Combinatorics of free Wick products

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Take a $*$-algebra $\mathcal{A}$ with a state $\langle\cdot\rangle$. For example,

$$\mathcal{A} = L^\infty[0,1],$$

$$\langle f\rangle = \int_0^1 f(x)\,dx.$$

Let $\Gamma(\mathcal{A})$ be the tensor algebra of $\mathcal{A}$. $\Gamma(\mathcal{A})$ is generated by

$$\{X(f)|f \in \mathcal{A}\}$$

where

$$X : \mathcal{A} \to \Gamma(\mathcal{A})$$

is a linear map. Make $\Gamma(\mathcal{A})$ into a $*$-algebra by requiring that $X(f) \in \Gamma(\mathcal{A})^{sa}$ if $f \in \mathcal{A}^{sa}$.

Define multilinear maps $W : \mathcal{A}^n \to \Gamma(\mathcal{A})$ by

$$W(f) = X(f) - \langle f \rangle$$

and for $f_1, \ldots, f_n \in \mathcal{A}^{sa}$,

$$\begin{aligned} W(f, f_1, f_2, \ldots, f_n) = X(f) W(f_1, f_2, \ldots, f_n) \\ - \langle f f_1 \rangle W(f_2, \ldots, f_n) \\ - W(f f_1, f_2, \ldots, f_n) \\ - \langle f \rangle W(f_1, f_2, \ldots, f_n). \end{aligned}$$

INTERPRETATIONS.

1) $\Gamma(\mathcal{A})$ has a natural representation on the full Fock space

$$\mathcal{F}\left(L^2(\mathcal{A}, \langle\cdot\rangle)\right).$$

Then $W(f_1, f_2, \ldots, f_n)$ is the Wick product

$$: X(f_1)X(f_2)\ldots X(f_n) : .$$

2) For $\mathcal{A} = L^\infty[0,1]$, define

$$W(t) = W(f_1\mathbf{1}_{[0,t)}, f_2\mathbf{1}_{[0,t)}, \ldots, f_n\mathbf{1}_{[0,t)}).$$

This is always a martingale, and can be represented as a multiple stochastic integral.

3) $W(f_1, f_2, \ldots, f_n)$ is a polynomial in

$$\left\{X(f_i), X(f_i f_j), X(f_i f_j f_k), \ldots\right\}.$$

QUESTIONS.

a) Write

$$X(f_1)X(f_2)\dots X(f_n) = \sum W(\dots).$$

b) Write

$$W(f_1, f_2, \dots, f_n) = \text{Polynomial}\ (X(f_i), X(f_i f_j), \dots).$$

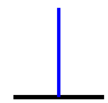
c) Write

$$\prod W(\dots) = \sum W(\dots).$$

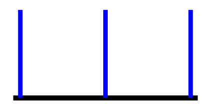
Interpretations: (1) Wick formula (2) Itô formula (3) linearization coefficient formula.

Use the language of diagrams.

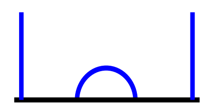
$$W(f_1) =$$



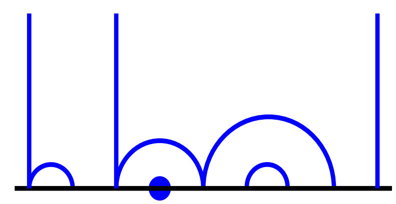
$$W(f_1, f_2, f_3) =$$



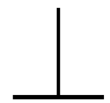
$$W(f_1, f_4)\,\langle f_2 f_3\rangle =$$



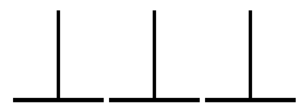
$$W(f_1 f_2, f_3 f_5 f_8, f_9)\,\langle f_4\rangle\,\langle f_6 f_7\rangle =$$



