

Other properties.

Polynomial recursion.

$$xP_0 = P_1,$$
$$xP_1 = P_2 + bP_1 + P_0,$$
$$xP_n = P_{n+1} + bP_n + (t+c)P_{n-1}.$$

"almost constant coefficients"

Stieltjes transform.

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z-x}.$$

Modified R-transform

$$G\left(\frac{1+R(z)}{z}\right) = z.$$

$$\frac{R(z)}{z^2} = 1 + b\frac{R(z)}{z} + c\left(\frac{R(z)}{z}\right)^2 .$$

cf.

$$(\log \mathcal{F})'' = 1 + b(\log \mathcal{F})' + c[(\log \mathcal{F})']^2$$

Second order difference equation, with $f'(z)$ replaced by $\frac{f(z)-f(0)}{z}$.

Rao, Edelman.

CONDITIONAL CHARACTERIZATION (Bożejko, Bryc 2006).

Let $\mathbb{X}$, $\mathbb{Y}$ be freely independent, with conditional expectation

$$\mathsf{E}[\mathbb{X}|\mathbb{X}+\mathbb{Y}] = \alpha(\mathbb{X}+\mathbb{Y}) + \alpha_0 I$$

and conditional variance

$$\mathsf{Var}[\mathbb{X}|\mathbb{X}+\mathbb{Y}] = C(I + b((\mathbb{X}+\mathbb{Y}) + c(\mathbb{X}+\mathbb{Y})^2).$$

Then both $\mathbb{X}$ and $\mathbb{Y}$ have free Meixner distributions.

Matrix version?

Gamma characterizations.

Classical: Suppose X, Y are non-degenerate, independent non-negative random variables, and $S = X + Y$ positive. Let $Z = X/S$. S and Z are independent if and only if X, Y have gamma-type distributions. (Lukacs 1955).

Wishart: Suppose $\mathbb{X}, \mathbb{Y}$ are non-degenerate, independent positive random matrices, and $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ is strictly positive. Let $\mathbb{Z} = \mathbb{S}^{-1/2}\mathbb{X}\mathbb{S}^{-1/2}$. $\mathbb{S}$ and $\mathbb{Z}$ are independent if and only if $\mathbb{X}$ and $\mathbb{Y}$ have Wishart distributions. (Olkin, Rubin 1962).

Symmetric cones: Casalis, Letac 1996.

Wishart: gamma-like but limit free Poisson.

Suppose $\mathbb{X}$, $\mathbb{Y}$ are non-degenerate, freely independent non-commutative random variables, and $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ is strictly positive. Let $\mathbb{Z} = \mathbb{S}^{-1/2}\mathbb{X}\mathbb{S}^{-1/2}$. $\mathbb{S}$ and $\mathbb{Z}$ are freely independent if and only if $\mathbb{X}$ and $\mathbb{Y}$ have free Poisson-type distributions.

Suppose $\mathbb{X}$, $\mathbb{Y}$ are non-degenerate, freely independent, identically distributed, and strictly positive non-commutative random variables. Let $\mathbb{Z} = \mathbb{S}^{-1}\mathbb{X}^2\mathbb{S}^{-1}$. If $\mathbb{Z}$ and $\mathbb{S}$ are free, $\mathbb{X}$ and $\mathbb{Y}$ are free gamma-type. Do not know if free gamma has this property.

No matrix version (Letac)

Questions.

Realize free Meixner distributions for general b, c by random matrices

Realize free Meixner distributions for general b, c in free probability.

Appearance in wireless communications?

Meixner distributions for matrices of finite size?