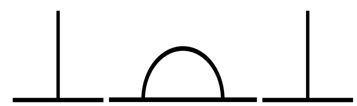
$$X(f_1) =$$



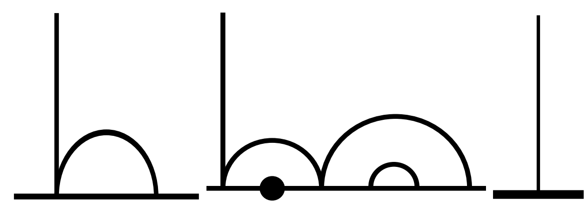
$$X(f_1)X(f_2)X(f_3) =$$



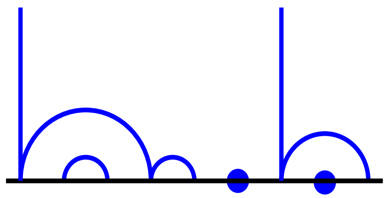
$$X(f_1)X(f_4)\langle f_2 f_3\rangle =$$



$$X(f_1 f_2)X(f_3 f_5 f_8)X(f_9)\langle f_4\rangle\langle f_6 f_7\rangle =$$



Conversely,



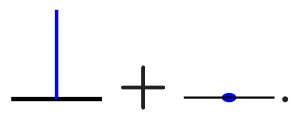
$$= W(f_1 f_4 f_5, f_7 f_9) \langle f_2 f_3 \rangle \langle f_6 \rangle \langle f_8 \rangle .$$

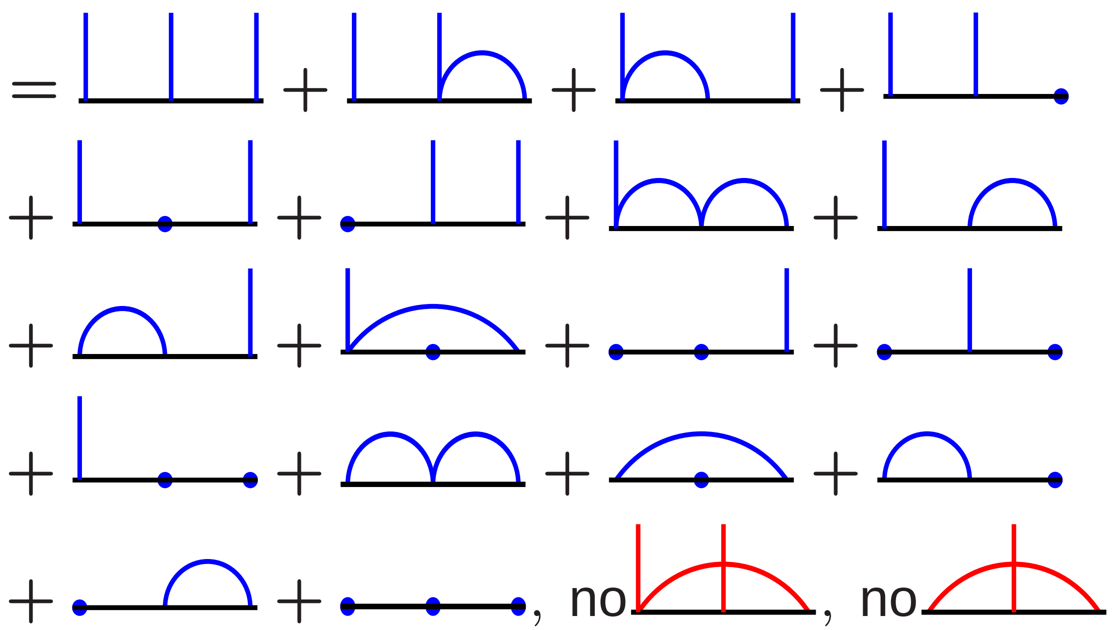
Connection to free probability: only non-crossing diagrams appear.

Extended partitions, have open and closed classes.

Answers.

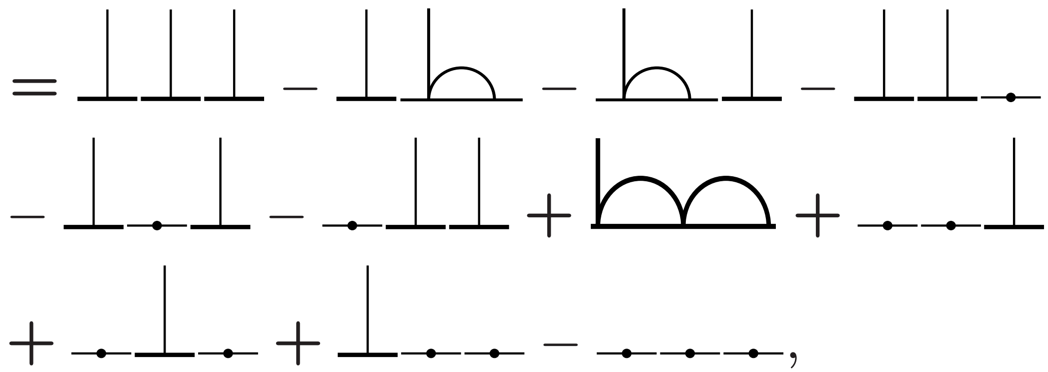
a) $$X(f_1)X(f_2)\dots X(f_n) = \sum_{\text{all diagrams } \pi} W_\pi.$$

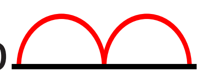
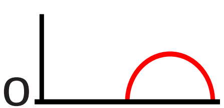
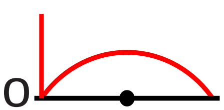
$X(f_1) =$ 

$X(f_1)X(f_2)X(f_3) =$ 

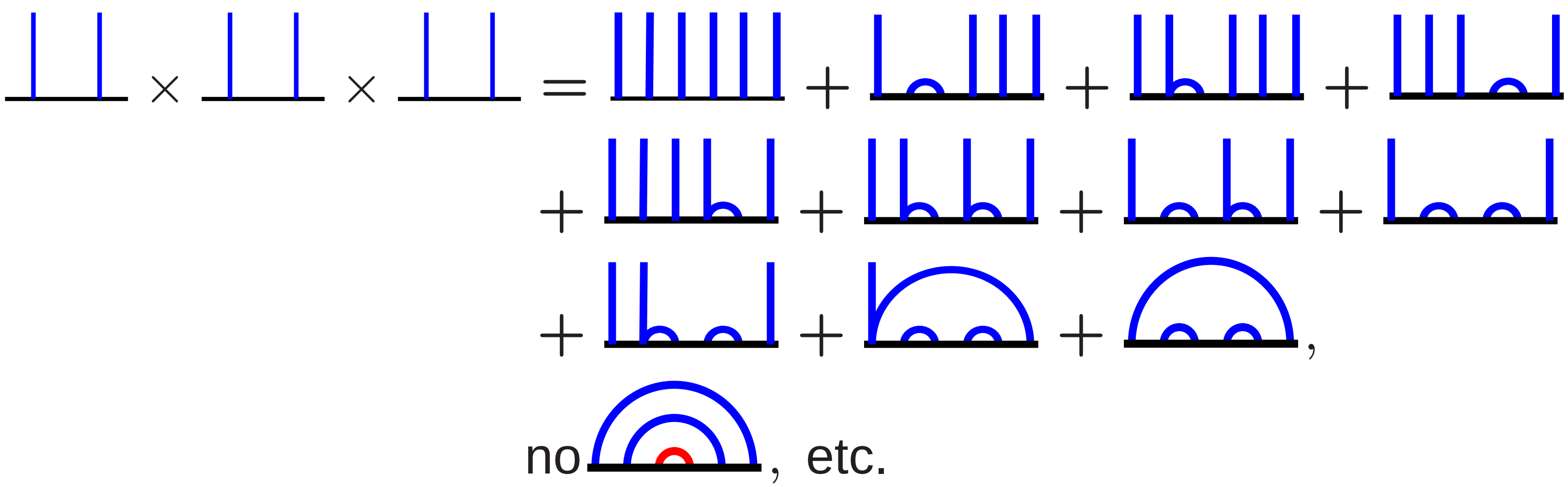
b) $W(f_1, f_2, \ldots, f_n) = \sum_{\substack{\text{interval diagrams} \\ \text{closed classes singletons}}} (-1)^{n-|\text{open classes}|} X_\pi.$

$W(f_1) =$ 

$W(f_1, f_2, f_3) =$ 

no , no , no , etc.

c) $\prod W(\ldots) = \sum_{\substack{\text{glue open classes of different components} \\ \text{in all possible non-crossing ways}}} W(\ldots).$

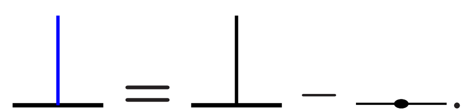


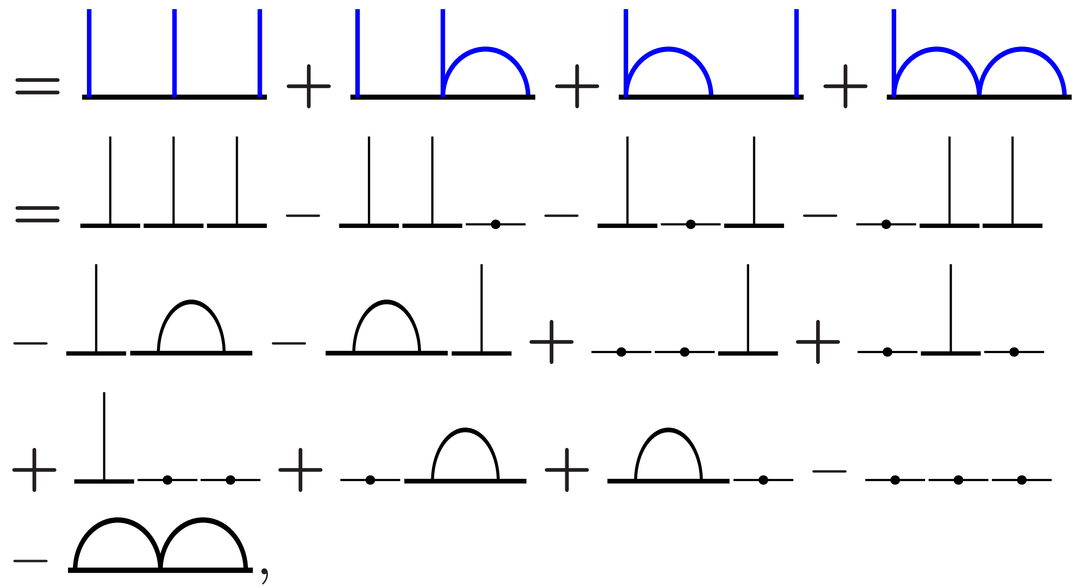
Recall: $W(f_1, f_2, \ldots, f_n)$ a polynomial in $\left\{X(f_i), X(f_i f_j), X(f_i f_j f_k), \ldots\right\}$. Even $W(f, f, \ldots, f)$ a polynomial in $\left\{X(f), X(f^2), X(f^3), \ldots\right\}$. Unless f a projection (Gaussian, Poisson case), multi-variate. However, define

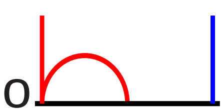
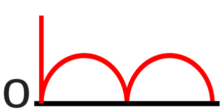
$$A(f_1, f_2, \ldots, f_n) = \sum_{\substack{\text{interval partitions} \\ \text{with open classes}}} W_\pi .$$

Then $A(f, f, \ldots, f)$ a polynomial in $X(f)$ only.

Free Appell polynomials.

$A(f_1) =$ 

$A(f_1, f_2, f_3) =$ 

no , no , etc.

Important connections to free probability:

a) If (say)

$$\{f_1, f_2\} \text{ are orthogonal to } \{f_3, f_4\},$$

then

$$\{X(f_1), X(f_2)\} \text{ are freely independent from } \{X(f_3), X(f_4)\},$$

and

$$A\Big(X(f_1), X(f_2), X(f_3), X(f_4)\Big) = A\Big(X(f_1), X(f_2)\Big) \cdot A\Big(X(f_3), X(f_4)\Big).$$

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b) $\partial_{X(f_2)} A\Big(X(f_1), X(f_2), X(f_3), X(f_4), X(f_5)\Big)$

$$= A\Big(X(f_1)\Big) \cdot A\Big(X(f_3), X(f_4), X(f_5)\Big),$$

where ∂_x is the free difference quotient.

c) Explicit generating function:

$$1 + \sum A(f_i)z_i + \sum A(f_i, f_j)z_i z_j + \ldots = \Big(1 - \sum X(f_i)z_i + R(\mathbf{z})\Big)^{-1},$$

where $R(\mathbf{z})$ is the free cumulant generating function of the joint distribution of $\{X(f_i)\}$.